数理解析研究所講究録
1136 巻 2000 年 45-55

Strong Convergence Theorems with Compact Domains

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ABSTRACT. In this paper, we prove a nonlinear strong ergodic theorem for nonexpansive mappings of a compact convex subset of a strictly convex Banach space into itself. Further, we prove a nonlinear strong ergodic theorem for a one-parameter nonexpansive semigroup.

1. INTRODUCTION

Let C be a nonempty closed convex subset of a real Banach space E. Then, a mapping $T: C\to C$ is called nonexpansive if $\|Tx-Ty\|\leq\|x-y\|$ for $x,y\in C$. We denote by $F(T)$ the set of fixed points of T. Let $\mathcal{S}=\{T(s):0\leq s<\infty\}$ be a family of nonexpansive mappings of C into itself such that $T(s+t)=T(s)T(t)$ for $s,t\in\mathbb{R}^{+}$, $t\mapsto T(t)x$ is continuous for each $x\in C$ and $T(0)=I$, where I is the identity mapping, which is called a one-parameter nonexpansive semigroup on C. Let $x\in C$. Then, for a nonexpansive mapping $T: C\to C$, the ω-limit set of x is defined by

$$\omega(x)=\{z\in C: z=\lim_{i\to\infty}T^{n_{i}}x \quad \text{with} \quad n_{i}\to\infty \quad \text{as} \quad i\to\infty\}.$$

Similarly, the ω-limit set of x for a one-parameter semigroup $\mathcal{S}$ on C is defined by

$$\omega(\mathcal{S},x)=\{z\in C: z=\lim_{i\to\infty}T(s_{i})x \quad \text{with} \quad s_{i}\to\infty \quad \text{as} \quad i\to\infty\}.$$

Edelstein [10] obtained the following nonlinear ergodic theorem for nonexpansive mappings with compact domains in a Banach space: Let C be a nonempty compact convex subset of a strictly convex Banach space and let T be a nonexpansive mapping of C into itself. Let $x\in C$. Then, for any $\xi\in\overline{co}\omega(x)$, the Cesàro mean $S_{n}(\xi)=(1/n)\sum_{k=0}^{n-1}T^{k}\xi$ converges strongly to some $y\in F(T)$, where $\overline{co}A$ is the closure of the convex hull of A. Dafermos and Slemrod [9] obtained the following theorem: Let C be a nonempty compact convex subset of a strictly convex Banach space and let $\mathcal{S}=\{T(t):0\leq t<\infty\}$ be a one-parameter nonexpansive semigroup on C. Let $x\in C$. Then, for any $\xi\in\overline{co}\omega(\mathcal{S},x)$, $(1/t)\int_{0}^{t}T(s)\xi ds$ converges strongly to some $y\in\bigcap_{0\leq t<\infty}F(T(t))$. On the other hand, the

2000 *Mathematics Subject Classification.* Primary 47H09, 47H10.
Key words and phrases. Nonlinear ergodic theorem, fixed point, nonexpansive mapping, strong convergence.

first nonlinear ergodic theorem for nonexpansive mappings with bounded domains was established in the framework of a Hilbert space by Baillon [5]: Let C be a nonempty bounded closed convex subset of a Hilbert space and let T be a nonexpansive mapping of C into itself. Then, for any $x \in C$, the Cesàro mean $S_n(x) = (1/n)\sum_{k=0}^{n-1} T^k x$ converges weakly to some $y \in F(T)$. Bruck [7] extended Baillon's theorem to a uniformly convex Banach space whose norm is Fréchet differentiable. Brézis and Browder [6] also proved a nonlinear strong ergodic theorem for nonexpansive mappings of odd-type in a Hilbert space (see also Reich [11]). In view of Edelstein's theorem, it is natural to ask the following question: For any $x \in C$, do the Cesàro mean $S_n(x)$ converges strongly to some $z \in F(T)$?

In this paper, we give an affirmative answer to the problem, that is, using Bruck [7, 8] and Atsushiba and Takahashi [1], we prove a nonlinear strong ergodic theorem for nonexpansive mappings of a compact convex subset of a strictly convex Banach space into itself. Further, we prove a nonlinear strong ergodic theorem for a one-parameter nonexpansive semigroup.

2. Strong Ergodic Theorem for nonexpansive mappings

Throughout the rest of this paper, we assume that a Banach space E is real and we denote by E^* the dual space of E. In addition, we denote by $\mathbb{R}^+$ and $\mathbb{N}$ the sets of all nonnegative real numbers and all positive integers, respectively. For a subset A of E, we denote by $\mathrm{co}A$ the convex hull of A.

A Banach space E is said to be strictly convex if $\|x+y\|/2 < 1$ for $x, y \in E$ with $\|x\| = \|y\| = 1$ and $x \neq y$. In a strictly convex Banach space, we have that if

$$\|x\| = \|y\| = \|(1-\lambda)x + \lambda y\|$$

for $x, y \in E$ and $\lambda \in (0,1)$, then $x = y$. Throughout the rest of this paper, we assume that E is a strictly convex Banach space.

In this section, we shall give a nonlinear strong ergodic theorem for nonexpansive mappings. First, we give two lemmas which play an important role in the proof (see also [3, 4, 7, 8]).

Lemma 2.1. Let C be a nonempty compact convex subset of E. Then,

$$\lim_{n\to\infty} \sup_{\substack{y \in C \\ T \in N(C)}} \left\| \frac{1}{n}\sum_{i=0}^{n-1} T^i y - T\left(\frac{1}{n}\sum_{i=0}^{n-1} T^i y\right) \right\| = 0,$$

where $N(C)$ denotes the set of all nonexpansive mappings of C into itself.

Lemma 2.2. Let C be a nonempty compact convex subset of E and let T be a nonexpansive mapping of C into itself. Let $x \in C$ and $n \in \mathbb{N}$. Then, for any $\varepsilon > 0$, there exists $l_0 = l_0(n, \varepsilon) \in \mathbb{N}$ such that

$$\sup_{k \in \mathbb{N}} \left\| \frac{1}{n} \sum_{l=0}^{n-1} T^{l+k+m} x - T^k \left(\frac{1}{n} \sum_{l=0}^{n-1} T^{l+m} x \right) \right\| < \varepsilon$$

for every $m \geq l_0$.

Using Lemma 2.2, we can prove the following lemma (see [3]).

Lemma 2.3. Let C be a nonempty compact convex subset of E and let T be a nonexpansive mapping of C into itself. Let $x \in C$. Then, there exists a sequence $\{i_n\}$ in $\mathbb{N}$ such that for each $z \in F(T)$,

$$\lim_{n \to \infty} \left\| \frac{1}{n} \sum_{j=0}^{n-1} T^{j+i_n} x - z \right\|$$

exists.

Sketch of the proof of Lemma 2.3. From [7], we have, for any $n, m \in \mathbb{N}$

$$\begin{aligned} &\frac{1}{m} \sum_{j=0}^{m-1} T^{j+i_m+i_n} x \\ &= \frac{1}{mn} \sum_{j=1}^{n-1} (n-j)(T^{j+i_m+i_n-1} x - T^{j+i_m+i_n+m-1} x) + \frac{1}{m} \sum_{j=0}^{m-1} \frac{1}{n} \sum_{h=0}^{n-1} T^{j+h+i_m+i_n} x. \end{aligned} \tag{1}$$

Fix $z \in F(T)$. From (1) and Lemma 2.2, we obtain

$$\begin{aligned} &\left\| \frac{1}{m} \sum_{j=0}^{m-1} T^{j+i_m+i_n} x - z \right\| \\ &\leq \left\| \frac{1}{mn} \sum_{j=1}^{n-1} (n-j)(T^{j+i_m+i_n-1} x - T^{j+i_m+i_n+m-1} x) \right\| \\ &\quad + \left\| \frac{1}{m} \sum_{j=0}^{m-1} \frac{1}{n} \sum_{h=0}^{n-1} T^{h+j+i_m+i_n} x - \frac{1}{m} \sum_{j=0}^{m-1} T^{j+i_m} \left(\frac{1}{n} \sum_{h=0}^{n-1} T^{h+i_n} x \right) \right\| \\ &\quad + \left\| \frac{1}{m} \sum_{j=0}^{m-1} T^{j+i_m} \left(\frac{1}{n} \sum_{h=0}^{n-1} T^{h+i_n} x \right) - z \right\| \\ &\leq \frac{1}{nm} \sum_{j=1}^{n-1} (n-j) \cdot 2M + \varepsilon + \left\| \frac{1}{n} \sum_{h=0}^{n-1} T^{h+i_n} x - z \right\| \leq \frac{Mn}{m} + \varepsilon + \left\| \frac{1}{n} \sum_{h=0}^{n-1} T^{h+i_n} x - z \right\|, \end{aligned}$$

where $M = \sup\{\|T^j x\| : j \in \mathbb{N} \cup \{0\}\}$. Therefore, we have

$$\overline{\lim_{m\to\infty}} \left\| \frac{1}{m} \sum_{j=0}^{m-1} T^{j+i_m} x - z \right\| = \overline{\lim_{m\to\infty}} \left\| \frac{1}{m} \sum_{j=0}^{m-1} T^{j+i_m+i_n} x - z \right\| \le \varepsilon + \left\| \frac{1}{n} \sum_{h=0}^{n-1} T^{h+i_n} x - z \right\|.$$

Then, we can show that

$$\lim_{n\to\infty} \left\| \frac{1}{n} \sum_{j=0}^{n-1} T^{j+i_n} x - z \right\|$$

exists. □

Remark 2.4. In Lemma 2.3, take a sequence $\{i_n'\}$ in $\mathbb{N}$ such that $i_n' \ge i_n$ for each $n \in \mathbb{N}$. Then, we can see that

$$\lim_{n\to\infty} \left\| \frac{1}{n} \sum_{j=0}^{n-1} T^{j+i_n} x - z \right\| = \lim_{n\to\infty} \left\| \frac{1}{n} \sum_{j=0}^{n-1} T^{j+i_n'} x - z \right\|.$$

for every $z \in F(T)$.

Now, we can show a nonlinear strong ergodic theorem for nonexpansive mappings (see [3]).

Theorem 2.5. Let E be a strictly convex Banach space and let D be a nonempty closed convex subset of E. Let T be a nonexpansive mapping of D into itself such that $T(D) \subset K$ for some compact subset K of D and let $x \in D$. Then, $(1/n)\sum_{i=0}^{n-1} T^{i+h} x$ converges strongly to a fixed point of T uniformly in $h \in \mathbb{N} \cup \{0\}$. In this case, if $Qx = \lim_{n\to\infty}(1/n)\sum_{i=0}^{n-1} T^i x$ for each $x \in D$, then Q is a nonexpansive mapping of D onto $F(T)$ such that $QT^k = T^k Q = Q$ for every $k \in \mathbb{N}$ and $Qx \in \overline{\mathrm{co}}\{T^k x : k \in \mathbb{N}\}$ for every $x \in D$.

Sketch of the proof of Theorem 2.5. From Mazur's theorem, $C = \overline{\mathrm{co}}(\{x\} \cup T(D))$ is a compact subset of D. We see that $C = \overline{\mathrm{co}}(\{x\} \cup T(D))$ is convex and invariant under T and contains $\overline{\mathrm{co}}\{T^k x : k \in \mathbb{N} \cup \{0\}\}$. Thus, we may assume that T is a nonexpansive mapping of a compact convex subset of D into itself.

From Lemma 2.3, there exists a sequence $\{i_n\}$ in $\mathbb{N}$ such that for each $z \in F(T)$,

$$\lim_{n\to\infty} \left\| \frac{1}{n} \sum_{j=0}^{n-1} T^{j+i_n} x - z \right\| \tag{2}$$

exists. From Lemma 2.1, we have

$$\lim_{n\to\infty} \left\| \frac{1}{n} \sum_{j=0}^{n-1} T^{j+i_n} x - T\left(\frac{1}{n} \sum_{j=0}^{n-1} T^{j+i_n} x \right) \right\| = 0. \tag{3}$$

Let $\{\Phi_n\} = \left\{(1/n)\sum_{j=0}^{n-1} T^{j+i_n}x\right\}$. From the compactness, $\{\Phi_n\}$ must contain a subsequence which converges strongly to a point in C. So, let $\{\Phi_{n_k}\}$ be a subsequence of $\{\Phi_n\}$ such that $\lim_{k\to\infty}\Phi_{n_k} = y_0$. From (3), we see that y_0 is a fixed point of T. From (2), we have $\Phi_n \to y_0$. In the above argument, take a sequence $\{i_n{}'\}$ in $\mathbb{N}$ such that $i_n{}' \geq i_n$ for each $n \in \mathbb{N}$. Then, repeating the above argument, we see that $\Phi_n{}' = (1/n)\sum_{j=0}^{n-1} T^{j+i_n{}'}x$ converges strongly to some $y_1 \in F(T)$. From Remark 2.4, we can show $y_0 = y_1$. Since $\{i_n{}'\}$ is any sequence in $\mathbb{N}$ such that $i_n{}' \geq i_n$ for each $n \in \mathbb{N}$, we see that $(1/n)\sum_{j=0}^{n-1} T^{j+h+i_n}x$ converges strongly to y_0 uniformly in $h \in \mathbb{N} \cup \{0\}$. Then, using an idea of (1), we can prove that $(1/n)\sum_{j=0}^{n-1} T^{j+h}x$ converges strongly to y_0 uniformly in $h \in \mathbb{N} \cup \{0\}$. If $Qx = \lim_{n\to\infty}(1/n)\sum_{i=0}^{n-1} T^i x$ for each $x \in D$, then Q is a nonexpansive mapping of D onto $F(T)$ such that $QT^k = T^kQ = Q$ for every $k \in \mathbb{N}$ and $Qx \in \overline{\mathrm{co}}\{T^k x : k \in \mathbb{N}\}$ for every $x \in D$ (for example, see [12, 13]). □

We also obtain the following corollary.

Corollary 2.6. Let E, C, T and x be as in Theorem 2.5. Then, $\{T^n x : n \in \mathbb{N}\}$ is strongly convergent if and only if

$$T^{n+1}x - T^n x \to 0.$$

In this case, the limit point of $\{T^n x : n \in \mathbb{N}\}$ is a fixed point of T.

3. Strong ergodic theorem for a one-parameter nonexpansive semigroup

A family $\mathcal{S} = \{T(s) : 0 \leq s < \infty\}$ of mappings of C into itself is called a one-parameter nonexpansive semigroup on C if it satisfies the following conditions:

(i) $T(0)x = x$ for all $x \in C$;
(ii) $T(s+t) = T(s)T(t)$ for all $s, t \in \mathbb{R}^+$;
(iii) $\|T(s)x - T(s)y\| \leq \|x - y\|$ for all $x, y \in C$ and $s \in \mathbb{R}^+$;
(iv) for each $x \in C$, $s \mapsto T(s)x$ is continuous.

We denote by $F(\mathcal{S})$ the set of common fixed points of $T(t), t \in \mathbb{R}^+$, that is, $F(\mathcal{S}) = \bigcap_{0\leq t<\infty} F(T(t))$.

In this section, we give a strong ergodic theorem for a one-parameter nonexpansive semigroup. For a compact subset of a strictly convex Banach space, we obtained the following two lemmas (see [3]):

Lemma 3.1. Let C be a nonempty compact convex subset of E and let $n \in \mathbb{N}$. Then, there exists a strictly increasing continuous, convex function $\gamma_n : \mathbb{R}^+ \to \mathbb{R}^+$ such that

$\gamma_n(0) = 0$ and

$$\gamma_n\left(\left\|\sum_{i=1}^{n}\lambda_i Ty_i - T\left(\sum_{i=1}^{n}\lambda_i y_i\right)\right\|\right) \leq \max_{1\leq i,j\leq n}(\|y_i - y_j\| - \|Ty_i - Ty_j\|)$$

for every nonexpansive mapping T of C into itself, every sequence $\{\lambda_i\}_{i=1}^n$ in $\mathbb{R}^+$ with $\sum_{i=1}^n \lambda_i = 1$ and $\{y_i\}_{i=1}^n$ in C.

Lemma 3.2. Let C be a nonempty compact convex subset of E. For any $\varepsilon > 0$, there exists $\delta > 0$ such that for any nonexpansive mapping T of C into itself,

$$\overline{\text{co}}F_\delta(T) \subset F_\varepsilon(T).$$

Using Lemmas 2.1 and 3.2, we obtain the following lemma (see [2, 4]).

Lemma 3.3. Let C be a nonempty compact convex subset of E and let $\mathcal{S} = \{T(t) : 0 \leq t < \infty\}$ be a one-parameter nonexpansive semigroup on C. Then, for any $h \in \mathbb{R}^+$,

$$\lim_{t\to\infty}\sup_{y\in C}\left\|\frac{1}{t}\int_0^t T(s)yds - T(h)\left(\frac{1}{t}\int_0^t T(s)yds\right)\right\| = 0.$$

Sketch of the proof of Lemma 3.3. Let $\varepsilon > 0$ and $h \in \mathbb{R}^+$. From Lemma 3.2, there exists $\delta > 0$ such that $\overline{\text{co}}F_\delta(T) \subset F_\varepsilon(T)$ for every nonexpansive mapping T of C into itself. From Lemma 2.1, there exits $n_1 \in \mathbb{N}$ such that

$$\sup_{\substack{y\in C\\ s\in\mathbb{R}^+}}\left\|\frac{1}{n}\sum_{i=0}^{n-1}T(hi+s)y - T(h)\left(\frac{1}{n}\sum_{i=0}^{n-1}T(hi+s)y\right)\right\| < \delta$$

for every $n \geq n_1$. Then, we obtain

$$\frac{1}{n}\sum_{i=0}^{n-1}T(hi+s)y \in F_\delta(T(h)) \subset \overline{\text{co}}F_\delta(T(h)) \tag{4}$$

for every $s \in \mathbb{R}^+, n \geq n_1$ and $y \in C$. Let $n \geq n_1$. Then, we have that for any $t \in \mathbb{R}^+$ with $t > h(n-1)$ and $y \in C$,

$$\begin{aligned}
&\left\|\frac{1}{t}\int_0^t T(s)yds - T(h)\left(\frac{1}{t}\int_0^t T(s)yds\right)\right\| \\
&\leq \frac{2}{n}\sum_{i=0}^{n-1}\left\|\frac{1}{t}\int_0^t T(s)yds - \frac{1}{t}\int_0^t T(hi+s)yds\right\| \\
&\quad + \left\|\frac{1}{n}\sum_{i=0}^{n-1}\frac{1}{t}\int_0^t T(hi+s)yds - T(h)\left(\frac{1}{n}\sum_{i=0}^{n-1}\frac{1}{t}\int_0^t T(hi+s)yds\right)\right\|
\end{aligned}$$

and

$$\frac{1}{n}\sum_{i=0}^{n-1}\left\|\frac{1}{t}\int_0^t T(s)yds - \frac{1}{t}\int_{hi}^{t+hi} T(s)yds\right\| \leq \frac{M_0 h(n-1)}{t},$$

where $M_0 = \sup_{z\in C}\|z\|$. Using (4) and the separation theorem, we can prove that there exists $t_0 \in \mathbb{R}^+$ with $t_0 > h(n-1)$ such that $\frac{1}{n}\sum_{i=0}^{n-1}\frac{1}{t}\int_0^t T(hi+s)yds \in \overline{\text{co}}F_\delta(T(h))$ for all $y \in C$ and $t \geq t_0$. From $\overline{\text{co}}F_\delta(T(h)) \subset F_\varepsilon(T(h))$, we have

$$\left\|\frac{1}{t}\int_0^t T(s)yds - T(h)\left(\frac{1}{t}\int_0^t T(s)yds\right)\right\| \leq \frac{2M_0 h(n-1)}{t} + \varepsilon$$

for $t \geq t_0$. Since $y \in C$ is arbitrary, we have

$$\lim_{t\to\infty}\sup_{y\in C}\left\|\frac{1}{t}\int_0^t T(s)yds - T(h)\left(\frac{1}{t}\int_0^t T(s)yds\right)\right\| = 0. \qquad \square$$

Lemma 3.4. Let C be a nonempty compact convex subset of E and let $\mathcal{S} = \{T(s) : 0 \leq s < \infty\}$ be a one-parameter nonexpansive semigroup on C. Let $x \in C$ and $t > 0$. Then, for any $\varepsilon > 0$, there exists $p_t = p_t(\varepsilon) \in \mathbb{R}^+$ such that

$$\sup_{h\in\mathbb{R}^+}\left\|\frac{1}{t}\int_0^t T(h+p+\tau)xd\tau - T(h)\left(\frac{1}{t}\int_0^t T(p+\tau)xd\tau\right)\right\| < \varepsilon$$

for every $p \geq p_t$.

Sketch of the proof of Lemma 3.4. Let $t > 0$ and $\varepsilon > 0$. We know that there exists $\delta_1 = \delta_1(\varepsilon) > 0$ such that $\|T(s_1)x - T(s_2)x\| < \varepsilon/3$ if $|s_1 - s_2| \leq \delta_1$. Choose $N = N(t,\varepsilon) \in \mathbb{N}$ such that $N > t/\delta_1$ and $\left\|\frac{1}{t}\int_0^t T(\tau)xd\tau - \frac{1}{t}\frac{t}{N}\sum_{i=1}^N T\left(\frac{it}{N}\right)x\right\| < \frac{\varepsilon}{3}$. Then, we can show, for each $h, p \in \mathbb{R}^+$,

$$\left\|\frac{1}{t}\int_0^t T(h+p+\tau)xd\tau - \frac{1}{t}\frac{t}{N}\sum_{i=1}^N T\left(h+p+\frac{it}{N}\right)x\right\| < \frac{\varepsilon}{3}. \tag{5}$$

Hence, for each $p \in \mathbb{R}^+$,

$$\left\|\frac{1}{t}\int_0^t T(p+\tau)xd\tau - \frac{1}{t}\frac{t}{N}\sum_{i=1}^N T\left(p+\frac{it}{N}\right)x\right\| < \frac{\varepsilon}{3}. \tag{6}$$

We see that for each $i, j \in \{1, 2, \dots, N\}$,

$$\lim_{s\to\infty}\left\|T\left(s+\frac{it}{N}\right)x - T\left(s+\frac{jt}{N}\right)x\right\| = \lim_{s\to\infty}\left\|T(s)T\left(\frac{t}{N}\right)^i x - T(s)T\left(\frac{t}{N}\right)^j x\right\|$$

exists. Let γ_N be as in Lemma 3.1. Since γ_N^{-1} is continuous and $\gamma_N^{-1}(0) = 0$, there exists $\delta_2 = \delta_2(\varepsilon) > 0$ such that $\gamma_N^{-1}(\delta) < \varepsilon/3$ for every δ with $0 \le \delta \le \delta_2$. Then, there exists $p_1 = p_1(\varepsilon, i, j, t) \in \mathbb{R}^+$ such that

$$0 \le \left\| T\Big(s + \frac{it}{N}\Big)x - T\Big(s + \frac{jt}{N}\Big)x \right\| - \left\| T\Big(q + s + \frac{it}{N}\Big)x - T\Big(q + s + \frac{jt}{N}\Big)x \right\| < \delta_2$$

for every $s \ge p_1$ and $q \in \mathbb{R}^+$. Let $p_t = \max\{p_1(\varepsilon, i, j, t) : 1 \le i, j \le N\}$. It follows from Lemma 3.1 that

$$\begin{aligned} &\left\| \frac{1}{N} \sum_{i=1}^{N} T(h) T\Big(p + \frac{it}{N}\Big)x - T(h) \left(\frac{1}{N} \sum_{i=1}^{N} T\Big(p + \frac{it}{N}\Big)x \right) \right\| \\ &\le \gamma_N^{-1} \left(\max_{1 \le i,j \le N} \left(\left\| T\Big(p+\frac{it}{N}\Big)x - T\Big(p+\frac{jt}{N}\Big)x \right\| - \left\| T\Big(h+p+\frac{it}{N}\Big)x - T\Big(h+p+\frac{jt}{N}\Big)x \right\| \right) \right) \\ &< \gamma_N^{-1}(\delta_2) < \frac{\varepsilon}{3} \end{aligned} \tag{7}$$

for every $i, j \in \{1, 2, \dots, N\}$, $h \in \mathbb{R}^+$ and $p \ge p_t$. Therefore, from (5), (6) and (7), we have

$$\left\| \frac{1}{t} \int_0^t T(h + p + \tau)x d\tau - T(h) \left(\frac{1}{t} \int_0^t T(p + \tau)x d\tau \right) \right\| < 3 \cdot \frac{\varepsilon}{3} = \varepsilon$$

for every $h \in \mathbb{R}^+$ and $p \ge p_t$. □

Using Lemma 3.4, we can show the following lemma (see [4]).

Lemma 3.5. Let C be a nonempty compact convex subset of E and let $\mathcal{S} = \{T(s) : 0 \le s < \infty\}$ be a one-parameter nonexpansive semigroup on C. Let $x \in C$. Then, there exists a net $\{p_t\}$ in $\mathbb{R}^+$ such that for each $z \in F(\mathcal{S})$,

$$\lim_{t \to \infty} \left\| \frac{1}{t} \int_0^t T(\tau + p_t)x d\tau - z \right\|$$

exists.

Sketch of the proof of Lemma 3.5. Let $\varepsilon > 0$. From Lemma 3.4, for any $t > 0$, there exists $p_t \in \mathbb{R}^+$ such that

$$\left\| T(h) \left(\frac{1}{t} \int_0^t T(p + \tau)x d\tau \right) - \frac{1}{t} \int_0^t T(h + p + \tau)x d\tau \right\| < \varepsilon \tag{8}$$

for every $p \geq p_t$ and $h \in \mathbb{R}^+$. From an idea of [7], we have, for any $t, s > 0$,

$$\frac{1}{t}\int_0^t T(\tau+p_t+p_s)xd\tau \tag{9}$$
$$= \frac{1}{st}\int_0^s (s-\eta)[T(\eta+p_t+p_s)x - T(\eta+p_t+p_s+t)x]d\eta + \frac{1}{t}\int_0^t \left(\frac{1}{s}\int_0^s T(\tau+\eta+p_t+p_s)xd\eta\right)d\tau.$$

Fix $z \in F(\mathcal{S})$ and $t, s > 0$. Put $M_0 = \sup\{\|v\| : v \in C\}$. Then, we have

$$\left\|\frac{1}{st}\int_0^s (s-\eta)[T(\eta+p_t+p_s)x - T(\eta+p_t+p_s+t)x]\,d\eta\right\| \leq \frac{2M_0}{st}\int_0^s ((s-\eta)d\eta \leq \frac{M_0 s}{t}. \tag{10}$$

From (8), we have, for $t > 0$ with $t \geq p_s$,

$$\left\|\frac{1}{t}\int_0^t \left(\frac{1}{s}\int_0^s T(\tau+\eta+p_t+p_s)xd\eta - z\right)d\tau\right\|$$
$$\leq \left\|\frac{1}{t}\int_0^t \left(\frac{1}{s}\int_0^s T(\tau+p_t+\eta+p_s)xd\eta\right)d\tau - \frac{1}{t}\int_0^t T(\tau+p_t)\left(\frac{1}{s}\int_0^s T(\eta+p_s)xd\eta\right)d\tau\right\|$$
$$+ \left\|\frac{1}{t}\int_0^t \left(T(\tau+p_t)\left(\frac{1}{s}\int_0^s T(\eta+p_s)xd\eta\right) - z\right)d\tau\right\|$$
$$< \varepsilon + \left\|\frac{1}{s}\int_0^s T(\eta+p_s)xd\eta - z\right\|. \tag{11}$$

Hence, from (9),(10) and (11), we have

$$\overline{\lim_{t\to\infty}}\left\|\frac{1}{t}\int_0^t T(\tau+p_t)xd\tau - z\right\| = \overline{\lim_{t\to\infty}}\left\|\frac{1}{t}\int_0^t T(\tau+p_t+p_s)xd\tau - z\right\|$$
$$\leq \varepsilon + \left\|\frac{1}{s}\int_0^s T(\eta+p_s)xd\eta - z\right\|.$$

Then, we can show that

$$\lim_{t\to\infty}\left\|\frac{1}{t}\int_0^t T(\tau+p_t)xd\tau - z\right\|$$

exists for each $z \in F(\mathcal{S})$. □

Remark 3.6. In Lemma 2.3, take a net $\{p_t'\}$ in $\mathbb{R}^+$ such that $p_t' \geq p_t$ for each $t > 0$. Then, we can see

$$\lim_{t\to\infty}\left\|\frac{1}{t}\int_0^t T(\tau+p_t)xd\tau - z\right\| = \lim_{t\to\infty}\left\|\frac{1}{t}\int_0^t T(\tau+p_t')xd\tau - z\right\|$$

for every $z \in F(\mathcal{S})$.

Now, we can shgow a nonlinear strong ergodic theorem for a one-parameter nonexpansive semigroup (see [4]).

Theorem 3.7. Let E be a strictly convex Banach space and let C be a nonempty compact convex subset of E. Let $\mathcal{S} = \{T(t) : 0 \le t < \infty\}$ be a one-parameter nonexpansive semigroup on C and let $x \in C$. Then, $(1/t)\int_0^t T(\tau + h)xd\tau$ converges strongly to a common fixed point of $T(t), t \in \mathbb{R}^+$ uniformly in $h \in \mathbb{R}^+$. In this case, if $Qx = \lim_{t\to\infty}(1/t)\int_0^t T(\tau)xd\tau$ for each $x \in C$, then Q is a nonexpansive mapping of C onto $F(\mathcal{S})$ such that $QT(q) = T(q)Q = Q$ for every $q \in \mathbb{R}^+$ and $Qx \in \overline{\text{co}}\{T(s)x : 0 \le s < \infty\}$ for every $x \in C$.

Sketch of the proof of Theorem 3.7. From Lemma 3.5, there exists a net $\{p_t\}$ in $\mathbb{R}^+$ such that for each $z \in F(\mathcal{S})$,

$$\lim_{t\to\infty}\left\|\frac{1}{t}\int_0^t T(\tau + p_t)xd\tau - z\right\| \tag{12}$$

exists. From Lemma 3.3, we have, for any $q \in \mathbb{R}^+$,

$$\lim_{t\to\infty}\sup_{y\in C}\left\|\frac{1}{t}\int_0^t T(\tau + p_t)yd\tau - T(q)\left(\int_0^t T(\tau + p_t)yd\tau\right)\right\| = 0. \tag{13}$$

Let $\{\Phi_t\} = \{(1/t)\int_0^t T(\tau + p_t)xd\tau\}$. From compactness of C, $\{\Phi_t\}$ must contain a subnet which converges strongly to a point in C. So, let $\{\Phi_{t_\alpha}\}$ be a subnet of $\{\Phi_t\}$ such that $\lim_\alpha \Phi_{t_\alpha} = y_0 \in C$. From (13), we can show that y_0 is a common fixed point of $T(t), t \in \mathbb{R}^+$. From (12), we can prove that $\Phi_t \to y_0 \in F(\mathcal{S})$. In the above argument, take a net $\{p_t'\}$ in $\mathbb{R}^+$ such that $p_t' \ge p_t$ for each $t > 0$. Then, repeating the above argument, we see that $\Phi_t' = (1/t)\int_0^t T(\tau + p_t')xd\tau$ converges strongly to some $y_1 \in F(\mathcal{S})$. Using Remark 3.6, we can show $y_0 = y_1$. Since $\{p_t'\}$ is any net in $\mathbb{R}^+$ such that $p_t' \ge p_t$ for each $t > 0$, we see that $(1/t)\int_0^t T(\tau + p_t + h)xd\tau$ converges strongly to y_0 uniformly in $h \in \mathbb{R}^+$. Then, using an idea of (9), we can prove that $(1/t)\int_0^t T(\tau + h)xd\tau$ converges strongly to y_0 uniformly in $h \in \mathbb{R}^+$. If $Qx = \lim_{t\to\infty}(1/t)\int_0^t T(\tau)xd\tau$ for each $x \in C$, then Q is a nonexpansive mapping of C onto $F(\mathcal{S})$ such that $QT(q) = T(q)Q = Q$ for every $q \in \mathbb{R}^+$ and $Qx \in \overline{\text{co}}\{T(s)x : 0 \le s < \infty\}$ for every $x \in C$. □

We also obtain the following corollary.

Corollary 3.8. Let E, C, x and $\mathcal{S} = \{T(t) : 0 \le t < \infty\}$ be as in Theorem 3.7. Then, $\{T(t)x : 0 \le t < \infty\}$ is strongly convergent if and only if

$$T(s + t)x - T(t)x \to 0 \quad \text{for every } s \in \mathbb{R}^+.$$

In this case, the limit point of $\{T(t)x : 0 \le t < \infty\}$ is a common fixed point of $T(t), t \in \mathbb{R}^+$.

References

1. S. Atsushiba and W. Takahashi, *Nonlinear ergodic theorems in a Banach space satisfying Opial's condition*, Tokyo J. Math., **21** (1998), 61–81.
2. S. Atsushiba and W. Takahashi, *A weak convergence theorem for nonexpansive semigroups by the Mann iteration process in Banach spaces,* Proceedings of the International Conference on Nonlinear Analysis and Convex Analysis (W. Takahashi and T. Tanaka, Eds.), World Scientific publishers, 1999, 102–109.
3. S. Atsushiba and W. Takahashi, *A nonlinear strong ergodic theorem for nonexpansive mappings with compact domains*, to appear in Math. Japon..
4. S. Atsushiba and W. Takahashi, *Strong convergence theorems for one-parameter nonexpansive semigroups with compact domains,* to appear in Nonlinear Analysis and Its Applications, (S.P. Singh and Bruce Watson, Eds.), Marcel Dekker.
5. J. B. Baillon, *Un théorème de type ergodique pour les contractions non linéaires dans un espace de Hilbert,* C. R. Acad. Sci. Paris Sér. A-B, **280** (1975), 1511–1514.
6. H. Brézis and F. E. Browder, *Nonlinear ergodic theorems,* Bull. Amer. Math. Soc., **82** (1976), 959–961.
7. R. E. Bruck, *A simple proof of the mean ergodic theorem for nonlinear contractions in Banach spaces*, Israel J. Math., **32** (1979), 107–116.
8. R. E. Bruck, *On the convex approximation property and the asymptotic behavior of nonlinear contractions in Banach spaces*, Israel J. Math., **38** (1981), 304–314.
9. C. M. Dafermos and M. Slemrod, Asymptotic behavior of nonlinear contraction semigroups, J. Funct. Anal., **13** (1973), 97–106.
10. M. Edelstein, On non-expansive mappings of Banach spaces, Proc. Camb. Phil. Soc., **60** (1964), 439–447.
11. S. Reich, *Almost convergence and nonlinear ergodic theorems,* J. Approx. Theory, **24** (1978), 269–272.
12. W. Takahashi, *Fixed point theorems for families of nonexpansive mappings on unbounded sets,* J. Math. Soc. Japan, **36** (1984), 543–553.
13. W. Takahashi, *Nonlinear Functional Analysis,* Kindai-kagakusha, Tokyo, 1988 (Japanese).

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