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Variational principles in Banach spaces and their parametrizations

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Ekeland's variational principle and its smooth analogues are now classical tools for investigations of many non-linear problems in various areas in mathematics (see for instance [8], [9], [1], [2], [5], [6]).

In this paper we present parametric versions of the Ekeland variational principle [8], [9], [1], stating that the minimum point of the perturbed function, under some conditions, can be chosen to depend continuously on a parameter. We introduce a new smooth variational principle involving bump functions, called here modified smooth variational principle, which unifies Borwein-Preiss' variational principle [2] and Deville-Godefroy-Zizler's variational principle [5] (concerning only existence of arbitrarily small smooth perturbations producing a point of minimum of the perturbed function). We present also a parametric variant of this principle.

The tool for proving the parametric analogue of the Ekeland variational principle is a parametric version of a Phelps' lemma [18]. This parametric version produces 'extremal selections': this is, in fact, a selection theorem for the efficient points set of images of a continuous mapping with respect to a convex closed pointed cone. As a corollary we prove existence of a continuous selection of the support points of a closed convex bounded set depending continuously (in the Hausdorff sense) on a parameter (existence of such support points is garanteed by Bishop-Phelps' theorem [18]).

As an application of this parametric Ekeland's variational principle we present an analogue of Ekeland's variational principle for minimax problems, which can be considered as a minimax variational principle.

We present some applications of the parametric modified smooth variational principle: the first one shows existence of a continuous selection of a subdifferential mapping depending on a parameter. The second application is about existence of a Nash equilibrium for convex functions after smooth convex perturbations, when one of the sets forming the domain of the involved functions is not compact. When $n = 2$ this theorem is a 'perturbed' version of Sion's [19] minimax theorem, showing that the perturbed function has a saddle point. As a third application we present a very easy proof of a variant of Ky Fan's inequality, in which smooth convex perturbations are involved.

An advantage of these smooth perturbations is the possibility to write second order optimality conditions, when the norm of the space is second order Fréchet differentiable (off 0).

We recall the following definitions.

A multivalued mapping $F : T \to M$, where T is a topological space and (M, d) is a metric space is said to be Hausdorff upper semicontinuous (resp. Hausdorff lower semicontinuous) at x_0, if for every $\varepsilon > 0$ there exists and open set $U \ni x_0$ such that $F(x) \subset \{z \in M : dist(z, F(x_0)) < \varepsilon\}$ (resp. $F(x_0) \subset \{z \in M : dist(z, F(x)) < \varepsilon\}$) for every $x \in U$, where $dist(., X)$ is the distance function to the set X. F is said to be Hausdorff continuous at x_0, if it is Hausdorff upper and Hausdorff lower semicontinuous at x_0.

F is said to be upper (resp. lower) semicontinuous at x_0, if for every open $V \supset F(x_0)$ (resp. every open V with $V \cap F(x_0) \neq \emptyset$) there exists an open $U \ni x_0$ such that $F(x) \subset V$ (resp. $F(x) \cap V \neq \emptyset$) for every $x \in U$.

Firstly we present a parametric version of the Phelps lemma [18], which is of independent interest, because it is a selection theorem for a multivalued mapping with non-convex images.

Let C be a closed, convex cone in a Banach space $(E, \|.\|)$. We shall say that C is a *strongly pointed cone*, if there exists $l \in S^*$, such that $\sup l(C) = 0$ and

$$c_n \to 0 \quad \text{whenever} \quad \{c_n\} \subset C \quad \text{and} \quad l(c_n) \to 0. \tag{1}$$

Recall that the set of all *weakly efficient points* of a set $Z \subset E$ with respect to C is

$$WEP_C(Z) = \{z \in Z : int(z + C) \cap Z = \emptyset\};$$

the set of all *efficient points* of Z is

$$EP_C(Z) = \{z \in Z : (z + C) \cap Z = \{z\}\}.$$

Define the set of all *strongly efficient points* of a set $Z \subset E$ with respect to C by

$$SEP_C(Z) = \{ \ y \in Z : (y + C) \cap Z = \{y\} \text{ and } x_n \to y \text{ whenever } \{x_n\} \subset (y + C) \text{ and } dist(x_n, Z) \to 0\}.$$

We shall say that the set $Z \subset E$ is *strongly bounded* with respect to C if there exist $z \in Z$ and $\varepsilon > 0$ such that the set $(z + C) \cap (Z + \varepsilon B)$ is bounded.

The proof of the following proposition is an interesting exercise, left to the reader.

Proposition 1 *Let C be a strongly pointed convex cone with non-empty interior. If the set Z is convex and strongly bounded with respect to C, then for every $y \in Z$ and for every $\varepsilon > 0$ the set $(y + C) \cap (Z + \varepsilon B)$ is bounded.*

Below we present the main result about extreme continuous selections.

Theorem 2 *Let X be a paracompact topological space, $F : X \to 2^E$ be a Hausdorff continuous multivalued mapping with closed, convex and non-empty images and C be a strongly pointed closed convex cone with non-empty interior. Assume that for every $x \in X$, $F(x)$ is strongly bounded with respect to C. Then the multivalued mapping $WEP_C(F(.))$ has a continuous selection. Something more, if $y' : X \to Y$ is a continuous selection of F, then there exists a continuous selection of the multivalued mapping $(y'(x) + C) \cap WEP_C F(x)$.*

If, in addition, for every $x \in X$, $F(x)$ is strongly bounded with respect to C_ε for some $\varepsilon > 0$, where $C_\varepsilon = \cup\{\lambda\overline{C \cap S + \varepsilon B} : \lambda \geq 0\}$, ($S$ is the unit sphere), then the multivalued mapping $(y'(x) + C_\varepsilon) \cap SEP_C F(x)$ has a continuous selection.

The proof of this theorem uses Michael's selection theorem [17] and the following geometrical lemma, whose proof can be found in [7].

Lemma 3 *Let $F : X \to 2^E, G : X \to 2^E$ be Hausdorff continuous multivalued mappings with convex and closed images. Define $H(x) := F(x) \cap G(x)$ and assume that $intH(x) \neq \emptyset$ for every $x \in X$. Then H is Hausdorff continuous.*

Proof of Theorem 2. Denote $D := C \cap S$ and $H = l^{-1}(0)$.

We shall prove that $dist(H, D) > 0$. Assume the contrary. Then there exists $b_n \in D$ such that $dist(b_n, H) \to 0$. It is well known and easy to prove that $dist(b_n, H) = -l(b_n)$, and by (1) we obtain a contradiction.

Let $\varepsilon \in (0, \frac{1}{2}dist(H, D))$. Obviously C_ε is a closed, strongly pointed cone with respect to the above definition.

Let $\{\varepsilon_n\}_{n=1}^{\infty}, \{\varepsilon'_n\}_{n=1}^{\infty}, \{\varepsilon''_n\}_{n=1}^{\infty}$ be sequences of positive numbers converging to 0 such that the series $\sum_{n=1}^{\infty} \varepsilon_n$ and $\sum_{n=1}^{\infty} \varepsilon'_n$ are convergent and

$$\varepsilon_{n-1} < \varepsilon_n + \varepsilon\varepsilon'_n \quad \forall n \geq 2. \tag{2}$$

Let $e \in D$.

The proof of the following Claim 1 is evident and is omitted.

Claim 1. *For every $\delta > 0$ we have $\delta\varepsilon B \subset C - \delta e$.*

Define inductively the mappings $H_n, F_n : X \to 2^E$ by $H_n(x) = (F(x) + \varepsilon_n B) \cap \{y_{n-1}(x) - \varepsilon'_n e + C\}$, $F_n(x) = \{y \in H_n(x) : l(y) \leq \inf l(H_n(x)) + \varepsilon''_n\}$, where $y_{n-1} : X \to Y$ is a continuous selection of $F_{n-1}, F_0 := F$.

We will prove by induction that such a definition is possible.

Assume that for some n, F_{n-1} is define as above and is lower semicontinuous with nonempty closed and convex images (for $n = 1$ this is true). By Michael's selection theorem there exists a continuous selection of F_{n-1}, denoted by y_{n-1} (if $n = 1$, then we take $y_0 = y'$ - the given selection by assumption). Define F_n as above with this y_{n-1} in the definition of $H_n(x)$ (here we use Proposition 2.1 to assure that H_n is bounded). We shall prove that F_n is lower semicontinuous , which will complete the induction, since obviously F_n has closed and convex images.

Let x_0 and $\alpha > 0$ be given.

By Claim 1 and by the choice of ε_n and ε'_n it follows that $intH_n(x_0) \neq \emptyset$. Indeed, assume that $intH_n(x_0) = \emptyset$. Then, by Claim 1 we have $y_{n-1}(x) + \varepsilon'_n\varepsilon B \subset y_{n-1}(x) - \varepsilon'_n e + C$, therefore $int\{(F(x_0) + \varepsilon_n B) \cap (y_{n-1}(x_0) + \varepsilon'_n \varepsilon B)\} = \emptyset$, whence $\varepsilon_n + \varepsilon\varepsilon'_n \leq \varepsilon_{n-1}$, a contradiction with (2).

By Proposition 1 it follows that $H_n(x_0)$ is bounded and since $intH_n(x_0) \neq \emptyset$, we have $intF_n(x_0) \neq \emptyset$. Let $z_0 \in F_n(x_0)$. There exists $z_1 \in intF_n(x_0)$ such that $\|z_0 - z_1\| < \alpha$. Then $l(z_1) < \inf l(H_n(x_0)) + \varepsilon''_n$. Let $\gamma \in (0, m(x_0) + \varepsilon''_n - l(z_1))$, where $m(x) = \inf l(H_n(x))$. By continuity of y_{n-1} and F it follows, applying Lemma 3, that H_n is Hausdorff continuous. So there exists $\delta > 0$ such that $H_n(x) \subset \{z : l(z) > m(x_0) - \gamma\}$ and $z_1 \in H_n(x)$ for every $x \in B(x_0; \delta)$. Hence $m(x) := \inf l(H_n(x)) \geq m(x_0) - \gamma$ and $l(z_1) < m(x_0) + \varepsilon''_n - \gamma < m(x) + \varepsilon''_n$ for every $x \in B(x_0; \delta)$.

Therefore $z_1 \in F_n(x)$ for every $x \in B(x_0; \delta)$, which proves the lower semicontinuity of F_n at x_0 and the corectness of the definition.

For every $x \in X, z \in H_n(x)$ we have $z = y_{n-1}(x) - \varepsilon'_n e + c$ for some $c \in C$. Hence

$$l(y_{n-1}(x) - z) = \varepsilon'_n l(e) - l(c) \geq \varepsilon'_n l(e) \tag{3}$$

We need the following.

Claim 2. *Let $s = \inf\{\|x-y\| : x \in C \cap S, y \in l^{-1}(0)\}$. Then the conditions $x \in C, r > 0,\ l(x) \geq -r$ imply $\|x\| \leq \frac{r}{s}$.*

Proof. Let $x \in C, r > 0$ and $l(x) \geq -r$. Then $s \leq dist\left(\frac{x}{\|x\|}, l^{-1}(0)\right) = l\left(\frac{-x}{\|x\|}\right) \leq \frac{r}{\|x\|}$, whence $\|x\| \leq \frac{r}{s}$. ■

By Claim 2 and by (3) it follows

$$\|y_{n-1}(x) - z\| \leq \frac{\varepsilon'_n l(-e)}{s}, \quad \forall x \in X, \forall z \in H_n(x) \tag{4}$$

whence

$$diam H_n(x) \leq \frac{2e'_n l(-e)}{s}, \quad \forall x \in X. \tag{5}$$

By (4) for $z = y_n(x)$ we obtain that $\{y_n(x)\}_{n=1}^{\infty}$ is a fundamental sequence. Let $v(x)$ be its limit. From (4) it follows that this limit is uniform with respect to x, i.e. $y_n(x)$ converges uniformly on $x \in E$ to $v(x)$, therefore v is a continuous mapping.

Since $y_n(x) \in F(x) + \varepsilon_n B$, we obtain that $v(x) \in F(x)$ for every $x \in E$.

We shall prove that $(v(x) + intC) \cap F(x) = \emptyset$ for every $x \in E$. Assume the contrary: there exists $z \in (v(x) + intC) \cap F(x)$ for some $x \in X$. Then for large n we have $[z, \frac{v(x)+z}{2}] \subset H_n(x)$ (here $[p, q]$ denotes the segment with ends p and q), and therefore, $diam H_n(x)$ does not converge to 0, which is a contradiction with (5) . Therefore $v(x)$ is a weakly efficient point of $(y'(x) + C) \cap F(x)$.

Assume that for every $x \in X$, $F(x)$ is strongly bounded with respect to C_ε for some $\varepsilon > 0$. Then by the proof above when C is replaced with C_ε, we conclude that the multivalued mapping $(y'(.) + C_\varepsilon) \cap WEP_{C_\varepsilon}(F(.))$ has a continuous selection. It is easy to see that $WEP_{C_\varepsilon}(F(.)) \subset SEP_C(F(.))$, which completes the proof. ∎

As a corollary of the above theorem we prove existence of a continuous selection of the support points of a closed convex bounded set depending continuously on a parameter (the existence of such support points is garanteed by Bishop-Phelps' theorem [18]).

Theorem 4 *Let $F : X \to 2^E$ be a Hausdorff continuous multivalued mapping with closed, convex, bounded and non-empty values from a paracompact topological space X to a Banach space E. Then for every $\varepsilon > 0$ and every $l \in E^*$ there exists a continuous selection of the multivalued mapping $F_{l,\varepsilon} : X \to 2^E$ defined by $F_{l,\varepsilon}(x) = \{y \in F(x) : \exists x^* \in B^*(l;\varepsilon) : \langle x^*, y\rangle = \max_{z\in F(x)}\langle x^*, z\rangle\}$. In particular the multivalued mappings which assigh to every $x \in X$ the support points and the boundary points of $F(x)$ have continuous selections.*

Proof. The same (using Theorem 2) as the proof of the Bishop-Phelps theorem in [18].∎

Now we present a parametric Ekeland's variational principle.

Theorem 5 *Let E be a Banach space, X be a paracompact topological space and Y be closed convex subset of E, $f : X \times Y \to \mathbf{R}$ be a function with the following properties:*

(a) the functions $\{f(.,y) : y \in Y\}$ are equi-continuous,

(b) $f(x,.)$ is convex and lower semicontinuous for every $x \in X$,

(c) $\inf_{y\in Y} f(x,y) > -\infty \quad \forall x \in X$.

Then

(d) for every $\varepsilon > 0$ there exists a continuous mapping $y_0 : X \to Y$ such that

$$f(x, y_0(x)) = \min_{y\in Y}\Big[f(x,y) + \varepsilon\|y - y_0(x)\|\Big] \quad \forall x \in X.$$

If, moreover, $f(x,.)$ is continuous for every $x \in X$, then we have the following localization property:

(d') for every continuous mapping $y' : X \to Y$, for every $\varepsilon > 0, \lambda > 0, \delta \in (0,\varepsilon)$ there exists a continuous mapping $y_0 : X \to Y$ such that $f(x, y_0(x)) = \min_{y\in Y}\Big[f(x,y) + \frac{\varepsilon}{\lambda}\|y - y_0(x)\|\Big] \quad \forall x \in X,$

and

(e) $\|y'(x) - y_0(x)\| < \lambda$ *whenever* $f(x, y'(x)) < \inf_{z \in X} f(x, z) + \varepsilon - \delta$,

(f) $y_0(x)$ *is the strong minimum point in (d) for every* $x \in X$ *(it means every minimizing sequence in d) is convergent).*

Proof. Let C be the following cone in $E \times \mathbf{R}$: $C = \{(x, -t) : t \geq 0, t\lambda \geq (\varepsilon - \delta)\|x\|\}$. It is easy to see that the multivalued mapping $F(x) = epif(x, .) := \{(y, t) \in E \times \mathbf{R} : t \geq f(x, y)\}$ (the epigraph of $f(x, .)$) is Hausdorff continuous and, in the case (d'), the mapping $s : X \to Y \times \mathbf{R}, s(x) = (y'(x), f(x, y'(x)))$ is a continuous selection of F. By Theorem 2 there exists a continuous selection (y_0, r_0) of $(s + C) \cap WEP_C(F(.))$. Therefore $int((y_0, r_0) + C) \cap epif(x, .) = \emptyset$ and $r_0(x) = f(x, y_0(x))$. This proves $f(x, y_\delta(x)) = \min_{y \in Y}\left[f(x, y) + \frac{\varepsilon - \delta}{\lambda}\|y - y_\delta(x)\|\right]$. The condition $(y_0, r_0) \in (s(x) + C)$ proves (e).

Let $\{y_n\}$ be a minimizing sequence for the function $g_2(x, .)$, where $g_2(x, y) = f(x, y) + \frac{\varepsilon}{\lambda}\|y - y_\delta(x)\|$. Putting $g_1(x, y) = f(x, y) + \frac{\varepsilon - \delta}{\lambda}\|y - y_\delta(x)\|$, we have $f(x, y_\delta(x)) \leq g_1(x, y_n) < g_2(x, y_n) \to f(x, y_\delta(x))$. Hence $g_2(x, y_n) - g_1(x, y_n) \to 0$, i.e. $\delta\|y_n - y_\delta(x)\| \to 0$ and (f) is proved. ■

The following theorem is an extension of Ekeland's variational principle to minimax problems and can be considered as a minimax variational principle. The proof is direct and uses Theorem 5 and Ekeland's variational principle.

Theorem 6 *Let* E_1 *and* E_2 *be Banach spaces,* X *and* Y *be closed non-empty subsets of* E_1 *and* E_2 *respectively,* Y *be convex, bounded and* $f : X \times Y \to \mathbf{R}$ *be a function with the following properties:*

a) the functions $\{f(., y) : y \in Y\}$ *are equi-continuous,*

b) $f(x, .)$ *is continuous and concave for every* $x \in X$,

c) $\sup_{y \in Y} f(x, y) < +\infty \quad \forall x \in X$,

d) $\inf\limits_{x \in X} \sup\limits_{y \in Y} f(x, y) > -\infty$.

Let $\varepsilon_1, \varepsilon_2, \lambda_1, \lambda_2 > 0$ *be given and* $x' \in X$ *and* $y' \in Y$ *be such that:*

e) $\sup\limits_{y \in Y} f(x', y) < \inf\limits_{x \in X} \sup\limits_{y \in Y} f(x, y) + \varepsilon_1$

f) $f(x', y') > \sup\limits_{y \in Y} f(x', y) - \varepsilon_2$.

Then there exist a continuous mapping $\tilde{y} : X \to Y$ *and a point* $x_0 \in X$ *such that for* $y_0 = \tilde{y}(x_0)$ *we have*

g) $f_2(x_0, y_0) = \max\limits_{y \in Y} f_2(x_0, y) = \min\limits_{x \in X} \sup\limits_{y \in Y} f_2(x, y)$, *where*

$f_2(x,y) = f(x,y) + \frac{\bar{\varepsilon}_1}{\lambda_1}\|x - x_0\| - \frac{\varepsilon_2}{\lambda_2}\|y - \tilde{y}(x)\|$, $\bar{\varepsilon}_1 = \varepsilon_1 + \frac{\varepsilon_2}{\lambda_2} diamY$,
h) x_0 and y_0 are the strong minimum and strong maximum points of the functions $\sup_{y \in Y} f_2(.,y)$ *and* $f_2(x_0,.)$ *respectively.*
i) $\|x_0 - x'\| < \lambda_1, \|y' - y_0\| \le \lambda_2 + \|\tilde{y}(x') - \tilde{y}(x_0)\|$.

Below we present a smooth variational principle involving bump functions, called here *modified smooth variational principle*, which unifies Borwein-Preiss' variational principle [2] and Deville-Godefroy-Zizler's variational principle [5] (concerning only existence of arbitrarily small smooth perturbations producing a point of minimum of the perturbed function). As an advantage it can be noted that this new variant is produced by Ekeland's variational principle [8] and has the same localization properties as the latter. Namely, the ratio $\frac{\varepsilon}{\lambda^p}, p \ge 1$, which appears in the Borwein-Preiss variational principle, is replaced here by $\frac{\varepsilon}{\lambda}$, as in the Ekeland variational principle, but the price for this is a new perturbation, which is also convex, if we work with norms instead of bumps. This refines also the localization given in [5]. It is worth to mention that the same precise localization for Deville-Godefroy-Zizler's variational principle, as well as the density part in the latter follows from [13], where a prototype of it was obtained, concerning δ-minimum point of the perturbed function. Another advantage of the presented here modified smooth variational principle is that the sequence involved in it can be forced to converge to the minimum of the perturbed function as fast as we like in each step of the construction after the first one. This idea allows more precise localization of the minimum point v of the perturbed function: namely, under additional assumptions, v can be arranged to belong to the complement of an arbitrary, given in the beginning, σ-porous set.

Variants of variational principles are obtained in [14] and [15] (without localization). The reader can consult with [16] for a discussion about the relationships between the variational principles, a complement to which is the new one presented here with respect to the localization and unification.

We present here also a parametric variant of this modified smooth variational principle, which is of the spirit of [10] and [11].

Let $(E, \|.\|)$ be a Banach space. A *bornology* β of E is a family of closed bounded and centrally symmetric subsets of E whose union is E, which is closed under multiplication by scalars and is directed upwards (that is, the union of any two members of β is contained in some member of β). We will denote by E^*_β the dual space of E endowed with the topology of uniform convergence on β-sets. The most important bornologies are those formed by all (symmetric) bounded sets (the Fréchet bornology, denoted by F), weak compact sets (the weak Hadamard bornology, denoted by WH), compact sets (the Hadamard bornology, denoted by H) and finite sets (the Gateaux bornology, denoted by G).

Given a function $f : E \to \mathbf{R} \cup \{+\infty\}$, we say that f is β-differentiable at x and has a β-derivative $\nabla^\beta f(x)$ if $f(x)$ is finite and

$$\frac{f(x+th)-f(x)}{t} - \langle \nabla^\beta f(x), h \rangle \to 0$$

as $t \to 0$ uniformly in $h \in V$ for every $V \in \beta$. We say that a function f is *β-smooth* at x if $\nabla^\beta f : E \to E^*_\beta$ is continuous in a neighborhood of x.

Recall that a function $b : E \to \mathbf{R}$ is called *bump function* if b is positive on a bounded set, called *suppb*, and zero on the complement of *suppb*.

We shall use the following lemma, which is presented in [3] and which is a straitforward generalization of [6, Section VIII, Lemma 1.3].

Lemma 7 *Let E be a Banach space that addmits a bump function which is Lipschitzian and β-smooth. Then there exist a function $d : W \to \mathbf{R}^+$ and a scalar $K > 1$ such that*

i) d is bounded, Lipschitzian on X and β-smooth on $X \setminus 0$.

ii) $\|x\| \le d(x) \le K\|x\|$ if $\|x\| \le 1$ and $d(x) = 2$ if $\|x\| \ge 1$.

The proof of the following lemma is straightforward and is omitted.

Lemma 8 *Let α be given. Then for every $\varepsilon > 0$ there exists $p \in (1,2)$ such that*

$$\alpha\|x\| < \alpha\|x\|^p + \varepsilon \quad \forall x \in E.$$

In what follows we use the notation $B(x;r)$ (resp. $B[x;r]$) for an open (resp. closed) ball with center x and radius r.

Theorem 9 *(Modified smooth variational principle). Let E be a Banach space that addmits a bump function which is Lipschitzian and β-smooth, $f : E \to \mathbf{R} \cup \{+\infty\}$ be a lower semicontinuous function bounded below and let $\varepsilon > 0$, $\lambda > 0$ be given. Suppose that x_0 satisfies the condition:*

$$f(x_0) < \inf f(E) + \varepsilon.$$

Then for the function $d : E \to \mathbf{R}_+$ produced by Lemma 1, we have: there exist $\lambda_0 \in (0, \lambda), \mu_0 \in (0,1)$ and $x_1 \in B(x_0; \lambda_0)$ such that for every $i = 1, 2, \ldots$, for every sufficiently small $\mu_i \in (0,1), \lambda_i \in (0, \lambda_0)$ (possible chosen after $x_1, \ldots, x_i$) there exist $x_{i+1} \in B(x_i; \lambda_i)$ and $p_i \in (1,2)$ such that $x_n \to v$, where $v \in B(x_0; \lambda)$, $\sum_{i=0}^{\infty} \lambda_i \leq \lambda$, $\sum_{i=0}^{\infty} \mu_i \leq 1$ and

$$f(v) + \Delta(v) \leq f(x) + \Delta(x) \quad \forall x \in E, \tag{6}$$

$$\Delta(x) = \frac{\varepsilon}{\lambda} \sum_{i=1}^{\infty} \mu_{i-1} [d(x - x_i)]^{p_i}. \tag{7}$$

Proof. Choose $\lambda_0 < \lambda$ and $\mu_0 < 1$ such that

$$f(x_0) < \inf f(E) + \lambda_0 \mu_0 \frac{\varepsilon}{\lambda}.$$

By Lemma 1 and Lemma 2 define inductively functions $f_n : E \to \mathbf{R}$ satisfying

$$f_n(x) = f_{n-1}(x) + \frac{\varepsilon}{\lambda} \mu_{n-1} [d(x - x_n)]^{p_n}, f_0 := f,$$

where $x_n \in B(x_{n-1}, \lambda_{n-1})$ is produced by Ekeland's variational principle:

$$f_{n-1}(x_n) < f_{n-1}(x) + \frac{\varepsilon}{\lambda} \mu_{n-1} \|x - x_n\| \quad \forall x \neq x_n,$$

$\lambda_n \in (0, \lambda - \sum_{i=0}^{n-1} \lambda_i), \mu_n \in (0, 1 - \sum_{i=0}^{n-1} \mu_i)$ are chosen possibly after x_n, and $p_n \in (1,2)$ is such that

$$\frac{\varepsilon}{\lambda} \mu_{n-1} \|x\| < \frac{\varepsilon}{\lambda} \mu_{n-1} \|x\|^{p_n} + \mu_n \lambda_n \frac{\varepsilon}{\lambda} \quad \forall x \in E.$$

It is a routine matter to prove that $\{x_n\}$ is a fundamental sequence and its limit $v \in B(x_0, \lambda)$ satisfies (6). ∎

It is clear that d in the previous theorem can be replaced by $\|.\|$. So we have a variant of Borwein-Preiss' variational principle [2].

Theorem 10 *(Parametric modified smooth variational principle). Suppose that T is a paracompact topological space, X is a convex closed and nonempty subset of a Banach space $E, \|.\|$ and the function $f : T \times X \to \mathbf{R}$ satisfies the conditions:*

(i) *the function* $f(t,.)$ *is convex and continuous for every* $t \in T$*;*
(ii) *the functions* $\{f(.,x) : x \in X\}$ *are equi-continuous.*
Given $\varepsilon > 0$, $\lambda > 0$, *let* $x_0 : T \to X$ *be a continuous mapping, such that*

$$f(t, x_0(t)) \leq \inf f(t, X) + \varepsilon, \quad \forall t \in T.$$

Then for every $\alpha > 0$, *there exist* $\lambda_0 \in (0, \lambda)$, $\mu_0 \in (0,1)$ *and a continuous mapping* $x_1 : T \to X$ *such that* $x_1(t) \in B(x_0(t), \lambda_0)$ *for every* $t \in T$ *and for every* $i = 1, 2, ...$, *for every sufficiently small* $\mu_i, \lambda_i > 0$ *(possibly chosen after* $x_1, ..., x_i$*) there exist* $p_i \in (1,2)$ *and a continuous mapping* $x_{i+1} : T \to X$ *with* $x_{i+1}(t) \in B(x_i(t); \lambda_i)$ *for every* $t \in T$ *such that* $x_i(t)$ *converges uniformly to a continuous mapping* $v : T \to X$ *with* $v(t) \in B(x_0(t); \lambda)$ *for every* $t \in T$, $\sum_{i=0}^{\infty} \lambda_i \leq \lambda$, $\sum_{i=0}^{\infty} \mu_i \leq 1$ *and*

$$f(t, v(t)) + \Delta(t, v(t)) \leq f(t,x) + \Delta(t,x) \quad \forall x \in x, \forall t \in T$$

where

$$\Delta(t,x) = \frac{\varepsilon + \alpha}{\lambda} \sum_{i=1}^{\infty} \mu_{i-1} [d(x - x_i(t))]^{p_i}.$$

Here d *is either the norm* $\|.\|$, *or the function produced by Lemma 7, if* E *has a* β*-smooth Lipschitz bump function.*

As an advantage of Theorem 10 comparing with the analogous parametrization of Borwein-Preiss variational principle in [11] we note that the assumptions on the boundedness of certain level sets in [11] (when $p > 1$) are missing in Theorem 10.

In the next theorem we establish a continuous selection theorem for the subdifferential of a convex function depending on a parameter.

Theorem 11 *Let the Banach space* E *have Fréchet differentiable norm off* 0. *Suppose that* X *is a paracompact topological space and the function* $f : X \times E \to \mathbf{R}$ *satisfies the conditions:*

(i) *for every* $x \in X$ *the function* $f(x,.)$ *is convex, continuous and bounded below on* E*;*

(ii) *the functions* $\{f(.,y) : y \in E\}$ *are equi-continuous.*

Then for every $\gamma > 0$ *there exists a continuous mapping* $v : X \to E$, *such that the multivalued mapping* $F(x) := \partial_y f(x, v(x)) \cap B[0,\gamma]$ *has a continuous selection, where* $\partial_y f$ *denotes the usual subdifferential with respect to the second variable.*

Proof. Let v and Δ be the mapping and function produced by Theorem 10, with $\lambda = 2$, $\varepsilon = \alpha = \frac{\gamma}{2}$, $Y = E, d = \|.\|$. Then by the necessary condition of a minimum, $0 \in \partial_y[f(x, v(x)) + \Delta(x, v(x))] = \partial_y f(x, v(x)) + \Delta'_y(x, v(x))$, which shows that $-\Delta'_y(x, v(x))$ is a continuous selection of F. Obviously $\|\Delta'_y(x, v(x))\| \leq \gamma$. ∎

In the next theorem we establish existence of a Nash equilibrium for convex functions after smooth perturbations, when one of the sets is non-compact. It can be regarded as a generalization of Sion's minimax theorem [19] for Nash equilibrium problems.

Theorem 12 *Let $X_2, \ldots, X_n$ be convex compact sets in Banach spaces, X_1 be a closed convex bounded subset of a Banach space. Denote $X = X_1 \times \ldots \times X_n$, $x = (x_1, \ldots, x_n)$, $x_{\hat{i}} = (x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_n)$, $X_{\hat{i}} = X_1 \times \ldots \times X_{i-1} \times X_{i+1} \times \ldots \times X_n$, $\forall i = 1, \ldots, n$. Let $f_i : X \to \mathbf{R}$ be convex lower semicontinuous functions with respect to the variable $x_i \in X_i$ and the functions $\{f_i(\ldots, x_i, \ldots) : x_i \in X_i\}$ be equicontinuous on $X_{\hat{i}}$ for every $i = 1, \ldots, n$. Then for every $\varepsilon > 0$ there exist convex Lipschitz functions $b_i : X_i \to \mathbf{R}$ with a Lipschitz constant less than ε, which are differentiable, if the norm of E_i is differentiable off 0, and there exist points $\bar{x}_i \in X_i$ such that the point $\bar{x} = (\bar{x}_1, \ldots, \bar{x}_n)$ is a Nash equilibrium for the functions*

$$\bar{f}_i(x) = f_i(x) + b_i(x_i), i = 1, \ldots, n$$

i.e.

$$f_i(\bar{x}_1, \ldots, \bar{x}_i, \ldots, \bar{x}_n) + b_i(\bar{x}_i) \leq f_i(\bar{x}_1, \ldots, x_i, \ldots, \bar{x}_n) + b_i(x_i)$$
$$\text{for every } \; x_i \in X_i \; \text{ and } \; i = 1, \ldots, n.$$

Proof. From Theorem 10 applied with d equal to the norm in X, for every $i = 1, \ldots, n$ there exists a continuous mapping $y_i : X_{\hat{i}} \to X_i$ and a function $\Delta_i : X \to \mathbf{R}$, which is convex and Lipschitz on $x_i \in X_i$ with a Lipschitz constant less than ε (and differentiable, if the norm of E_i is differentiable out of 0) such that

$$(f_i + \Delta_i)(x_1, \ldots, y_i(x_{\hat{i}}), \ldots, x_n) \leq (f_i + \Delta_i)(x), \quad \forall x \in X, \; \forall i = 1, \ldots, n.$$

The composition mapping

$$X_2 \times \ldots \times X_n \ni x_{\hat{1}} \to (y_2(\varphi_2(x_{\hat{1}})), \ldots, y_n(\varphi_n(x_{\hat{1}}))) \in X_2 \times \ldots \times X_n,$$

where $\varphi_i(x_{\hat{1}}) = (y_1(x_{\hat{1}}), x_2, \ldots, x_{i-1}, x_{i+1}, \ldots, x_n)$, $i = 2, \ldots, n$ is a continuous mapping from the compact convex set $X_2 \times \ldots \times X_n$ to itself and from Schauder's fixed point theorem it has a fixed point $\bar{x}_{\hat{1}} = (\bar{x}_2, \ldots, \bar{x}_n)$. If we put $\bar{x}_1 = y_1(\bar{x}_{\hat{1}})$ and $b_i(x_i) := \Delta(\bar{x}_1, ..., x_i, ..., \bar{x}_n)$, then $\bar{x}_i = y_i(\bar{x}_{\hat{i}})$ for every $i = 2, \ldots, n$, and the proof is completed. ∎

As an advantage of the smooth perturbations in the above theorem, we would mention the possibility to write second order optimality conditions at the Nash equilibrium point for the perturbed functions, when the norm of the space is second order Fréchet differentiable (off 0). For example, in the setting of [12], such optimality conditions can be written in terms of second order subdifferentials, if the sets X_i are defined by equalities and inequalities, and all involved functions are of class $C^{1,1}$.

As a next application we give a short proof of a variant of the Ky Fan inequality considered in [1, Theorem 6.3.2], when a perturbation of the function is involved.

Theorem 13 *Let X be convex, compact and nonempty subset of a Banach space $(E, \|.\|)$, $f : X \times X \to \mathbf{R}$ be a function such that*

a) $f(., y)$ is lower semicontinuous for every $y \in X$;

b) $f(x, .)$ is concave for every $x \in X$.

c) the functions $\{f(., y) : y \in X\}$ are lower semicontinuous and equi-upper semicontinuous.

Then for every $\varepsilon > 0$ there exists $x_\varepsilon \in X$ and a function $\Delta : X \times Y \to \mathbf{R}$ which, with respect to the first variable, is continuous and with respect to the second variable is convex, Lipschitz with a Lipschitz constant less that ε, and differentiable, if the norm of E is differentiable off 0, such that for the function $f_\varepsilon(x, y) = f(x, y) - \varepsilon\Delta(x, y)$ we have

$$f_\varepsilon(x_\varepsilon, y) < f_\varepsilon(x_\varepsilon, x_\varepsilon) \quad \forall y \in X, y \neq x_\varepsilon,$$

and every maximizing sequence for the function $f_\varepsilon(x_\varepsilon, .)$ is convergent to x_ε.

Proof. For given $\varepsilon > 0$, by Theorem 10 applied for $-f$ and $\varepsilon/2$ with $\lambda = 1, \alpha = \varepsilon/2, d = \|.\|$ we obtain: there exists a continuous mapping $\tilde{y}_\varepsilon : X \to X$ and a function $\Delta : X \times Y \to \mathbf{R}$ of type (7) such that

$$-f(x, \tilde{y}_\varepsilon(x)) + \Delta(x, \tilde{y}_\varepsilon(x)) \leq -f(x, y) + \Delta(x, y) \qquad \forall x \in X, \forall y \in X.$$

By Schauder's fixed point theorem, there exists a fixed point $z_\varepsilon \in X$ of $\tilde{y}_\varepsilon$, i.e. $\tilde{y}_\varepsilon(z_\varepsilon) = z_\varepsilon$. Therefore

$$f(z_\varepsilon, y) - \Delta(z_\varepsilon, y) \leq f(z_\varepsilon, z_\varepsilon) + \Delta(z_\varepsilon, z_\varepsilon) \ \ \forall y \in X.$$

and the proof is completed. ∎

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