<table>
<thead>
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<th>Title</th>
<th>NORM ARCHIVED TOEPLITZ AND HANKEL OPERATORS (Analytic Function Spaces and Operators on these Spaces)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Author(s)</td>
<td>Yoshino, Takashi</td>
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</tbody>
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Kyoto University
NORM ACHIEVED TOEPLITZ AND HANKEL OPERATORS

東北大学・理学研究科 吉野 崇 (Takashi Yoshino)

Let $\mu$ be the normalized Lebesgue measure on the Borel sets of the unit circle in the complex plane $\mathbb{C}$. For a $\varphi \in L^\infty$ the Laurent operator $L_\varphi$ is given by $L_\varphi f = \varphi f$ for $f \in L^2$ as the multiplication operator on $L^2$. And the Laurent operator induces, in a natural way, twin operators on $H^2$ called the Toeplitz operator $T_\varphi$ given by $T_\varphi f = PL_\varphi f$ for $f \in H^2$ where $P$ is the orthogonal projection from $L^2$ onto $H^2$ and the Hankel operator $H_\varphi$ given by $H_\varphi f = J(I-P)L_\varphi f$ for $f \in H^2$ where $J$ is the unitary operator on $L^2$ defined by $J(z^{-n}) = z^{n-1}, \ n = 0, \pm 1, \pm 2, \cdots$.

The following results are known.

**Proposition 1.** If $\varphi$ is a non-constant function in $L^\infty$, then $\sigma_p(T_\varphi) \cap \overline{\sigma_p(T_{\varphi^*})} = \emptyset$ where $\sigma_p(T_\varphi)$ denotes the point spectrum of $T_\varphi$ and the bar denotes the complex conjugate.

**Proposition 2.** If $\varphi$ and $\psi$ are in $H^\infty$, then $T_\varphi H^2 \subseteq T_\psi H^2$ if and only if there exists a $g \in H^\infty$ uniquely, up to a unimodular constant, such that $T_\varphi = T_\psi T_g = T_{\psi g}$. And then $\varphi = \psi g$. Particularly, if $\varphi$ and $\psi$ are inner, then $g$ is also inner.

**Proposition 3.** $H_\varphi$ has the following properties.
1. $T_z H_\varphi = H_\varphi T_z$
2. $H_{\varphi^*} = H_{\varphi^*}$ where $\varphi^*(z) = \overline{\varphi(\overline{z})}$
3. $H_{\alpha \varphi + \beta \psi} = \alpha H_\varphi + \beta H_\psi$, $\alpha, \beta \in \mathbb{C}$
4. $H_\varphi = 0$ if and only if $(I-P)\varphi = 0$ (i.e., $\varphi \in H^\infty$)
5. $\|H_\varphi\| = \min\{\|\varphi + \psi\|_\infty : \psi \in H^\infty\}$

**Proposition 4.** $H_{\psi^*} H_\varphi = T_{\psi^*} - T_{\psi^*} T_\varphi$. 
Proposition 5. For any $\psi \in H^\infty$, $H_\varphi T_\psi = H_\varphi \psi$.

Lemma 1. The following assertions are equivalent.

1. $\mathcal{N}_{H_\varphi} \neq \{0\}$.
2. $[H_\varphi H^2]^{\sim L^2} \neq H^2$.
3. $\varphi = \overline{g}h$ for some inner function $g$ and $h \in H^\infty$ such that $g$ and $h$ have no common non-constant inner factor.

Proof. (1) $\Rightarrow$ (2); 

\[ H_\varphi f = 0 \quad \Rightarrow \quad \varphi^* f^* \in H^2 \quad \Rightarrow \quad H_\varphi^* f^* = H_\varphi^* f^* = 0 \quad \Rightarrow \quad f^* \perp [H_\varphi H^2]^{\sim L^2} . \]

(1) $\Rightarrow$ (3); Since $\mathcal{N}_{H_\varphi}$ is a non-zero invariant subspace of $T_z$ by Proposition 3, $\mathcal{N}_{H_\varphi} = T_g H^2$ for some inner function $g$. Hence, by Proposition 5, $O = H_\varphi T_g = H_{\varphi g}$ and $\varphi g = h \in H^\infty$ by Proposition 3(4). Therefore $\varphi = \overline{g}h$. If $g = g_1g_2$ and $h = g_1h_1$ for some non-constant inner function $g_1$ and $g_2$, $h_1 \in H^\infty$, then, by Propositions 2 and 5,

\[ T_{g_2} H^2 \supseteq T_g H^2 = \mathcal{N}_{H_\varphi} = N_{H_{g_1}} \supseteq T_{g_2} H^2 \]

and this is a contradiction. Therefore $g$ and $h$ have no common non-constant inner factor.

(3) $\Rightarrow$ (1); By Propositions 5 and 3(4), we have $H_\varphi T_g H^2 = H_{\varphi g} H^2 = H_h H^2 = \{0\}$ and $\mathcal{N}_{H_\varphi} \supseteq T_g H^2 \neq \{0\}$. \square

Theorem 1. The Toeplitz operator $T_\varphi$ is norm-achieved (i.e., $\{f \in H^2 : \|T_\varphi f\|_2 = \|T_\varphi\| \|f\|_2 \neq \{0\}\}$ if and only if $\frac{\varphi}{\|T_\varphi\|} = g$ for some $g \in L^\infty$ such that $|g| = 1$ a.e. and that $0 \in \sigma_p(H_g)$.

And, in this case, $\{f \in H^2 : \|T_\varphi f\|_2 = \|T_\varphi\| \|f\|_2\} = \mathcal{N}_{H_g}$ and it is invariant under $T_z$ by Proposition 3(1).

Proof. $\Rightarrow$; If $\|T_\varphi f\|_2 = \|T_\varphi\| \|f\|_2$ for some non-zero $f \in H^2$, then we have, for $g = \frac{\varphi}{\|T_\varphi\|}$,

\[ \|f\|_2 = \|T_\varphi^* f\|_2 = \|T_g f\|_2 \leq \|L_g f\|_2 \leq \|f\|_2 \]
because $\|L_g\| = \|T_g\| = \|T_{\varphi}\| / \|T_{\varphi}\| = 1$. Hence $T_g^*T_gf = f$ and $PL_gf = L_gf$ and hence $H_gf = J(I - P)L_gf = 0$ (i.e., $0 \in \sigma_p(H_g)$). Since, by Proposition 4, $H_g^*H_g = T|g|^2 - Tg_Tg$, we have $T_g^*T_gf = f$ (i.e., $1 \in \sigma_p(T|g|^2)$) and, by Proposition 1, $|g|^2$ is constant and $|g| = 1 \text{ a.e.}$.

$(\leftarrow)$: Since $\|T_g\| = \|T_{\varphi}\| / \|T_{\varphi}\| = 1$ and since, by Proposition 4, $H_g^*H_g = I - Tg_Tg$, we have $T_g^*T_gf = f$ for all $f \in \mathcal{N}_{H_g}$ and hence $\|T_gf\|_2 = \|f\|_2$. Therefore $\|T_{\varphi}f\|_2 = \|T_{\varphi}\| \|T_gf\|_2 = \|T_{\varphi}\| \|f\|_2$.

The last assertion is clear. In fact, $(\rightarrow)$ implies that $\{f \in H^2 : \|T_{\varphi}f\|_2 = \|T_{\varphi}\| \|f\|_2\} \subseteq \mathcal{N}_{H_g}$ and $(\leftarrow)$ implies the converse inclusion. 

**Corollary 1.** $T_{\varphi}$ is norm-achieved if and only if $T_{\varphi} = g$ for some inner functions $q$ and $h$ such that $q$ and $h$ have no common non-constant inner factor.

And, in this case, $\emptyset \neq \sigma(T_{\varphi}) \cap \{\lambda \in \mathbb{C} : \|T_{\varphi}\| = |\lambda|\} \subseteq \sigma_c(T_{\varphi})$ where $\sigma_c(T_{\varphi})$ denotes the continuous spectrum of $T_{\varphi}$.

**Proof.** By Theorem 1, $T_{\varphi}$ is norm-achieved if and only if $\|T_{\varphi}\| = g$ for some $g \in L^\infty$ such that $|g| = 1 \text{ a.e.}$ and that $0 \in \sigma_p(H_g)$. And then, by Lemma 1, $\mathcal{N}_{H_g} \neq \{0\}$ if and only if $g = \overline{q}h$ for some inner function $q$ and $h \in H^\infty$ such that $q$ and $h$ have no common non-constant inner factor. Since $|g| = 1 \text{ a.e.}$ if and only if $|h| = 1 \text{ a.e.}$ and $h$ is also an inner function.

It is known that $\sigma(L_g) \subseteq \sigma(T_{\varphi})$ and since $L_g$ is unitary because $|g| = 1 \text{ a.e.}$, we have $\sigma(T_{\varphi}) \cap \{\lambda \in \mathbb{C} : \|T_{\varphi}\| = |\lambda|\} \neq 0$. If $T_gx = e^{i\theta}x$ for some $\theta \in [0, 2\pi)$ and non-zero $x \in H^2$, then $\|x\| = \|T_gx\| = \|T_q^*T_hx\| \leq \|T_hx\| = \|x\|$ and $e^{i\theta}T_qx = T_qT_gx = T_qT_q^*T_hx = T_hx$. Since $T_h - e^{i\theta}T_q$ is hyponormal, $(T_h - e^{i\theta}T_q)x = o$ implies $(T_h - e^{i\theta}T_q)^*x = o$ and this contradicts Proposition 1 and hence $\sigma(T_{\varphi}) \cap \{\lambda \in \mathbb{C} : \|T_{\varphi}\| = |\lambda|\} \subseteq \sigma_c(T_{\varphi})$ because $\sigma_r(T_{\varphi}) \cap \{\lambda \in \mathbb{C} : \|T_{\varphi}\| = |\lambda|\} = 0$.

where $\sigma_r(T_{\varphi})$ denotes the residual spectrum of $T_{\varphi}$. 

\[\square\]
In the case of Hankel operators, we have the following.

**Theorem 2.** The Hankel operator $H_{\varphi}$ is norm-achieved (i.e., \( \{ f \in H^2 \mid ||H_{\varphi}f||_2 = ||H_{\varphi}||f||_2 \} \neq \{ o \} \)) if and only if \( \frac{\varphi}{||H_{\varphi}||} = g + \psi \) for some \( \psi \in H^\infty \) and \( g \in L^\infty \) such that \( |g| = 1 \) a.e. and that \( 0 \in \sigma_p(T_g) \).

And, in this case, \( \{ f \in H^2 \mid ||H_{\varphi}f||_2 = ||H_{\varphi}||f||_2 \} = N_{T_g} \).

**Proof.** \( \rightarrow \); By Proposition 3, there exists a \( g \in L^\infty \) such that \( H_{\frac{\varphi}{||H_{\varphi}||}} = H_g \) and \( ||H_g|| = ||g||_\infty \). And then \( H_{\frac{\varphi}{||H_{\varphi}||}} - g = O \) and \( \psi = \frac{\varphi}{||H_{\varphi}||} - g \in H^\infty \) by Proposition 3. If \( ||H_{\varphi}f||_2 = ||H_{\varphi}||f||_2 \) for some non-zero \( f \in H^2 \), then we have

\[
||f||_2 = ||H_{\frac{\varphi}{||H_{\varphi}||}}f||_2 = ||H_gf||_2 = ||(I - P)L_{\varphi}f||_2 \leq ||L_gf||_2 \leq ||f||_2
\]

because \( ||L_g|| = ||g||_\infty = ||H_g|| = ||H_{\frac{\varphi}{||H_{\varphi}||}}|| = \frac{||H_{\varphi}||}{||H_{\varphi}||} = 1 \). Hence \( H_g^*H_gf = f \) and \( (I - P)L_gf = L_gf \) and hence \( T_gf = PL_gf = o \) (i.e., \( 0 \in \sigma_p(T_g) \)). Since, by Proposition 4, \( H_g^*H_g = T_{|g|^2} - T_gT_g \), we have \( T_{|g|^2}f = f \) (i.e., \( 1 \in \sigma_p(T_{|g|^2}) \)) and, by Proposition 1, \( |g|^2 \) is constant and \( |g| = 1 \) a.e.

\( \leftarrow \); By Proposition 3, \( ||H_g|| = ||H_{\frac{\varphi}{||H_{\varphi}||}}|| = \frac{||H_{\varphi}||}{||H_{\varphi}||} = 1 \). Since, by Proposition 4, \( H_g^*H_g = I - T_g^*T_g \), we have \( H_g^*H_gf = f \) for all \( f \in N_{T_g} \) and hence \( ||H_gf||_2 = ||f||_2 \). Therefore, by Proposition 3,

\[
||H_{\varphi}f||_2 = ||H_{\varphi}||H_gf||_2 = ||H_{\varphi}||||H_gf||_2 = ||H_{\varphi}||f||_2.
\]

The last assertion of the theorem is clear. In fact, \( \rightarrow \) implies that

\[
\{ f \in H^2 \mid ||H_{\varphi}f||_2 = ||H_{\varphi}||f||_2 \} \subseteq N_{T_g}
\]

and \( \leftarrow \) implies the converse inclusion. \( \Box \)