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Kyoto University
BOUNDED HARMONIC FUNCTIONS ON UNLIMITED COVERING SURFACES

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1. Introduction

Let $W$ be an open Riemann surface possessing a Green's function. Consider a $p$-sheeted unlimited covering surface $\bar{W}$ of $W$ with projection map $\pi$. It is easily seen that $\bar{W}$ also possesses a Green's function (cf. e.g. [A-S]). We denote by $HP(R)$ ($HB(R)$, resp.) the class of positive (bounded, resp.) harmonic functions on an open Riemann surface $R$. It is obvious that the inclusion relation

$$HX(W) \circ \pi := \{h \circ \pi : h \in HX(W)\} \subset HX(\bar{W})$$

holds for $X = P, B$. The main purpose of this paper is to give a necessary and sufficient condition, in terms of Martin boundary, in order that the relation $HX(W) \circ \pi = HX(\bar{W})$ holds for $X = P, B$.

For an open Riemann surface $R$, we denote by $R^*, \Delta^R$ and $\Delta^R_1$ the Martin compactification, the Martin boundary and the minimal Martin boundary of $R$, respectively. It is known that the projection map $\pi$ of $\bar{W}$ to $W$ is extended to $\bar{W}^*$ continuously and $\pi(\Delta^\bar{W}) = \Delta^W$ (cf. [M-S2]). For each $\zeta \in \Delta^W$, put

$$\Delta^\bar{W}_1(\zeta) = \Delta^\bar{W}_1 \cap \pi^{-1}(\zeta) = \{\tilde{\zeta} \in \Delta^\bar{W}_1 : \pi(\tilde{\zeta}) = \zeta\},$$

which is the set of minimal boundary points of $\bar{W}$ lying over $\zeta \in \Delta^W$. Our main results are the followings.

**Theorem 1.** In order that the relation $HP(W) \circ \pi = HP(\bar{W})$ holds, it is necessary and sufficient that $\Delta^\bar{W}_1(\zeta)$ consists of a single point for every $\zeta \in \Delta^W_1$.

**Theorem 2.** In order that the relation $HB(W) \circ \pi = HB(\bar{W})$ holds, it is necessary and sufficient that $\Delta^\bar{W}_1(\zeta)$ consists of a single point for $\omega^W_z$ almost all $\zeta \in \Delta^W_1$, where $\omega^W_z$ is a harmonic measure on $\Delta^W$ with respect to $W$ and $z \in W$.

Proofs of Theorems 1 and 2 will be given in §3 and §4, respectively.

Let $D$ be the unit disc $\{|z| < 1\}$. In §5, we will be concerned with $p$-sheeted unlimited covering surfaces of $D$ which illustrate Theorems 1 and 2. We will prove the following.
Proposition. Set \( A = \{(1 - 2^{-n-1})e^{i2\pi k/2^{n+2}} : n = 1, 2, \ldots, k = 1, \ldots, 2^{n+2}\} \). If \( \overline{D} \) is a \( p \)-sheeted unlimited covering surface of \( D \) with projection map \( \pi \) such that there is a branch point of \( \overline{D} \) of order \( p - 1 \) (or multiplicity \( p \)) over every \( z \in A \) and there are no branch points of \( \overline{D} \) over \( D \setminus A \), then \( HP(D) \circ \pi = HP(\overline{D}) \).

We will show a bit more (cf. Theorem 5.1). Modifying the above \( \overline{D} \), we will also give a \( p \)-sheeted unlimited covering surface \( \overline{D}_1 \) of \( D \) with projection map \( \pi \) such that \( HP(D) \circ \pi \neq HP(\overline{D}_1) \) and \( HB(D) \circ \pi = HB(\overline{D}_1) \).

2. Martin boundary of \( p \)-sheeted unlimited covering surfaces

Let \( W \) be an open Riemann surface possessing a Green's function and \( \overline{W} \) a \( p \)-sheeted unlimited covering surface of \( W \) with projection map \( \pi \). Since the pullback of a Green's function on \( W \) by \( \pi \) is a nonconstant positive superharmonic function on \( \overline{W} \), we see that \( \overline{W} \) possesses a Green's function (cf. e.g. [A-S], [S-N]). For Martin compactification, Martin boundary and minimal Martin boundary, we follow the notation in Introduction. We first note the following (cf. [M-S2]).

**Proposition 2.1.** The projection map \( \pi \) of \( \overline{W} \) onto \( W \) is extended to the Martin compactification \( \overline{W}^* \) of \( \overline{W} \) continuously and \( \pi(\Delta_{\overline{W}}) = \Delta_{W} \).

We recall the definition of \( \Delta_{\overline{W}}(\zeta) \) \((\zeta \in \Delta_{W})\) in Introduction:

\[
\Delta_{\overline{W}}(\zeta) = \Delta_{\overline{W}} \cap \pi^{-1}(\zeta) = \{\tilde{\zeta} \in \Delta_{\overline{W}} : \pi(\tilde{\zeta}) = \zeta\}.
\]

We denote by \( \nu_{\overline{W}}(\zeta) \) the (cardinal) number of \( \Delta_{\overline{W}}(\zeta) \). We next fix a point \( a \in W \) and a point \( \tilde{a} \in \overline{W} \) with

\[
\pi(\tilde{a}) = a.
\]

We consider the Martin kernel \( k^W(\cdot) \) \((k_{\tilde{\zeta}}^{\tilde{W}}(\cdot), \text{ resp.})\) on \( W \) \((\tilde{W}, \text{ resp.})\) with pole at \( \zeta \) \((\tilde{\zeta}, \text{ resp.})\) and with reference point \( a \) \((\tilde{a}, \text{ resp.})\), that is,

\[
k^W(\zeta) = \frac{g^W(z, \zeta)}{g^W(a, \zeta)} \quad (k_{\tilde{\zeta}}^{\tilde{W}}(\tilde{z}) = \frac{g^{\tilde{W}}(\tilde{z}, \tilde{\zeta})}{g^{\tilde{W}}(\tilde{a}, \tilde{\zeta})}, \text{ resp.})
\]

for \( \zeta \in W \) \((\tilde{\zeta} \in \tilde{W}, \text{ resp.})\), where \( g^W(\cdot, \zeta) \) \((g^{\tilde{W}}(\cdot, \tilde{\zeta}), \text{ resp.})\) is a Green's function on \( W \) \((\tilde{W}, \text{ resp.})\) with pole at \( \zeta \) \((\tilde{\zeta}, \text{ resp.})\). Note that

\[
k^W(\zeta)(a) = k_{\tilde{\zeta}}^{\tilde{W}}(\tilde{a}) = 1.
\]

In our previous paper [M-S2], we proved the following.
Proposition 2.2. Suppose $\zeta \in \Delta^W$. Then

(i) If $\zeta \in \Delta^W \setminus \Delta_1^W$, then $\nu_{\tilde{W}}(\zeta) = 0$;
(ii) If $\zeta \in \Delta_1^W$, then $1 \leq -\nu_{\tilde{W}}(\zeta) \leq p$;
(iii) If $\zeta \in \Delta_1^W$ and $\Delta_1^W(\zeta) = \{\tilde{\zeta}_1, \ldots, \tilde{\zeta}_n\}_f$, then there exist positive numbers $c_1, \ldots, c_n$ such that

\[
(2.3) \quad k^W_{\zeta} \circ \pi = c_1 k^W_{\tilde{\zeta}_1} + \cdots + c_n k^W_{\tilde{\zeta}_n}.
\]

In the relation (2.3) above, by (2.1) and (2.2), we have

\[
(2.4) \quad \sum_{i=1}^n c_n = 1.
\]

Let $s$ be a positive superharmonic function on $W$ and $E$ is a subset of $W$. We denote by $w\tilde{R}_E^F$ the balayage of $s$ with respect to $E$ on $W$. We here give the definitions of minimal thinness and minimal fine neighborhood (cf. [B]).

DEFINITION 2.1. Let $\zeta$ be a point of $\Delta_1^W$ and $E$ a subset of $W$. We say that $E$ is minimally thin at $\zeta$ if $w\tilde{R}_E^F \neq k^W_\zeta$.

DEFINITION 2.2. Let $\zeta$ be a point of $\Delta_1^W$ and $U$ a subset of $W$. We say that $U \cup \{\zeta\}$ is a minimal fine neighborhood of $\zeta$ if $W \setminus U$ is minimally thin at $\zeta$.

The following is easily verified from Proposition 3.1 of our previous paper [M-S2] (see also [M]).

Proposition 2.3. Let $\tilde{\zeta} \in \Delta_1^W$ and $\tilde{U}$ a subset of $\tilde{W}$. Then $\tilde{U} \cup \{\tilde{\zeta}\}$ is a minimal fine neighborhood of $\tilde{\zeta}$ if and only if $\pi(\tilde{U}) \cup \{\pi(\tilde{\zeta})\}$ is a minimal fine neighborhood of $\pi(\tilde{\zeta})$.

For $\zeta \in \Delta_1^W$, we denote by $\mathcal{M}_W(\zeta)$ the class of connected open sets $M$ such that $W \setminus M$ is minimally thin at $\zeta$. Moreover, for $M \in \mathcal{M}_W(\zeta)$ and a $p$-sheeted unlimited covering surface $\tilde{W}$ of $W$ with projection map $\pi$, we denote by $n_{\tilde{W}}(M)$ the number of connected components of $\pi^{-1}(M)$. Then $\nu_{\tilde{W}}(\zeta)$ is characterized by $n_{\tilde{W}}(M)$ as follows, which is a main result of our previous paper [M-S2].

Proposition 2.4. Suppose $\zeta \in \Delta_1^W$. Then $\nu_{\tilde{W}}(\zeta) = \max_{M \in \mathcal{M}_W(\zeta)} n_{\tilde{W}}(M)$.

3. Proof of Theorem 1
In this section, we give the proof of Theorem 1. For the sake of simplicity, we introduce the following notation:

\[ \Delta = \Delta^W, \quad \Delta_1 = \Delta_1^W, \quad \check{\Delta} = \check{\Delta}^W, \quad \check{\Delta}_1 = \check{\Delta}_1^W, \quad \check{\Delta}(\zeta) = \Delta^W(\zeta) \]

and

\[ k_\zeta = k_\zeta^W, \quad \check{k}_\zeta = \check{k}_\zeta^W. \]

**Proof of Theorem 1.** Assume that \( HP(W) \circ \pi = HP(\check{W}) \). Let \( \zeta \) be an arbitrary point in \( \Delta_1 \). We need to show that \( \check{\Delta}_1(\zeta) \) consists of a single point. Take a point \( \check{\zeta} \in \check{\Delta}_1(\zeta) \). By Proposition 2.2 (iii), there exists a positive constant \( c \) such that

\[ (3.1) \quad c \check{k}_\zeta \leq k_\zeta \circ \pi \]

on \( \check{W} \). By assumption, there exists an \( h \in HP(W) \) such that

\[ (3.2) \quad \check{k}_\zeta = h \circ \pi \]

on \( \check{W} \). Hence, by (3.1), we see that \( ch \leq k_\zeta \) on \( W \). This with minimality of \( k_\zeta \) implies that there exists a positive constant \( c_1 \) such that

\[ (3.3) \quad h = c_1 k_\zeta \]

on \( W \). Hence, by (3.2), we see that \( \check{k}_\zeta = c_1 k_\zeta \circ \pi \) on \( \check{W} \). From this with (2.1) and (2.2), it follows that \( c_1 = 1 \). Therefore we obtain

\[ (3.4) \quad \check{k}_\zeta = k_\zeta \circ \pi \]

on \( \check{W} \). This yields that \( \check{\Delta}_1(\zeta) = \{ \check{\zeta} \} \).

Conversely, assume that \( \nu_{\check{W}}(\zeta) = 1 \) for every \( \zeta \in \Delta_1 \). We only need to show \( HP(\check{W}) \subset HP(W) \circ \pi \), since the reversed inclusion is trivial. By assumption, we set \( \check{\Delta}_1(\zeta) = \{ \check{\zeta} \} \) for each \( \zeta \in \Delta_1 \). By Proposition 2.2 (iii) and (2.4), we have

\[ (3.5) \quad \check{k}_\zeta = k_\zeta \circ \pi \]

for every \( \zeta \in \Delta_1 \). Take an arbitrary \( \check{h} \) in \( HP(\check{W}) \). By the Martin representation theorem (cf. e.g. [ ], [ ] and [ ]), there exists a Radon measure \( \check{\mu} \) on \( \check{\Delta} \) with \( \check{\mu}(\check{\Delta} \setminus \check{\Delta}_1) = 0 \) such that

\[ (3.6) \quad \check{h} = \int \check{k}_\zeta d\check{\mu}(\zeta). \]

Choose arbitrary two points \( \check{z}_1 \) and \( \check{z}_2 \) in \( \check{W} \) with \( \pi(\check{z}_1) = \pi(\check{z}_2) \). In view of (3.5) and (3.6), we obtain

\[ \check{h}(\check{z}_1) = \int \check{k}_\zeta(\check{z}_1) d\check{\mu}(\zeta) = \int \check{k}_\zeta(\check{z}_2) d\check{\mu}(\zeta) = \check{h}(\check{z}_2). \]
Therefore we deduce that \( \tilde{h} \in HP(W) \circ \pi \) for every \( \tilde{h} \in HP(\tilde{W}) \), and hence \( HP(\tilde{W}) \subset HP(W) \circ \pi \).

The proof is herewith complete. \( \square \)

4. Proof of Theorem 2

In this section, we give the proof of Theorem 2. Let \( \omega_z(\cdot) \) (\( \tilde{\omega}_{\overline{z}}(\cdot) \), resp.) be the harmonic measure on \( \Delta \) (\( \tilde{\Delta} \), resp.) with respect to \( W \) (\( \tilde{W} \), resp.) and \( z \in W \) (\( \tilde{z} \in \tilde{W} \), resp.). It is well-known that harmonic measure is a Radon measure (cf. e.g. [C-C]). It is also well-known that \( \omega_z(\cdot) \) (\( \tilde{\omega}_{\overline{z}}(\cdot) \), resp.) can be extended to the outer measure on \( \Delta \) (\( \tilde{\Delta} \), resp.) by

\[
\omega_z(E) = \inf \{ \omega_z(B) : B \text{ is a Borel set with } E \subset B \} \\
(\tilde{\omega}_{\overline{z}}(\tilde{E}) = \inf \{ \tilde{\omega}_{\overline{z}}(\tilde{B}) : \tilde{B} \text{ is a Borel set with } E \subset B \}, \text{ resp.)}
\]

for a subset \( E \) (\( \tilde{E} \), resp.) of \( \Delta \) (\( \tilde{\Delta} \), resp.). It is known that \( h(z) = \omega_z(E) \) is a nonnegative harmonic function on \( W \) for every \( E \subset \Delta \). By minimum principle, it is obvious that, for an arbitrary \( E(\subset \Delta) \) (\( \tilde{E} \subset \tilde{\Delta} \), resp.), \( \omega_z(E) = 0 \) (\( \tilde{\omega}_{\overline{z}}(\tilde{E}) = 0 \), resp.) for a \( z \in W \) (\( \tilde{z} \in \tilde{W} \), resp.) if and only if \( \omega_z(E) = 0 \) (\( \tilde{\omega}_{\overline{z}}(\tilde{E}) = 0 \), resp.) for all \( z \in W \) (\( \tilde{z} \in \tilde{W} \), resp.). Let \( f \) be a real-valued function on the Martin boundary \( \Delta^R \) of an open Riemann surface \( R \). We denote by \( H^R_W \) (\( \overline{H}^R_f \), resp.) the solution (upper solution, resp.) of Dirichlet problem on \( R (= W \) or \( \tilde{W} \) with boundary values \( f \) in the sense of Perron-Wiener-Brelot. We first prove the following.

**Lemma 4.1.** Let \( \tilde{E} \) be a subset of \( \tilde{\Delta} \). Then \( \tilde{\omega}_{\overline{z}}(\tilde{E}) = 0 \) if and only if \( \omega_z(\pi(\tilde{E})) = 0 \).

**Proof.** Suppose that \( \tilde{\omega}_{\overline{z}}(\tilde{E}) = 0 \). By definition, there exists a Borel set \( \tilde{B} \subset \tilde{\Delta} \) with \( \tilde{E} \subset \tilde{B} \) such that

\[
(4.1) \quad \tilde{\omega}_{\overline{z}}(\tilde{B}) = \overline{H}^W_{\tilde{B}}(\tilde{z}) = 0,
\]

where \( 1_{\tilde{B}} \) is the characteristic function of \( \tilde{B} \) on \( \tilde{\Delta} \). Let \( \tilde{s} \) be an arbitrary positive superharmonic function on \( \tilde{W} \) such that \( \liminf_{\tilde{z} \to \zeta} \tilde{s}(\tilde{z}) \geq 1 \) for every \( \zeta \in \tilde{B} \). Set

\[
s(\zeta) := \sum_{\tilde{z} \in \pi^{-1}(\zeta)} m(\tilde{z})\tilde{s}(\tilde{z}),
\]

where \( m(\tilde{z}) \) is multiplicity of \( \pi \) at \( \tilde{z} \). Then \( s(\zeta) \) is a positive superharmonic function on \( W \) and \( \liminf_{\zeta \to \pi(\tilde{B})} s(\zeta) \geq 1 \) for every \( \zeta \in \pi(\tilde{B}) \). Hence \( s(\zeta) \geq \overline{H}^W_{\pi(\tilde{B})}(\zeta) \). From this and the fact \( \overline{H}^W_{\pi(\tilde{B})}(\zeta) \geq \omega_z(\pi(\tilde{B})) \) (cf. e.g. [C-C]), it follows that

\[
s(\zeta) \geq \omega_z(\pi(\tilde{B})) \geq \omega_z(\pi(\tilde{E})).
\]
Therefore, by letting $s(z)$ arbitrarily small in view of (4.1), we obtain $\omega_z(\pi(\tilde{E})) = 0$

Suppose $\omega_z(\pi(\tilde{E})) = 0$. By definition, there exists a Borel set $B \subset \Delta$ with $B \supset \pi(\tilde{E})$ such that

\[(4.2) \quad \omega_z(B) = H^W_{1_B}(z) = 0.\]

Let $s$ be an arbitrary positive superharmonic function on $W$ such that $\liminf_{z \to \zeta} s(z) \geq 1$ for every $\zeta \in B$. Then $s \circ \pi(\tilde{z})$ is a positive superharmonic function on $\tilde{W}$ and $\liminf_{\zeta \to \tilde{\zeta}} s \circ \pi(\tilde{\zeta}) \geq 1$ for every $\tilde{\zeta} \in \pi^{-1}(B)$. Hence $s \circ \pi(\tilde{\zeta}) \geq H_{1_B}^{\tilde{W}}(\tilde{\zeta})$. From this and the fact $H_{1_B}^{\tilde{W}}(\tilde{\zeta}) \geq \tilde{\omega}_z(\pi^{-1}(B))$, it follows that

\[s \circ \pi(\tilde{\zeta}) \geq \tilde{\omega}_z(\pi^{-1}(B)) \geq \tilde{\omega}_z(\pi(\tilde{E})).\]

Therefore, letting $s \circ \pi(\tilde{z})$ arbitrarily small in view of (4.2), we obtain $\tilde{\omega}_z(\tilde{E}) = 0$.

The proof is herewith complete. \(\square\)

We next consider the sets

\[N_1 := \{\zeta \in \Delta_1 : \nu_{\tilde{W}}(\zeta) = 1\}\]

and

\[N_2 := \Delta_1 \setminus N_1 = \{\zeta \in \Delta_1 : \nu_{\tilde{W}}(\zeta) \geq 2\}.\]

Put $\tilde{N}_1 = \pi^{-1}(N_1) \cap \Delta_1$ and $\tilde{N}_2 = \pi^{-1}(N_2) \cap \Delta_1$. By means of Proposition 2.2, it is easily seen that $\tilde{N}_1 \cup \tilde{N}_2 = \Delta_1$ and $\pi(\tilde{N}_i) = N_i (i = 1, 2)$. We denote by $d(\cdot, \cdot)$ the metric on $\tilde{W}$ defined by

\[d(z, \zeta) = \sum_{n=1}^{\infty} \frac{1}{2^n} \left| \frac{k_z(z_n)}{1 + k_z(z_n)} - \frac{k_{\zeta}(z_n)}{1 + k_{\zeta}(z_n)} \right|,\]

where $\{z_n : n = 1, 2, \ldots\}$ is a dense subset of $\tilde{W}$. Set $\tilde{U}_r(\tilde{z}_0) = \{\tilde{z} \in \tilde{W}^* : d(\tilde{z}, \tilde{z}_0) < r\}$ for $\tilde{z}_0 \in \tilde{W}^*$ and $r > 0$.

**Lemma 4.2.** Suppose $\omega_z(N_2) > 0$. Then there exists a $\tilde{\zeta}_0 \in \tilde{N}_2$ such that $\tilde{\omega}_z(\tilde{N}_2 \cap \tilde{U}_r(\tilde{z}_0)) > 0$ for every $r > 0$.

**Proof.** By virtue of Lemma 4.1, we have $\tilde{\omega}_z(\tilde{N}_2) > 0$, since $\pi(\tilde{N}_2) = N_2$. Contrary to the assertion, assume that, for every $\tilde{\zeta} \in \tilde{N}_2$, there exists an $r_{\tilde{\zeta}} > 0$ such that $\tilde{\omega}_z(\tilde{N}_2 \cap \tilde{U}_{r_{\tilde{\zeta}}}(\tilde{\zeta})) = 0$. Then, by the Lindelöf covering theorem, there exists a sequence $\{\tilde{\zeta}_j\}_{j=1}^{\infty}$ in $\tilde{N}_2$ such that $\tilde{N}_2 \subset \cup_{j=1}^{\infty} \tilde{U}_{r_{\tilde{\zeta}_j}}(\tilde{\zeta}_j)$. Hence we have

\[\tilde{\omega}_z(\tilde{N}_2) \leq \sum_{j=1}^{\infty} \tilde{\omega}_z(\tilde{N}_2 \cap \tilde{U}_{r_{\tilde{\zeta}_j}}(\tilde{\zeta}_j)) = 0,\]
which is a contradiction.

Here, we again recall the definition of $\tilde{\Delta}_1(\zeta)$:

$$\tilde{\Delta}_1(\zeta) = \Delta_1 \cap \pi^{-1}(\zeta) = \{\tilde{\zeta} \in \tilde{\Delta}_1 : \pi(\tilde{\zeta}) = \zeta\}.$$ 

**Lemma 4.3.** Let $\tilde{\xi}$ be a point in $\overline{N}_2$. Then there exists a $\rho > 0$ such that $\tilde{\Delta}_1(\zeta) \setminus \tilde{U}_\rho(\tilde{\xi})$ is not empty for every $\zeta \in N_2 \cap \pi(\tilde{U}_\rho(\tilde{\xi}))$.

**Proof.** Set $\pi(\tilde{\xi}) = \xi$. Then, by definition, $\xi \in N_2$. Assume that the assertion is false. Then there exists a sequence $\{\zeta_j\}_{j=1}^\infty$ in $N_2 \setminus \{\pi(\tilde{\xi})\}$ such that

$$\tilde{d}(\tilde{\Delta}_1(\zeta_j), \tilde{\xi}) < 1/j.$$ 

From this it follows that

$$\lim_{j \to \infty} k_{\zeta_j} = k_{\xi}.$$ 

By Proposition 2.2 and (2.4), for each $j$, there exist positive constants $c_{j1}, \ldots, c_{jn_j}$ with $\sum_{i=1}^{n_j} c_{ji} = 1$ such that

$$k_{\zeta_j} \circ \pi = \sum_{i=1}^{n_j} c_{ji} \tilde{k}_{\tilde{\zeta}_{ji}},$$

where $\tilde{\Delta}_1(\zeta_j) = \{\tilde{\zeta}_{j1}, \ldots, \tilde{\zeta}_{jn_j}\}$. Then, in view of (4.3), we see that

$$\lim_{j \to \infty} \tilde{k}_{\tilde{\zeta}_{ij}} = \tilde{k}_{\tilde{\xi}}$$

independently of choice of $i_j$ in $\{1, \ldots, n_j\}$. This with (4.4) and (4.5) implies that

$$k_{\xi} \circ \pi = \tilde{k}_{\tilde{\xi}}.$$ 

Therefore, by means of Proposition 2.2, we obtain $\tilde{\Delta}_1(\xi) = \{\tilde{\xi}\}$, which contradicts $\xi \in N_2$. This completes the proof. \hfill \square

We can restate Theorem 2, in terms of the set $N_2$, as follows: The relation $HB(W) \circ \pi = HB(\tilde{W})$ holds if and only if $\omega_z(N_2) = 0$.

**Proof of Theorem 2.** We first prove 'if' part. Suppose $\omega_z(N_2) = 0$. Then, by Lemma 4.1,

$$\check{\omega}_z(\overline{N}_2) = 0.$$ 

Take an arbitrary $\tilde{h} \in HB(\tilde{W})$. We only need to show $\tilde{h} \in HB(W) \circ \pi$. Adding a constant to $\tilde{h}$, we may assume that $\tilde{h} > 0$ on $\tilde{W}$. Let $c(> 0)$ be the supremum of $\tilde{h}$ on
By the Martin representation theorem, there exist Radon measures \( \tilde{\mu} \) and \( \tilde{\chi} \) on \( \tilde{\Delta} \) with 
\[
\tilde{\mu}(\tilde{\Delta} \setminus \tilde{\Delta}_1) = 0 \quad \text{and} \quad \tilde{\chi}(\tilde{\Delta} \setminus \tilde{\Delta}_1) = 0
\]
such that

\[
(4.7) \quad \tilde{h}(\tilde{z}) = \int \tilde{k}_\zeta(\tilde{z})d\tilde{\mu}(\zeta)
\]

and

\[
(4.8) \quad 1 = \int \tilde{k}_\zeta(\tilde{z})d\tilde{\chi}(\zeta).
\]

Then

\[
c \int \tilde{k}_\zeta(\tilde{z})d\tilde{\chi}(\zeta) = c \geq \tilde{h}(\tilde{z}) = \int \tilde{k}_\zeta(\tilde{z})d\tilde{\mu}(\zeta).
\]

Hence, by uniqueness of representing measure, we have

\[
(4.9) \quad c\tilde{\chi} \geq \tilde{\mu}.
\]

Note that \( \tilde{k}_\zeta(\tilde{z})d\tilde{\chi}(\zeta) = d\tilde{\omega}_\zeta(\tilde{\zeta}) \) (cf. [C-C, p.140]). From this and (4.9) it follows that

\[
\int_{\tilde{N}_2} \tilde{k}_\zeta(\tilde{z})d\tilde{\mu}(\zeta) \leq c \int_{\tilde{N}_2} \tilde{k}_\zeta(\tilde{z})d\tilde{\chi}(\zeta) = c \int_{\tilde{N}_2} d\tilde{\omega}_\zeta(\zeta) = c\tilde{\omega}(\tilde{N}_2).
\]

This with (4.6) yields that

\[
\int_{\tilde{N}_2} \tilde{k}_\zeta(\tilde{z})d\tilde{\mu}(\zeta) = 0.
\]

Therefore, by (4.7) and the fact \( \tilde{N}_1 \cup \tilde{N}_2 = \tilde{\Delta}_1 \), we have

\[
\tilde{h}(\tilde{z}) = \int_{\tilde{N}_1} \tilde{k}_\zeta(\tilde{z})d\tilde{\mu}(\zeta).
\]

Since \( \tilde{k}_\zeta \in HP(W) \circ \pi \) for every \( \zeta \in \tilde{N}_1 \), this implies that \( \tilde{h} \in HP(W) \circ \pi \cap HB(\tilde{W}) \subset HB(W) \circ \pi \).

We next prove 'only if' part. Suppose \( \omega_\zeta(N_2) > 0 \). Then, by Lemma 4.2, there exists a \( \tilde{\xi} \in \tilde{N}_2 \) such that

\[
(4.10) \quad \tilde{\omega}_\xi(\tilde{N}_2 \cap \tilde{U}_r(\tilde{\xi})) > 0
\]

for every \( r > 0 \). Moreover, by Lemma 4.3, there exists \( \rho > 0 \) such that

\[
(4.11) \quad \tilde{\Delta}_1(\zeta) \setminus \tilde{U}_\rho(\tilde{\xi}) \neq \emptyset
\]

for every \( \zeta \in N_2 \cap \pi(\tilde{U}_\rho(\tilde{\xi})) \). Set

\[
\tilde{E}_1 = \tilde{N}_2 \cap \tilde{U}_\rho(\tilde{\xi}).
\]

Then, by (4.10) and Lemma 4.1, we have

\[
(4.12) \quad \omega_\zeta(\pi(\tilde{E}_1)) > 0.
\]
Set
\[ \tilde{E}_2 = \overline{N}_2 \cap \pi^{-1}(\pi(\tilde{U}_{\rho/2}(\tilde{\xi})) \setminus \tilde{U}_\rho(\tilde{\xi}). \]

Inview of (4.11), we find that
\[ \pi(\tilde{E}_1) = \pi(\tilde{E}_2). \]

Put \( \tilde{h}(\tilde{z}) = \tilde{\omega}_z(\tilde{E}_1). \) Then \( \tilde{h}(\tilde{z}) \) is a bounded harmonic function on \( \overline{W}. \) We only need to show \( \tilde{h} \not\in HB(W) \circ \pi \). By the Fatou-Naïm-Dood theorem (cf. [C-C, p.152]), \( \tilde{h}(\tilde{z}) \) has fine limit 1 \((0, \text{ resp.})\) at almost all \( \tilde{\zeta} \) in \( \tilde{E}_1 \) (\( \tilde{E}_2 \), resp.) with respect to \( \tilde{\omega}_z \), since \( \tilde{E}_1 \cap \tilde{E}_2 = \emptyset \). Accordingly there exists a subset \( \tilde{F}_1 \) (\( \tilde{F}_2 \), resp.) of \( \tilde{E}_1 \) (\( \tilde{E}_2 \), resp.) with \( \tilde{\omega}_z(\tilde{F}_1) = 0 \) \((\tilde{\omega}_z(\tilde{F}_2) = 0, \text{ resp.})\) such that, for every \( \tilde{\zeta} \) in \( \tilde{E}_1 \setminus \tilde{F}_1 \) (\( \tilde{E}_2 \setminus \tilde{F}_2 \), resp.),
\[ (4.14) \quad \mathcal{F} - \lim_{\tilde{z} \to \zeta} \tilde{h}(\tilde{z}) = 1 \quad (\mathcal{F} - \lim_{\tilde{z} \to \zeta} \tilde{h}(\tilde{z}) = 0, \text{ resp.}) \]

Then, by Lemma 4.1, \( \omega_z(\pi(\tilde{F}_1) \cup \pi(\tilde{F}_2)) = 0 \). Hence, by (4.12) and (4.13), there exist points \( \tilde{\zeta}_1 \in \tilde{E}_1 \setminus \tilde{F}_1 \) and \( \tilde{\zeta}_2 \in \tilde{E}_2 \setminus \tilde{F}_2 \) with \( \pi(\tilde{\zeta}_1) = \pi(\tilde{\zeta}_2) \). This with (4.14) implies that there exists an open subset \( \tilde{O}_1 \) (\( \tilde{O}_2 \), resp.) of \( \tilde{W} \) such that \( \tilde{O}_1 \cup \{\tilde{\zeta}_1\} \) (\( \tilde{O}_2 \cup \{\tilde{\zeta}_2\} \), resp.) is a minimal fine neighborhood of \( \tilde{\zeta}_1 \) (\( \tilde{\zeta}_2 \), resp.) and that
\[ (4.15) \quad \inf_{\tilde{z} \in \tilde{O}_1} \tilde{h}(\tilde{z}) \geq \frac{2}{3} \quad (\sup_{\tilde{z} \in \tilde{O}_2} \tilde{h}(\tilde{z}) \leq \frac{1}{3}, \text{ resp.}) \]

Then, by virtue of Proposition 2.3, we see that \( \pi(\tilde{O}_1) \cap \pi(\tilde{O}_2) \cup \{\pi(\tilde{\zeta}_1)\} \) is a minimal fine neighborhood of \( \pi(\tilde{\zeta}_1) = \pi(\tilde{\zeta}_2) \), and hence \( \pi(\tilde{O}_1) \cap \pi(\tilde{O}_2) \neq \emptyset \). Therefore, by (4.15), there exists a subset \( \tilde{U}_j \) of \( \tilde{O}_j \) \((j = 1, 2)\) with \( \pi(\tilde{U}_j) = \pi(\tilde{U}_2) \) such that
\[ \inf_{\tilde{z} \in \tilde{U}_1} \tilde{h}(\tilde{z}) \geq \frac{2}{3} \quad (\sup_{\tilde{z} \in \tilde{U}_2} \tilde{h}(\tilde{z}) \leq \frac{1}{3}, \text{ resp.}) \]

This means that \( \tilde{h} \not\in HB(W) \circ \pi \).

The proof is herewith complete. \( \square \)

5. Harmonic functions on covering surfaces of the unit disc

Let \( D \) be the unit disc \( \{|z| < 1\} \). In this section, we are concerned with application of Theorem 1 and Theorem 2 in case base surface is \( D \). As is wellknown, the Martin compactification \( D^* \) of \( D \) is identified with the closure \( \overline{D} \) of \( D \) with respect to Euclidian topology and the Martin boundary \( \Delta^D \) of \( D \) consists of only minimal points. In this view, we regard \( \partial D = \{|z| = 1\} \) as the (minimal) Martin boundary of \( D \).

To state our main result of this section, we introduce some notations. For a discrete subset \( A \) of \( D \), we denote by \( B_p(A) \) the class of \( p \)-sheeted unlimited covering surface \( \overline{D} \) of \( D \) such that there exists a branch point in \( \overline{D} \) of order \( p - 1 \) (or multiplicity \( p \)) over every
If \( z \in A \) and there exist no branch points in \( \overline{D} \) over \( D \setminus A \). In addition to the Euclidean metric, we consider the pseudohyperbolic metric on \( D \) given by

\[
\rho(z, w) = \frac{|z - w|}{1 - \overline{w}z}.
\]

For \( \zeta \in \partial D \) and a positive number \( C(<1) \), we also consider the Stolz type domain with vertex \( \zeta \) given by

\[
S_C(\zeta) = \{ z \in D : C|z - \zeta| < 1 - |z| \}.
\]

**Theorem 5.1.** Let \( A = \{ a_n : n \in \mathbb{N} \} \) be a discrete subset of \( D \) and \( \overline{D} \) belong to \( B_p(A) \). Suppose that there exists a positive constant \( C(<1) \) satisfying the following two conditions

(i) for every pair \( (a_m, a_n) \) in \( A \) with \( a_m \neq a_n \), \( \rho(a_m, a_n) \geq C \);

(ii) for every \( \zeta \in \partial D \), there exists a subset \( B_\zeta = \{ b_n : n \geq n_0 \} \) \((n_0 = n_0(\zeta))\) of \( A \) such that \( b_n \in \{ z : 2^{-n-1} \leq |z - \zeta| \leq 2^{-n} \} \cap S_C(\zeta) \) for every \( n \geq n_0 \).

Then \( HP(\overline{D}) = HP(D) \circ \pi \), where \( \pi \) is the projection map.

For a bounded Borel subset \( K \) of \( C \), we denote by \( \lambda(K) \) the logarithmic capacity. As a necessary condition for minimal thinness, the following is available (cf. [LF],[J]).

**Lemma 5.1.** Let \( \zeta \) be in \( \partial D = \Delta^D_1 \) and \( E \) a relatively closed subset of \( S_C(\zeta) \). If \( E \) is minimally thin at \( \zeta \), then

\[
\sum_{n=1}^{\infty} \frac{1}{\log \frac{1}{\lambda(E_n)}} < \infty,
\]

where \( E_n = E \cap \{ z : 2^{-n-1} \leq |z - \zeta| \leq 2^{-n} \} \).

**Proof of Theorem 5.1.** Let \( \zeta \) be an arbitrary point in \( \partial D \). By virtue of Theorem 1, we only have to show that \( \Delta^D_1(\zeta) \) consists of a single point. Take an arbitrary \( M \in \mathcal{M}_D(\zeta) \). Our goal is to show that \( \pi^{-1}(M) \) is connected. In fact, in view of Proposition 2.4, connectivity of \( \pi^{-1}(M) \) for all \( M \in \mathcal{M}_D(\zeta) \) implies \( \Delta^D_1(\zeta) \) consists of a single point.

We first assume that there exists an \( a_n \in M \cap A \neq \emptyset \). Then, it is easily seen that \( \pi^{-1}(M) \) is connected, since \( \overline{D} \) has a branch point of order \( p - 1 \) over \( a_n \in M \) and \( M \) is connected.

We next assume \( M \cap A = \emptyset \). Put \( F = D \setminus M \). Note that \( F \) is minimally thin at \( \zeta \) and relatively closed in \( D \). For each \( n(\geq n_0) \), let \( F_n \) be the connected component of \( F \) which contains \( b_n \in B_\zeta \). We also assume that there exists an \( F_n \) \((n \geq n_0)\) such that

\[
d(F_n) < C^22^{-n-1},
\]

where \( d(F_n) \) indicates the diameter of \( F_n \). Then there exists a closed Jordan curve \( \gamma_n \) in \( M \setminus A \) such that \( \gamma_n \) surrounds \( F_n \) and

\[
d(F_n) < d(\gamma_n) < C^22^{-n-1}.
\]
By (i) and (ii), we have

$$|a_m - b_n| \geq C|1 - \overline{b}a| \geq C(1 - |b_n|) \geq C^22^{-n-1},$$

for every $a_m \in A \setminus \{b_n\}$. Hence, by means of (5.2), we see that $\gamma_n$ surrounds only one point $b_n$ in $A$. Therefore, $\pi^{-1}(\gamma_n)$ is connected, since $D$ has a branch point of order $p - 1$ over $b_n$. This with $\gamma_n \in M$ and connectivity of $M$ yields that $\pi^{-1}(M)$ is connected. Accordingly, we completes the proof if we show that there exists an $F_n (n \geq n_0)$ satisfying (5.1).

We assume that

$$(5.3) \quad d(F_n) \geq C^22^{-n-1}$$

for every $n(\geq n_0)$. Set $E = F \cap S_{\frac{c}{2}}(\zeta)$. Note that $E$ is minimally thin at $\zeta$. We denote by $F_n^*$ the connected component of $E$ which contains $b_n$. Then, in view of (ii) and (5.3), we find that there exists a positive constant $C_1(\leq C^2/2)$ such that

$$(5.4) \quad d(F_n^*) \geq C_12^{-n}$$

for every $n(\geq n_0)$. Set $E_n = E \cap \{z : 2^{-n-1} \leq |z - \zeta| \leq 2^{-n}\}$. Note that $b_n \in E_n$. Then, by (5.4), we see that, for every $n \geq n_0$, at least one of $\{E_{n-1}, E_n, E_{n+1}\}$ contains a continuum whose diameter is equal to or greater than $C_12^{-n-1}$. From this it follows that

$$\max\{\lambda(E_{n-1}), \lambda(E_n), \lambda(E_{n+1})\} \geq C_12^{-n-3}$$

for every $n(\geq n_0)$ (cf.[T]). Hence we see that

$$\frac{1}{\log \frac{1}{\lambda(E_{n-1})}} + \frac{1}{\log \frac{1}{\lambda(E_n)}} + \frac{1}{\log \frac{1}{\lambda(E_{n+1})}} \geq \frac{n \log 2 + \log(8/C_1)}{n \log 2 + \log(8/C_1)}$$

for every $n(\geq n_0)$. Therefore we deduce

$$\sum_{n=n_0-1}^{\infty} \frac{1}{\log \frac{1}{\lambda(E_n)}} \geq \frac{1}{3} \sum_{n=n_0}^{\infty} \left( \frac{1}{\log \frac{1}{\lambda(E_{n-1})}} + \frac{1}{\log \frac{1}{\lambda(E_n)}} + \frac{1}{\log \frac{1}{\lambda(E_{n+1})}} \right) \geq \frac{1}{3} \sum_{n=n_0}^{\infty} \frac{1}{n \log 2 + \log(8/C_1)} = \infty.$$

By Lemma 5.1, this is absurd, since $E$ is minimally thin at $\zeta$.

The proof is herewith complete. \qed

Using the notation above, we restate Proposition in Introduction as follows:

**Corollary 5.1.** Let $A = \{(1 - 2^{-n-1})e^{2\pi k/2^{n+2}} : n = 1, 2, ..., k = 1, ..., 2^{n+2}\}$ and $D$ belong to $B_p(A)$. Then $HP(D) \circ \pi = HP(D)$, where $\pi$ is the projection map.
Proof. It is easily seen that $A$ and a positive constant $C$ satisfy the condition (i) of Theorem 5.1. Let $\zeta$ be an arbitrary point in $\partial D$. For every positive integer $n$, we can choose a positive integer $k_n$ with $1 \leq k_n \leq 2^{n+2}$ such that

$$|\arg \zeta - \frac{2\pi k_n}{2^{n+2}}| \leq \frac{\pi}{2^{n+2}}.$$  \hfill (5.5)

Set

$$b_n = (1 - 2^{-n-1})e^{i2\pi k_n/2^{n+2}} \quad (n = 1, 2, \ldots).$$

Then, by (5.5), we have

$$(2^{-n-1})^2 \leq |b_n - \zeta|^2 \leq (2^{-n-1})^2 + 4\sin^2 \frac{\pi}{2^{n+3}}.$$

In view of this with (5.5), it is easily seen that $B_\zeta := \{b_n : n \geq 1\}$ and a positive constant $C$ satisfy the condition (ii) of Theorem 5.1. $\square$

At the last, we give a $p$-sheeted unlimited covering surface $\bar{D}_1$ of $D$ with projection map $\pi$ such that $HB(D) \circ \pi = HB(\bar{D}_1)$ and $HP(D) \circ \pi \neq HP(\bar{D}_1)$. Let $A$ be the same as in Corollary 5.1. Set $M = \{|z - \frac{1}{2}| < \frac{1}{2}\}$ and $A_1 = A \setminus M$. Consider a covering surface $D_1 \in B_p(A_1)$ with projection map $\pi$. We now show that $HB(D) \circ \pi = HB(\bar{D}_1)$ and $HP(D) \circ \pi \neq HP(\bar{D}_1)$. As is proved in the proof of Corollary 5.1, $A_1$ and a positive constant $C$ satisfy the following two conditions:

(i) for every pair $(a_m, a_n)$ in $A_1$ with $a_m \neq a_n$, $\rho(a_m, a_n) \geq C$;

(ii) for every $\zeta \in \partial D \setminus \{1\}$, there exist a subset $B_\zeta = \{b_n : n \geq n_0\}$ ($n_0 = n_0(\zeta)$) of $A_1$ such that $b_n \in \{z : 2^{-n-1} \leq |z - \zeta| \leq 2^{-n}\} \cap S_C(\zeta)$ for every $n \geq n_0$.

Therefore the proof of Theorem 5.1 yields that $\nu_{\bar{D}_1}(\zeta) = 1$ for every $\zeta \in \partial D \setminus \{1\}$. Hence, by virtue of Theorem 2, we have $HB(D) \circ \pi = HB(\bar{D}_1)$. On the other hand, it is easily seen that $M$ belongs to $\mathcal{M}_D(1)$ and $\pi^{-1}(M)$ consists of $p$ components. Hence, by Proposition 2.2 and 2.4, $\nu_{\bar{D}_1}(1) = p(> 1)$. Therefore, by Theorem 1, we see that $HP(D) \circ \pi \neq HP(\bar{D}_1)$. 

References


