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Subnormal operators regarded as generalized observables and compound-system-type normal extension related to $\mathfrak{su}(1,1)$

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Abstract

In this paper, subnormal operators, not necessary bounded, are discussed as generalized observables. In order to describe not only the information about the probability distribution of the output data of their measurement but also a framework of their implementations, we introduce a new concept 'compound-system-type normal extension', and we derive the 'compound-system-type normal extension' of a subnormal operator defined by an irreducible unitary representation of the algebra $\mathfrak{su}(1,1)$. The squeezed states are characterized as the eigenvectors of an operator from this viewpoint, and the squeezed states in multi-particle systems are shown to be the eigenvectors of the adjoints of these subnormal operators under a representation. The affine coherent states are discussed in the same context as well.

Key Words: Subnormal operator, Positive operator-valued measure(POVM), Normal extension, Algebra $\mathfrak{su}(1,1)$, Eigenvector system, Generalized coherent state, Squeezed state, Affine group.

1 Introduction

In quantum mechanics, observables are described by self-adjoint operators and the probability distributions of the output data of their measurement are described by the spectral measures of those self-adjoint operators and the density operators of states.

When a linear operator has its spectral measure, it is a *normal operator* where its self-adjoint part and its skew-adjoint part commute each other (Lemma 3). In a broader sense, therefore, it can be regarded as a complexified observable. (NB: From this viewpoint, in the following, we will use the expression 'measurement a normal operator' in this wider sense, even if the normal operator is not always self-adjoint. The 'measurement of a normal operator' is regarded as the simultaneous measurement of the self-adjoint part and the skew-adjoint part of the normal operator, as well.) However, the measurements in quantum systems, which are not necessarily the measurements of any observables, are described by the *positive operator-valued measures* (POVM), which are a generalization of spectral measures (Definition 5 and Lemma 4). In this paper, from these viewpoints, we try to treat the observables generalized even for the class of *subnormal operators*[3], which is known as a wider class including the class of normal

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[3]The concept of subnormality was introduced by Halmos [7, 8].

operators. A subnormal operator is defined as the restriction of the normal operator into a narrower domain. The pair of the normal operator and the wider domain is called its *normal extension* (Definition 2). We can define the POVM of a subnormal operator uniquely in a similar sense that we can define the spectral measure of a normal operator uniquely (Definition 15). In other words, the POVM is corresponding to the normal extension of the subnormal operator as will be shown in the proof of Lemma 10. By this correspondence, we will formulate the measurements of the subnormal operators which are not necessary bounded. In this paper, we will not only investigate the POVMs of the subnormal operators but also give some examples of frameworks of their implementations in a physical sense.

There are many cases where the adjoint operator of a subnormal operator has eigenvectors with continuous potency and an over-complete eigenvector system. In these cases, the POVM constructed from the over-complete eigenvector system is just the POVM of the subnormal operator (Lemma 17). Thus the subnormal operator is closely related to eigenvectors with continuous potency and to over-complete function systems, and these relations are important for the discussions on the properties of the subnormal operator. This fact may give us an illusion that the adjoint of any operator with a point spectrum with continuous potency would be a subnormal operator. However, the subnormality is not necessarily guaranteed only by the condition that its adjoint has point a spectrum with continuous potency.

For example, an implementation of the measurement of a subnormal operator has been already known for an actual system in quantum optics. Let Q and P be the multiplication operator and the $(-i)$-times differential operator on the Hilbert space $L^2(\mathbb{R})$. A POVM is constructed from the over-complete eigenvector system of the boson annihilation operator $a_b := \sqrt{1/2}(Q+iP)$ (i.e. known as the coherent states system). Then the POVM is just the POVM of the boson creation operator a_b^* which is a subnormal operator. The measurement of this POVM has been implemented as is shown in the following (see §3, in detail), and is called the *heterodyne measurement*; This implementation is performed by the measurement of a normal operator on the compound system between the basic system (i.e. the system of interest where the measurement is originally discussed) and an additional ancillary system prepared appropriately. Note that this operation, of measuring a normal operator on the compound system by preparing an additional ancillary system, gives a kind of normal extension of creation operator a_b^*. But, only giving the definition of the normal extension is not sufficient for discussing such a physical operation. For clarifying such a physical operation, in §3, we will introduce a new concept *compound-system-type normal extension* which describes not only the normal extension but also a framework of a physical operation (given in Definition 21).

In §4, under the circumstance where an irreducible unitary representation of the algebra $\mathfrak{su}(1,1)$ is given, we will construct two types of operators which have point spectra with continuous potency, and will investigate what condition guarantees the subnormality of these operators. The coherent states of the algebra $\mathfrak{su}(1,1)$ introduced by Perelomov [17], will be reinterpreted as the eigenvectors of these operators. Moreover, in §6, we will derive the compound-system-type normal extensions of these operators when they are subnormal operators.

In §5.2, from the relationship between the irreducible unitary representations of the algebra $\mathfrak{su}(1,1)$ and those of the affine $(ax+b)$ group, we will discuss what subnormal operators are related to the irreducible unitary representations of the affine group. Moreover, we will discuss the correspondence between the eigenvectors of this subnormal operator (or the coherent states of the the algebra $\mathfrak{su}(1,1)$) and the coherent states of the affine group. Hence we will show a relationship between our problem and the irreducible unitary representation of the affine group which is closely related to the continuous wavelet transform.

Next, in §5.3, from the relationship between the representation of the algebra $\mathfrak{su}(1,1)$ and the squeezed states, it will be confirmed that the squeezed states can be described as the coherent states of the algebra $\mathfrak{su}(1,1)$ in our context. In other words, the squeezed states are characterized as the eigenvectors of the operators (with point spectra with continuous potency) which are canonically constructed from an irreducible unitary representations of the algebra $\mathfrak{su}(1,1)$. However, the adjoints of these operators are not necessary subnormal operators and are not directly regarded as generalized observables.

We can easily confirm that the squeezed states are the eigenvectors of a operator with a point spectrum with continuous potency as follows; According to Yuen [21], let $b_{\mu,\nu} := \mu a_b + \nu a_b^*$ with $|\mu|^2 - |\nu|^2 = 1$, and characterize the squeezed state by the eigenvector $|\alpha; \mu, \nu\rangle$ of the operator $b_{\mu,\nu}$ associate with the eigenvalue $\alpha \in \mathbb{C}$. In special cases where $\alpha = 0$, the vector $|0; \mu, \nu\rangle$ can be obtained by operating the action of the group with the generators Q^2, P^2 and $(PQ + QP)$ upon the boson vacuum vector $|0; 1, 0\rangle$. The algebra with these generators satisfies the commutation relations of the algebra $\mathfrak{su}(1,1)$. By operating Q^{-1} (or $(a_b^*)^{-1}$) upon the characteristic equation $b_{\mu,\nu}|0; \mu, \nu\rangle = 0$ from the left, we have the characteristic equations

$$Q^{-1}P|0; \mu, \nu\rangle = i\frac{\mu + \nu}{\mu - \nu}|0; \mu, \nu\rangle \tag{1}$$

$$(a_b^*)^{-1}a_b|0; \mu, \nu\rangle = \frac{\nu}{\mu}|0; \mu, \nu\rangle. \tag{2}$$

In §5.3, we will derive these two equations again and reinterpret them from the viewpoint of the representation theory. In this framework, the operators $Q^{-1}P$ and $(a_b^*)^{-1}a_b$ have point spectra with continuous potency and they are constructed from an irreducible unitary representation of the algebra $\mathfrak{su}(1,1)$ naturally. While the adjoints of these operators are not subnormal operators in the case of one-particle system, the adjoints of these operators are subnormal operators in the cases of two-particle system and multi-particle systems. Hence we can characterize a type of physically interpretable states by tensor-product, as the eigenvectors of the adjoints of subnormal operators in the cases of two-particle and multi-particle systems.

From a more general viewpoint, our investigation in this paper is regarded as a problem of the joint measurement between the self-adjoint part and the skew-adjoint part of a subnormal operator which do not always commute each other. However, we should be careful about the difference between self-adjoint operators and symmetric operators in these discussions, because there are many delicate problems when unbounded operators are treated (§6.1).

2 Subnormal operator and POVM

In this section, we will summarize several well-known lemmas and will modify them for the discussion in the following sections. Some of the well-known lemmas will be extended for unbounded operators, and the proofs of the extended version will be given, as well. In this paper, only densely defined linear operator will be discussed. In the following, $D(X)$ denotes the domain of a linear operator X. and that of subnormal operator in the unbounded case. A densely defend operator X is called *closed* if the domain $D(X)$ is complete with respect to the graph norm $\|\phi\|_{D(X)} := \sqrt{\|\phi\|^2 + \|X\phi\|^2}$. In operator theory, for two densely defined operator X, Y, the operator XY is defined as $\phi \mapsto X(Y(\phi))$ for any vector ϕ belonging to the domain $D(XY) := \{\phi \in D(X) | X(\phi) \in D(Y)\}$. We will begin with reviewing the definition of normal operator

Definition 1 *A closed operator T on $\mathcal{H}$ is called normal if it satisfies the condition $T^*T = TT^*$.*

Remark that the operator X^*X is defined on its domain $D(X^*X) := \{\phi \in D(X) | X\phi \in D(X^*)\}$ and it is self-adjoint and non-negative.

Definition 2 *A closed operator S is called subnormal if there exists a Hilbert space $\mathcal{K}$ including $\mathcal{H}$ and a normal operator T on $\mathcal{K}$ such that $S = TP_{\mathcal{H}}$, where $P_{\mathcal{H}}$ denotes the projection from $\mathcal{K}$ to $\mathcal{H}$. In the following, we call the pair $(\mathcal{K}, T)$ a normal extension of the subnormal operator S.*

For a spectral measure E over $\mathbb{C}$, $\int_{\mathbb{C}} zE(dz)$ denotes the operator $\phi \mapsto \lim_{n\to\infty} \left(\int_{|z|<n} zE(dz)\phi\right)$ with the domain $\left\{\phi \in \mathcal{H} \,\middle|\, \int_{\mathbb{C}} |z|^2 \langle \phi, E(dz)\phi\rangle < \infty \right\}$. Concerning normal operators, the following lemma is well-known. See Chapter 13 of Rudin [19].

Lemma 3 *For a normal operator T, there uniquely exists a spectral measure E_T over $\mathbb{C}$ such that $T = \int_{\mathbb{C}} zE_T(dz)$.*

Lemma 3 tells that a normal operator corresponds to a spectral measure by one to one. Next, we will discuss measurements in a quantum system in order to investigate what is corresponding to Lemma 3 in the case of subnormal operators.

Let $\mathcal{H}$ be a Hilbert space representing a physical system of interest. Then, the state is denoted by a non-negative operator ρ on $\mathcal{H}$ whose trace is 1. It is called a density operator on $\mathcal{H}$, and the set of density operators on $\mathcal{H}$ is denoted by $\mathcal{S}(\mathcal{H})$. Let P_ρ be the probability distribution given by a density ρ and a measurement. Then, the probabilistic property of the measurement is described by the map $P : \rho \mapsto P_\rho$. We can naturally assume that the map P satisfies the following condition from the formulation of quantum mechanics:

$$\lambda P_{\rho_1} + (1-\lambda) P_{\rho_2} = P_{\lambda\rho_1 + (1-\lambda)\rho_2}, \quad 0 < \forall \lambda < 1. \tag{3}$$

Lemma 4 *For a map P satisfying (3), there uniquely exists a positive operator valued measure (POVM) defined in the following M satisfying the condition*

$$P_\rho(B) = \operatorname{tr} M(B)\rho, \quad \forall B \in \mathcal{F}(\Omega), \forall \rho \in \mathcal{S}(\mathcal{H}).$$

This lemma was proved by Ozawa [16] in a more general framework. For an easily proof of a finite-dimensional case, see §6 in Chapter I of Holevo [11]. The lemma guarantees that we have only to discuss POVMs in order to describe probabilistic properties.

Definition 5 *Let M be a map from a σ-field $\mathcal{F}(\Omega)$ over Ω to the set $\mathcal{B}_{sa}^+(\mathcal{H})$ of bounded, self-adjoint and non-negative operators on $\mathcal{H}$. The map M is called a positive operator valued measure (POVM) on $\mathcal{H}$ over Ω if it satisfies the following:*

- $M(\emptyset) = 0, \quad M(\Omega) = I$ *(I: indentity op.)*

- $\sum_i M(B_i) = M(\cup_i B_i)$ *for* $B_i \cap B_j = \emptyset, \quad (i \neq j)$.

A POVM M is a spectral measure if $M(B)$ is a projection for any B. The following Lemma 6 is called Naǐmark's extension theorem. For a proof, see Naǐmark [15], §5 in Chapter II in Holevo [11] or Theorem 6.2.18 in Hiai and Yanagi [10]. It tells that the set of spectral measures is an important class in POVMs.

Lemma 6 *Let M be a POVM over a σ-field $\mathcal{F}(\Omega)$ on a Hilbert space $\mathcal{H}$. There exist a Hilbert space $\mathcal{K}$ including $\mathcal{H}$ and a spectral measure E on the Hilbert space $\mathcal{K}$ such that*

$$M(B) = P_{\mathcal{H}} E(B) P_{\mathcal{H}}, \quad \forall B \in \mathcal{F}(\Omega),$$

where $P_{\mathcal{H}}$ denotes the projection from $\mathcal{K}$ to $\mathcal{H}$. We call such a pair $(\mathcal{K}, E)$ a Naǐmark extension of the POVM M.

In the following, we will treat only POVMs over the complex numbers $\mathbb{C}$ whose σ-field is a family of Borel sets.

We will give the following definition with respect to the inequalities among linear operators not necessarily bounded.

Definition 7 *For non-negative and self-adjoint operators X, Y on $\mathcal{H}$, we denote $X \geq Y$ if they satisfy*

$$\langle \phi, X\phi \rangle \geq \langle \phi, Y\phi \rangle, \quad \forall \phi \in D(q(X)) \subset D(q(Y)).$$

where $q(S)$ denotes the closed non-negative quadratic form defined by a non-negative self-adjoint operator S and $D(q)$ denotes the domain of a closed non-negative quadratic form q.

We introduce the operators $E(M)$ and $V(M)$ which are represent formally $\int_{\mathbb{C}} zM(dz)$ and $\int_{\mathbb{C}} |z|^2 \langle M(dz)$, respectively. Later, by using Lemma 8, we will give more rigorous definition of $E(M)$ and $V(M)$. Then, for $\phi \in D(M), \|\phi\| = 1$ and a POVM M, the expectation is $\langle \phi | E(M) | \phi \rangle$ and the variance is $\langle \phi | V(M) | \phi \rangle - |\langle \phi | E(M) | \phi \rangle|^2$. It is sufficient to evaluate the operator $V(M)$, in order to evaluate the variance. But, when they are unbounded, we should be more careful with respect to their domains. We define the closed non-negative quadratic form $q(M)$ with the domain $D(q(M))$ by

$$\begin{aligned} q(M)(\phi, \phi) &:= \int_{\mathbb{C}} |z|^2 \langle \phi, M(dz)\phi \rangle, \quad \phi \in D(q(M)). \\ D(q(M)) &:= \left\{ \phi \in \mathcal{H} \middle| \int_{\mathbb{C}} |z|^2 \langle \phi, M(dz)\phi \rangle < \infty \right\}. \end{aligned}$$

We assume the condition that the set $D(q(M))$ is a dense subset of $\mathcal{H}$. Let $V(M)$ be the self-adjoint operator defined by the closed non-negative quadratic form $q(M)$. Next, we will define the operator $\tilde{E}(M)$. Define $E_R(M) := \int_{|z|<R} zM(dz)$. Then, for any $\phi \in D(q(M))$, the sequence $\{E_n(M)\phi\}$ is a Cauchy sequence, because for $n < m$, we have $\|E_n(M)\phi - E_m(M)\phi\|^2 = \int_{n \leq |z| < m} |z|^2 \langle \phi, M(dz)\phi \rangle$. Therefore, we can define the vector $\tilde{E}(M)\phi := \lim_{n \to \infty} E_n(M)\phi$. Thus, we can define the operator $\tilde{E}(M)$ on the domain $D(q(M))$.

Lemma 8 *The operator $\tilde{E}(M)$ has a closed extension.*

From this lemma, we can define the closed operator $E(M)$ by the closure of the operator $\tilde{E}(M)$.
Proof Let $(E, \mathcal{K})$ and $P_{\mathcal{H}}$ be a Naĭmark extension of M and the projection from $\mathcal{K}$ to $\mathcal{H}$. The operator $T := \int zE(dz)$ is normal. From the definition of T, we have $D(T) = \{\phi \in \mathcal{K} | \int |z|^2 \langle \phi, E(dz)\phi\rangle < \infty\}$. Then the domain $D(q(M))$ equals $D(T) \cap \mathcal{H}$. Let $T = U|T|$ be a polar decomposition of T. Since T is normal, we have $U|T| = |T|U$. This equation implies that the domain of $|T|$ is invariant for the action of U.

In general, for a closed operator X on $\mathcal{K}$ and closed subset $\mathcal{H}$ of $\mathcal{K}$, the operator $XP_{\mathcal{H}}$ with the domain $D(X) \cap \mathcal{H}$ is closed if $D(X) \cap \mathcal{H}$ is dense in $\mathcal{H}$. We can define the closed operator $T^*P_{\mathcal{H}}$ on its domain $D(T^*P_{\mathcal{H}}) := D(T^*) \cap \mathcal{H} = D(T) \cap \mathcal{H} = D(q(M))$. Then, we have the relation $D((T^*P_{\mathcal{H}})^*) \supset D(T)$. Define the closed operator $(T^*P_{\mathcal{H}})^*P_{\mathcal{H}}$ on its domain $D((T^*P_{\mathcal{H}})^*P_{\mathcal{H}}) := D((T^*P_{\mathcal{H}})^*) \cap \mathcal{H} \supset D(T) \cap \mathcal{H} = D(q(M))$. Then, we obtain $(T^*P_{\mathcal{H}})^*P_{\mathcal{H}} \supset \tilde{E}(M)$. We proved that the operator $\tilde{E}(M)$ has a closed extension. □

Lemma 9 *Let X and M be an operator on a Hilbert space $\mathcal{H}$ and a POVM on the Hilbert space $\mathcal{H}$, respectively. If $X \supset E(M)$, then we have $V(M) \geq X^*X$.*

Proof For a vector $\phi \in D(q(M))$, we have

$$\begin{aligned} \int_{\mathbb{C}} \langle \phi | (\bar{z} - X^*) M(dz) (z - X) | \phi \rangle &\geq 0 \\ q(M)(\phi, \phi) - \langle \phi | X^* X | \phi \rangle &\geq 0. \end{aligned}$$

Since $D(q(M)) \subset D(E(M)) \subset D(X)$, we obtain Lemma 9. □
The bounded version of this lemma is proved by Helstrom [9].

Lemma 10 *Let S be an operator defined on the dense subset $D(S)$. The operator S is subnormal if and only if there exists a POVM M satisfying the conditions*

$$S = E(M) \tag{4}$$
$$S^*S = V(M). \tag{5}$$

Proof Let $(\mathcal{K}, T)$ and $P_{\mathcal{H}}$ be a normal extension of the operator S and the projection from $\mathcal{K}$ to $\mathcal{H}$, respectively. By defining a POVM M by $M(B) := P_{\mathcal{H}} E_T(B) P_{\mathcal{H}}$, the equation (4) is trivial. Since $V(M) = P_{\mathcal{H}} T^* T P_{\mathcal{H}} = S^*S$, we have (5). Assume the equations (4) and (5). From Naĭmark's extension theorem (Lemma 6) there exists a Naĭmark extension $(\mathcal{K}, E)$ of the POVM M. Define a normal operator $T := \int_{\mathbb{C}} zE(dz)$. Then we have $V(M) = P_{\mathcal{H}} T^* T P_{\mathcal{H}}, E(M) = P_{\mathcal{H}} T P_{\mathcal{H}}$. From the equations (4), (5) and Lemma 11, we can prove that S is subnormal. □
The bounded version of this lemma is proved by Bram [3]. We will prove Lemma 11 applied in the proof of Lemma 10.

Lemma 11 *Let S, $\mathcal{K}$ and $P_{\mathcal{H}}$ be an operator on a Hilbert space $\mathcal{H}$, a Hilbert space including the Hilbert space $\mathcal{H}$ and the projection from $\mathcal{K}$ to $\mathcal{H}$, respectively. For an operator T on $\mathcal{K}$, the following are equivalent:*

(A) $S = TP_{\mathcal{H}}$.

(B) $S^*S = P_{\mathcal{H}}T^*TP_{\mathcal{H}}, \quad S = P_{\mathcal{H}}TP_{\mathcal{H}}$.

Proof It is easy to derive the condition (**B**) from the condition (**A**). Assume the condition (**B**). We have $(TP_{\mathcal{H}})^*TP_{\mathcal{H}} = (P_{\mathcal{H}}TP_{\mathcal{H}})^*P_{\mathcal{H}}TP_{\mathcal{H}} + ((I-P_{\mathcal{H}})TP_{\mathcal{H}})^*(I-P_{\mathcal{H}})TP_{\mathcal{H}}$ and $(P_{\mathcal{H}}TP_{\mathcal{H}})^*P_{\mathcal{H}}TP_{\mathcal{H}} = S^*S = (TP_{\mathcal{H}})^*TP_{\mathcal{H}}$. Therefore, we obtain $(I-P_{\mathcal{H}})TP_{\mathcal{H}} = 0$. Then we get the condition (**A**). □

A closed subspace $\mathcal{H}'$ of $\mathcal{H}$ *reduces* a normal operator T on $\mathcal{H}$, if the projection $P_{\mathcal{H}'}$ to $\mathcal{H}'$ commutes $E_T(B)$ for any Borel set B. This condition is equivalent with the projection $P_{\mathcal{H}'}$ commutes the operators U, U^* and $e^{it|T|^2}$ for any real number t. A normal extension $(T,\mathcal{K})$ of a subnormal operator S on $\mathcal{H}$ is called *minimal* if $\mathcal{K}$ has no proper subspace which reduces the normal operator T.

Lemma 12 *A normal extension $(T,\mathcal{K})$ of a subnormal operator S on $\mathcal{H}$ is minimal if and only if $\mathcal{K} = \overline{\mathcal{L}}$ where the subspace $\mathcal{L}$ of $\mathcal{K}$ is defined as*

$$\mathcal{L} := \left\{ \sum_{k=1}^{n} U^{m_k} e^{it_k|T|^2}\psi_k \,\middle|\, \psi_k \in \mathcal{H}, t_k \in \mathbb{R}, m_k, n \in \mathbb{Z} \right\}$$

and $T = U|T|$ is a polar decomposition of T such that U is unitary.

Proof Assume that a closed subspace $\mathcal{K}'$ of $\mathcal{K}$ including $\mathcal{H}$ reduces the normal operator T. Then, for any $h \in \mathcal{H}$, any integer m and any real number t, we have $U^m e^{it|T|^2}h \in \mathcal{K}$. Since the closed subspace $\mathcal{K}$ includes $\mathcal{L}$, the closed subspace $\mathcal{K}$ includes $\overline{\mathcal{L}}$.

From the definition of $\mathcal{L}$, we have the relation $U\mathcal{L} \subset \mathcal{L}$. Since U is bounded, we have the relation $U\overline{\mathcal{L}} \subset \overline{\mathcal{L}}$. It implies that $[P_{\overline{\mathcal{L}}}, U] = 0$. Similarly, we have the relations $[P_{\overline{\mathcal{L}}}, U^*] = 0$ and $[P_{\overline{\mathcal{L}}}, e^{it|T|^2}] = 0$ for any real number t. It implies that the closed subspace $\mathcal{L}$ reduces the normal operator T. The lemma is now immediate. □

Lemma 13 *Let $(T,\mathcal{K})$ be a minimal normal extension of a subnormal operator S on $\mathcal{H}$. If a Hilbert space $\mathcal{K}'$ including $\mathcal{H}$ and a normal operator T' satisfy the condition that $T'P_{\mathcal{H}} \supset A$, there exists an isometric map V from $\mathcal{K}$ to $\mathcal{K}'$ such that $V\mathcal{H} = \mathcal{H}$ and $VTV^* = T'P_{\mathrm{Im}\,V}$. It implies that $T'P_{\mathcal{H}} = A$ i.e. the pair $(T',\mathcal{K}')$ is a normal extension of S.*

This lemma guarantees the uniqueness of the minimal normal extension and that if a normal operator T on $\mathcal{K}$ including $\mathcal{H}$ satisfies $TP_{\mathcal{H}} \supset S$, the pair $(T,\mathcal{H})$ is a normal extension of S.

Proof Define the subspace $\mathcal{C}$ of $\mathcal{K}$ and the subspace $\mathcal{C}'$ of $\mathcal{K}'$ by

$$\mathcal{C} := \left\{ \sum_{k=1}^{n} e^{it_k|T|^2}\psi_k \,\middle|\, \psi_k \in \mathcal{H}, t_k \in \mathbb{R}, n \in \mathbb{Z} \right\}, \quad \mathcal{C}' := \left\{ \sum_{k=1}^{n} e^{it_k|T'|^2}\psi_k \,\middle|\, \psi_k \in \mathcal{H}, t_k \in \mathbb{R}, n \in \mathbb{Z} \right\}.$$

Similarly to Proof of Lemma 12, we can prove that the closure $\overline{\mathcal{C}}$ reduces $|T|^2$ and the closure $\overline{\mathcal{C}'}$ reduces $|T'|^2$. Since the operator S^*S is self-adjoint, the operator e^{tS^*S} is densely defined on $\mathcal{H}$ $\phi(x) = f(x)x^{-k}$. for any real number t. Thus, we have

$$\sum_{k=0}^{\infty} \frac{t^k}{k!}\|S^k\phi_j\|^2 = \left\langle \phi_j, \sum_{k=0}^{\infty} \frac{t^k(S^*S)^k}{k!}\phi_j \right\rangle = \langle \phi_j, e^{tS^*S}\phi_j \rangle < \infty, \qquad \text{for } \phi_1, \phi_2 \in D(e^{tS^*S}).$$

Using Schwarz's inequality, we obtain

$$\sum_{k=0}^{\infty}\frac{t^k}{k!}\langle S^k\phi_1, S^k\phi_2\rangle < \sqrt{\sum_{k=0}^{\infty}\frac{t^k}{k!}\|S^k\phi_1\|^2}\sqrt{\sum_{k=0}^{\infty}\frac{t^k}{k!}\|S^k\phi_2\|^2} < \infty.$$

From Fubini's Theorem,

$$\begin{aligned}\langle \phi_1, e^{it|T|^2}\phi_2\rangle &= \left\langle \phi_1, \sum_{k=0}^{\infty}\frac{(it|T|^2)^k}{k!}\phi_2\right\rangle = \left\langle \phi_1, \sum_{k=0}^{\infty}\frac{(itS^*S)^k}{k!}\phi_2\right\rangle = \sum_{k=0}^{\infty}\frac{(it)^k}{k!}\langle S^k\phi_1, S^k\phi_2\rangle \\ &= \langle \phi_1, e^{it|T'|^2}\phi_2\rangle. \end{aligned}\tag{6}$$

From (6) and the fact that the operators $e^{it|T|^2}$ and $e^{it|T'|^2}$ are bounded and $D(e^{tS^*S})$ is dense in $\mathcal{H}$, we have the equation $\langle \phi_1, e^{it|T|^2}\phi_2\rangle = \langle \phi_1, e^{it|T'|^2}\phi_2\rangle$ for any $\phi_1, \phi_2 \in \mathcal{H}$. Thus, we can define the unitary map $V_{\mathcal{C}}$ from $\overline{\mathcal{C}}$ to $\overline{\mathcal{C}}'$ by

$$V_{\mathcal{C}}\left(\sum_{k=1}^{n} e^{it_k|T|^2}\phi_k\right) = \sum_{k=1}^{n} e^{it_k|T'|^2}\phi_k.$$

Then we have $V_{\mathcal{C}}|T|^2V_{\mathcal{C}}^* = |T'|^2$ on $\overline{\mathcal{C}'}$. It implies that $V_{\mathcal{C}}|T|V_{\mathcal{C}}^* = |T'|$ on $\overline{\mathcal{C}'}$. We can define the inverses $|T|^{-1}$ and $|T'|^{-1}$ on $\operatorname{Im}|T|\cap\overline{\mathcal{C}}$ and $\operatorname{Im}|T'|\cap\overline{\mathcal{C}'}$, respectively. Then we have $V_{\mathcal{C}}|T|^{-1}V_{\mathcal{C}}^* = |T'|^{-1}$ on $\operatorname{Im}|T'|\cap\overline{\mathcal{C}'}$.

Let $T = U|T|$ and $T' = U'|T'|$ be the polar decomposition of T and the one of T' satisfying that U and U' are unitary, respectively. The image $\operatorname{Im}|T|$ is invariant under the unitary transformation U, and the image $\operatorname{Im}|T'|$ is invariant under U'. Then, we have $\operatorname{Im}A \subset \operatorname{Im}|T|\cap\mathcal{H} \subset \operatorname{Im}|T|\cap\overline{\mathcal{C}}$. Similarly, we have $\operatorname{Im}S \subset \operatorname{Im}|T'|\cap\overline{\mathcal{C}'}$. Thus, for any $\phi \in \operatorname{Im}A$, we have $V_{\mathcal{C}}|T|^{-1}\phi = |T'|^{-1}\phi$. For any $\phi_1, \phi_2 \in \mathcal{H}$, we have

$$\begin{aligned}&\langle e^{it|T|^2}\phi_1, U^k\phi_2\rangle = \langle e^{it|T|^2}\phi_1, |T|^{-k}U^k|T|^k\phi_2\rangle = \langle e^{it|T|^2}\phi_1, |T|^{-k}S^k\phi_2\rangle \\ =\ &\langle V_{\mathcal{C}}e^{it|T|^2}\phi_1, V_{\mathcal{C}}|T|^{-k}S^k\phi_2\rangle = \langle e^{it|T'|^2}\phi_1, |T'|^{-k}S^k\phi_2\rangle = \langle e^{it|T'|^2}\phi_1, U'^k\phi_2\rangle.\end{aligned}$$

Then, we can define the isometric map V from $\mathcal{K}$ to $\mathcal{K}'$ by

$$V\left(\sum_{k=1}^{n} U^{m_k}e^{it_k|T|^2}\psi_k\right) = \sum_{k=1}^{n} U'^{m_k}e^{it_k|T'|^2}\psi_k.$$

The isometric map V satisfies that $V\mathcal{H} = \mathcal{H}$. From the definition of V, we obtain that $VUV^* = U', VU^*V^* = U'^*, Ve^{it|T|^2}V^* = Ve^{it|T'|^2}V^*$ on $\overline{\mathcal{L}'}$ for any real number t, where $\mathcal{L}' := \left\{\sum_{k=1}^{n} U'^{m_k}e^{it_k|T'|^2}\psi_k \,\middle|\, \psi_k \in \mathcal{H}, t_k \in \mathbb{R}, m_k, n \in \mathbb{Z}\right\}$. Then, we obtain that $VTV^* = T'$ on $\overline{\mathcal{L}'}$. It implies that $T'P_{\mathcal{H}} = A$. □

For a simple proof in the bounded case, see §2 in Chapter II in Conway [4]. The following lemma shows the one-to-one correspondence between POVMs satisfying the condition $V(M) = E(M)^*E(M)$ and subnormal operators, in a similar sense to Lemma 3 which shows the one-to-one correspondence between spectral measures and normal operators.

Corollary 14 *For any subnormal operator S, there uniquely exists the POVM M satisfying the equations (4) and (5).*

Proof It is sufficient to show the uniqueness. Let M be a POVM satisfying the equations (4) and (5), and let $(\mathcal{K}, E)$ be a Naĭmark extension of M. Then, the pair of the Hilbert space $\mathcal{K}$ and the normal operator $T := \int zE(dz)$ is a Normal extension of S. Define $\mathcal{L}$ as Lemma 12. Therefore, the pair of the closure $\overline{\mathcal{L}}$ and the normal operator $TP_{\overline{\mathcal{L}}}$ is the minimal normal extension of S. Then, we have $M(B) = P_{\mathcal{H}}E(B)P_{\mathcal{H}} = P_{\mathcal{H}}E_{TP_{\overline{\mathcal{L}}}}(B)P_{\mathcal{H}}$ for any Borel set B. From the uniqueness of the minimal normal extension (Lemma 13), we obtain the uniqueness of the POVM satisfying the equations (4) and (5). □

Definition 15 *If a POVM satisfies the conditions(4) and (5) with respect to the subnormal operator S, we call it the POVM of the subnormal operator S.*

The proof of Lemma 10 shows that Naĭmark's extension theorem (Lemma 6) guarantees the one to one correspondence between the normal extension of a subnormal operator and its POVM.

Subnormal operators have the following properties:

Lemma 16 *Let S be a subnormal operator on $\mathcal{H}$. Then*

$$S^*S \geq SS^*. \tag{7}$$

Proof Since $I \geq P_{\mathcal{H}}$, we have $P_{\mathcal{H}}TT^*P_{\mathcal{H}} \geq P_{\mathcal{H}}TP_{\mathcal{H}}T^*P_{\mathcal{H}}$. From the normality of T, we get $S^*S = P_{\mathcal{H}}T^*TP_{\mathcal{H}} \geq P_{\mathcal{H}}TP_{\mathcal{H}}T^*P_{\mathcal{H}} = TP_{\mathcal{H}}T^* = SS^*$. □

Operators satisfying (7) is called hyponormal operators[4] and the class of these operators is important in the operator theory. The following lemma 17 tells a relation between the POVM of a subnormal operator and an over-complete eigenvector system.

Lemma 17 *Let B and K be an operator on $\mathcal{H}$ and a subset of complex numbers $\mathbb{C}$, respectively. Assume that there exists a vector $|z\rangle \in D(B)$ satisfying $B|z\rangle = z|z\rangle$ for any complex number $z \in K$, and there exists a measure μ on K satisfying $\int_K |z\rangle\langle z|\mu(dz) = I$. Then, B^* is subnormal and the POVM $|z\rangle\langle z|\mu(dz)$ is the POVM of the subnormal operator B^*.*

Proof From the assumption, we have

$$B = B\int_K |z\rangle\langle z|\mu(dz) = \int_K z|z\rangle\langle z|\mu(dz).$$

Thus, The POVM $M(dz) := |z\rangle\langle z|\mu(dz)$ satisfies the condition (4). Therefore, we obtain

$$BB^* = \int_K B|z\rangle\langle z|B^*\mu(dz) = \int_K |z|^2|z\rangle\langle z|\mu(dz) = V(M).$$

Then the POVM M satisfies the condition (5) and the operator B^* is subnormal from Lemma 10. We can confirm that the POVM $|z\rangle\langle z|\mu(dz)$ is the POVM of the subnormal operator B^*. □

In the following of this section, we treat a relation between a subnormal operator and its spectrum.

[4]This class is introduced by Halmos [7].

Lemma 18 *Let S and ϕ be a subnormal operator and an eigenvector of S, respectively. Then, a vector ϕ is an eigenvector of the adjoint operator S^* of S.*

Proof Let $(\mathcal{K}, T)$ and $P_{\mathcal{H}}$ be a normal extension of S and the projection from $\mathcal{K}$ to $\mathcal{H}$, respectively. Assume that $\phi \in D(S)$ is the eigenvector of S associated with an eigenvalue c such that $\|\phi\| = 1$. Since $T\phi = c\phi$, we have $T^*\phi = \bar{c}\phi$. Then $S^* = P_{\mathcal{H}}T^*$. Therefore we get $S^*\phi = \bar{c}\phi$. Thus, we obtain the Lemma. □

Definition 19 *A subnormal operator S is called pure subnormal if it satisfies the following condition; If a subspace $\mathcal{I}$ of $\mathcal{H}$ satisfies that $SP_{\mathcal{I}}$ is subnormal, then the subspace $\mathcal{I}$ is $\{0\}$.*

Lemma 20 *Any pure subnormal operator S has no point spectrum.*

Proof Let $(\mathcal{K}, T)$, $P_{\mathcal{H}}$ and ϕ be defined in Proof of Lemma 18. Since, we have $S^*\phi = \bar{c}\phi$, the operator $|\phi\rangle\langle\phi|$ commutes the pure subnormal operator S. The fact contradicts the definition of pure subnormal operators. □

According to Conway [4], it is sufficient to assume the purity and hyponormality in Lemma 20.

3 Compound-system-type normal extension

Now, as an example of a subnormal operator and its normal extension, we will treat the boson creation operator a_b^* and the *heterodyne measurement* in quantum optics. The pair $(L^2(\mathbb{R}) \otimes L^2(\mathbb{R}), a_b^* \otimes I + I \otimes a_b)$ is a normal extension of the subnormal operator a_b^* under the isometric embedding $L^2(\mathbb{R}) \to L^2(\mathbb{R}) \otimes L^2(\mathbb{R})$ defined by $\psi \mapsto \psi \otimes |0\rangle$, where $|0\rangle$ denotes the boson vacuum vector. Here, $a_b^* \otimes I + I \otimes a_b$ is a normal operator, and we have $(a_b^* \otimes I + I \otimes a_b)\phi \otimes |0\rangle = (a_b^*\phi) \otimes |0\rangle$ for any $\phi \in L^2(\mathbb{R})$. By substituting a_b for B in Lemma 17 and and by letting $|\alpha\rangle$ be the boson coherent state, we can confirm that $|\alpha\rangle\langle\alpha|\, d^2\alpha$ is the POVM of the subnormal operator a_b^*.

The heterodyne measurement is implemented by the measurement of $a_b^* \otimes I + I \otimes a_b$ (i.e. the joint measurement between $Q \otimes I + I \otimes Q$ and $P \otimes I - I \otimes P$ which commute each other) under the circumstance where the state of the basic system is ϕ and the state of the ancillary system is controlled to be the vacuum states $|0\rangle\langle 0|$. In detail, see §6 in Chapter III in Holevo [11] or §6 in Chapter V in Helstrom [9]. We will generalize normal extensions of similar type to this, by the name of *compound-system-type normal extensions*, as follows;

Definition 21 *Let S be a subnormal operator defined on a dense linear subspace $D(S)$ of $\mathcal{H}$ and let $\mathcal{H}'$, T and ψ be a Hilbert space, a normal operator defined on a dense subspace $D(T)$ of the Hilbert space $\mathcal{H} \otimes \mathcal{H}'$ and an element of $\mathcal{H}'$ whose norm is 1, respectively. We call the triple $(\mathcal{H}', T, \psi)$ a "compound-system-type normal extension" of the subnormal operator S if it satisfies the condition*

$$D(S) \otimes \psi \subset D(T), \quad (S\phi) \otimes \psi = T(\phi \otimes \psi), \quad \text{for any } \phi \in D(S). \tag{8}$$

Thus the definition of the compound-system-type normal extension describes not only the probability distribution but also a framework of the concrete implementation process, while the definition of the normal extension given in §2 describes only the probability distribution. Therefore, a compound-system-type normal extension contains more informations than the corresponding POVM.

In the following of this section, we discuss compound-system-type normal extensions of isometric operators and symmetric operators. Let $\{|\uparrow\rangle, |\downarrow\rangle\}$ be a CONS of $\mathbb{C}^2$.

Lemma 22 *An isometric operator U on $\mathcal{H}$ is subnormal. Define the operator $T := U \otimes |\uparrow\rangle\langle\uparrow| + U^* \otimes |\downarrow\rangle\langle\downarrow| + P_{\mathrm{Im}\, U^\perp} \otimes |\uparrow\rangle\langle\downarrow|$, where $\mathrm{Im}\, U^\perp$ denotes the orthogonal complementary space of $\mathrm{Im}\, U$. Then, the operator T is unitary on $\mathcal{H} \otimes \mathbb{C}^2$ and the triple $(\mathbb{C}^2, T, |\uparrow\rangle)$ is a compound-system-type normal extension of U.*

Proof From the definition, we have

$$
\begin{aligned}
& T^*T \\
= \; & (U^* \otimes |\uparrow\rangle\langle\uparrow| + U \otimes |\downarrow\rangle\langle\downarrow| + P_{\mathrm{Im}\, U^\perp} \otimes |\downarrow\rangle\langle\uparrow|)\,(U \otimes |\uparrow\rangle\langle\uparrow| + U^* \otimes |\downarrow\rangle\langle\downarrow| + P_{\mathrm{Im}\, U^\perp} \otimes |\uparrow\rangle\langle\downarrow \\
= \; & I_{\mathcal{H}} \otimes |\uparrow\rangle\langle\uparrow| + P_{\mathrm{Im}\, U} \otimes |\downarrow\rangle\langle\downarrow| + P_{\mathrm{Im}\, U^\perp} \otimes |\downarrow\rangle\langle\downarrow| = I_{\mathcal{H}} \otimes I_{\mathbb{C}^2}.
\end{aligned}
$$

Then, the operator T is unitary. Moreover, we have $T(\phi \otimes |\uparrow\rangle) = (U\phi) \otimes |\uparrow\rangle$. Therefore, the triple $(\mathbb{C}^2, T, |\uparrow\rangle)$ is a compound-system-type normal extension of U. □

A closed symmetric operator X is called *maximal symmetric*, if there exists no symmetric operator Y such that $X \subsetneqq Y$.

Lemma 23 *A closed symmetric operator X is subnormal if and only if X is maximal symmetric. Define the operator $T := X^* \otimes |-\rangle\langle+| + X \otimes |+\rangle\langle-|$ on the domain $D(T) := D(X^*) \otimes |+\rangle \oplus D(X) \otimes |-\rangle$ with $|\pm\rangle := \frac{1}{\sqrt{2}}(|\uparrow\rangle \pm |\downarrow\rangle)$, for the maximal symmetric operator X on $\mathcal{H}$. Then, T is a self-adjoint operator and the triple $(\mathbb{C}^2, T, |\uparrow\rangle)$ is a compound-system-type normal extension of X.*

For this lemma, the proof that the maximal-symmtric property guarantees the subnormality has been given by Naĭmark. (Naĭmark [13, 14], Ahkiezer and Glazman [1].)

Proof Define the isometric map $I_{\mathcal{H}}$ from $\mathcal{H}$ to $\mathcal{K} := \mathcal{H} \otimes \mathbb{C}^2$ by $I_{\mathcal{H}}\phi = \phi \otimes |\uparrow\rangle$. Therefore, for $\phi \in D(X)$, we have $I_{\mathcal{H}}\phi \in D(X) \otimes \mathbb{C}^2 \subset D(T)$ and $T(\phi \otimes |\uparrow\rangle) = (X\phi) \otimes (|-\rangle\langle+| + |+\rangle\langle-|)|\uparrow\rangle = (X\phi) \otimes |\uparrow\rangle = I_{\mathcal{H}} X\phi$. Then, we obtain $X \subset I_{\mathcal{H}}^* T I_{\mathcal{H}}$. It implies that the operator $I_{\mathcal{H}}^* T I_{\mathcal{H}}$ is subnormal. Since T is normal, the operator $I_{\mathcal{H}}^* T I_{\mathcal{H}}$ is symmetric. When X is maximal symmetric, we have $X = I_{\mathcal{H}}^* T I_{\mathcal{H}}$ and the operator X is subnormal. If the operator X is symmetric and subnormal, there exists a maximal symmetric operator Y such that $X \subset Y$ and a normal extension $(T, \mathcal{K})$ of Y. Then, we have $X \subset TP_{\mathcal{H}}$. From Lemma 13 and the subnormality of X, we have that $X = TP_{\mathcal{H}} = Y$. It implies that X is maximal symmetric. □

For example, we apply the inequalities (7) in Lemma 16 to a symmetric operator X which doesn't necessary have a self-adjoint extension. If X is self-adjoint, we have $X^*X = XX^*$. But, if the operator X have no self-adjoint extension, we have $X^*X \subsetneqq XX^*$. This fact isn't contradictly to the inequalities (7).

4 Irreducible unitary representations of the algebra $\mathfrak{su}(1,1)$ and their coherent states

In this section, from the discrete-series-type unitary representations of the algebra $\mathfrak{su}(1,1)$ (defined in this section), we will canonically construct the corresponding subnormal operators, and will investigate the relationship between the coherent states defined by Perelomov [17] and these subnormal operators.

When a triplet (E_0, E_+, E_-) of skew-adjoint operators on a Hilbert space satisfies the following commutation relations, then it is called a unitary representation of the algebra $\mathfrak{su}(1,1)$:

$$[E_0, E_\pm] = \pm 2E_\pm, \quad [E_+, E_-] = E_0. \tag{9}$$

In the following, for a prescribed unitary representation (E_0, E_+, E_-), we will discuss the relations under the representation. For a unitary representation (E_0, E_+, E_-) of the algebra $\mathfrak{su}(1,1)$, define another triplet (L_0, L_+, L_-) by

$$L_0 := i(E_- - E_+), \quad L_\pm := \frac{1}{2}\big(E_0 \pm i(E_+ + E_-)\big). \tag{10}$$

Then, this triplet satisfies the commutation relations of the same type

$$[L_0, L_\pm] = \pm 2L_\pm, \quad [L_+, L_-] = L_0. \tag{11}$$

For this triplet,

$$L_0^* = L_0, \quad L_+^* = -L_- \tag{12}$$

hold, where L_+ and L_- are neither self-adjoint nor skew-adjoint. Conversely, from the triplet (L_0, L_+, L_-) satisfying the conditions (11) and (12), a unitary representation (E_0, E_+, E_-) of the algebra $\mathfrak{su}(1,1)$ can be constructed by

$$E_0 = L_+ + L_-, \quad E_\pm = \pm\frac{i}{2}(L_0 \mp L_+ \pm L_-). \tag{13}$$

The Casimir operator of the algebra $\mathfrak{su}(1,1)$ is given by

$$C := E_0^2 + 2(E_+E_- + E_-E_+) = L_0^2 + 2(L_+L_- + L_-L_+). \tag{14}$$

The relation (14) can be written in another form

$$C = L_0^2 - 2L_0 + 4L_+L_- \tag{15}$$

by using (11).

In the following, only non-trivial irreducible unitary representations will be discussed. From this assumption, it is easily shown that the dimensions of the kernels of L_- and L_+ are not more than one, and that either of them should be zero. In this paper, we discuss the irreducible unitary representations of the algebra $\mathfrak{su}(1,1)$ only in the cases where the dimension of the kernel of L_- is one. Let v_0 be a unit vector belonging to the Kernel of L_-. Then, because the Casimir operator should be scalar-valued, we can show that v_0 is the eigenvector of L_0 from (15). Let λ be the eigenvalue of L_0 with which v_0 is associated and let $v_n := (L_+)^n v_0$. From the commutation relations (11), we have the following relations;

$$\begin{array}{llll} L_0 v_n = (\lambda + 2n)v_n & n \geq 0 & L_- v_n = -n(\lambda + n - 1)v_{n-1} & n \geq 1 \\ L_+ v_n = v_{n+1} & n \geq 0 & L_- v_0 = 0, & \end{array}$$

whence we can confirm that the eigenvalue λ, with which v_0 is associated, specifies the representation uniquely. From the above assumptions, we can confirm that the basis $\{v_n\}_{n=1}^{\infty}$ is complete and orthogonal. In the following discussions, $\mathcal{H}_\lambda$ denotes the representation space of the irreducible unitary representation of $\mathfrak{su}(1,1)$ characterized by the minimum eigenvalue λ. We call such a representation a representation of *discrete-series type.* Especially, when λ is an integer, the representation of the Lie group SU(1,1) can be constructed and it is well-known as the *discrete series* [12, 6]. From the above relations, the Casimir operator C is calculated to be the scalar $\lambda(\lambda-2)$. The unitarity of the representation guarantees $\lambda > 0$ [12]. From the commutation relations (11), we have

$$\langle v_n, v_n\rangle = n(\lambda+n-1)\langle v_{n-1}, v_{n-1}\rangle.$$

Define the number operator by $N := \frac{1}{2}(L_0-\lambda)$. Then its eigenvector $|n\rangle_N$ associated with the eigenvalue n is given by

$$|n\rangle_N := \sqrt{\frac{\Gamma(\lambda)}{n!\Gamma(\lambda+n)}}v_n.$$

Next, we will construct the $\mathfrak{su}(1,1)$-*annihilation operator* in the following way; Since $N|n\rangle_N$ belongs to the range of L_+ for any n and the kernel of L_+ is $\{0\}$, the operator $a := L_+^{-1}N = \frac{1}{2}L_+^{-1}(L_0-\lambda)$ can be defined as a bounded operator by the relation

$$a|n\rangle_N = \sqrt{\frac{n}{n+\lambda-1}}|n-1\rangle_N, \tag{16}$$

where we mean $a|0\rangle_N = 0$ by (16) in the exceptional case where $\lambda = 1, n = 0$, as a convention. From (16), the commutation relation $[a, N] = a$ is derived. The $\mathfrak{su}(1,1)$-*creation operator* is defined by the adjoint a^* of a, and

$$a^*|n\rangle_N = \sqrt{\frac{n+1}{n+\lambda}}\;|n+1\rangle_N. \tag{17}$$

From (16) and (17), we have

$$a^*a = (N+\lambda-1)^{-1}N, \quad aa^* = (N+\lambda)^{-1}(N+1), \quad [a,a^*] = (\lambda-1)(N+\lambda)^{-1}(N+\lambda-1)^{-1} \tag{18}$$

for $\lambda \neq 1$, and

$$aa^* = I, \quad a^*a = I - |0\rangle_N\,{}_N\langle 0|, \quad [a,a^*] = |0\rangle_N\,{}_N\langle 0| \tag{19}$$

instead of (18) for $\lambda = 1$. In the case where $\lambda \neq 1$, from (18), we have

$$aa^* = -(a^*a+\lambda-2)^{-1}(\lambda a^*a-1), \quad a^*a = (aa^*-\lambda)^{-1}\big((2-\lambda)aa^*-1\big).$$

Next we will construct the eigenvector system of a. Introduce the unitary operator $D(\xi) := \exp\left(\xi L_+ - \bar{\xi}L_+^*\right)$ for a complex number ξ, according to Perelomov [17]. For the complex number ζ such that $|\zeta| < 1$, define the coherent state $|\zeta\rangle_a$ of the algebra $\mathfrak{su}(1,1)$ by

$$\begin{aligned} |\zeta\rangle_a &:= D\left(\frac{1}{2}e^{i\arg\zeta}\ln\frac{1+|\zeta|}{1-|\zeta|}\right)\;|0\rangle_N = \exp(\zeta L_+)\exp\left(\frac{1}{2}\ln(1-|\zeta|^2)\;L_0\right)\exp(\bar{\zeta}L_-)\;|0\rangle_N && (20)\\ &= (1-|\zeta|^2)^{\lambda/2}\exp(\zeta L_+)\;|0\rangle_N, && (21)\end{aligned}$$

where see pp.73-74 of Perelomov [17] for the derivation of the second equation (20). Because we can show that $[a, L_+] = I$, we obtain the commutation relation $[a, \exp(\zeta L_+)] = \zeta \exp(\zeta L_+)$. Moreover, from $\exp\left(\frac{1}{2}\ln(1-|\zeta|^2)\ L_0\right)\exp(\bar{\zeta}L_-)\ |0\rangle_N = (1-|\zeta|^2)^{\lambda/2}|0\rangle_N$, we have

$$a|\zeta\rangle_a = \exp(\zeta L_+)\ a\ |0\rangle_N + \zeta\exp(\zeta L_+)|0\rangle_N = \zeta|\zeta\rangle_a. \tag{22}$$

Therefore, the coherent states of the algebra $\mathfrak{su}(1,1)$ are characterized as the eigenvectors of the $\mathfrak{su}(1,1)$-annihilation operator a. Here the point spectrum $\sigma_p(a)$ is $D := \{z \in \mathbb{C}|\ |z| < 1\}$, the continuous spectrum $\sigma_c(a)$ is $S := \{z \in \mathbb{C}||z| = 1\}$ and the residual spectra is the empty set. Moreover, from Lemma 26, the point spectrum $\sigma_p(a^*)$, the continuous spectrum $\sigma_c(a^*)$ and the residual spectrum $\sigma_r(a^*)$ of a^* are the empty set, S and D, respectively. When $0 < \lambda$, a is not subnormal from Lemma 18 and the fact that it has eigenvectors. When $\lambda < 1$, it is shown that a^* is not subnormal from Lemma 16 and the fact that $[a, a^*] \geq 0$ does not hold.

Moreover, when $\lambda > 1$, the resolution of identity by the system of the coherent states holds, as

$$(\lambda - 1)\int_{\mathrm{D}} |\zeta\rangle_a\ {}_a\langle\zeta|\mu(d\zeta) = I \quad \text{with} \quad \mu(d\zeta) := \frac{d^2\zeta}{\pi(1-|\zeta|^2)^2}. \tag{23}$$

From this resolution of identity and Lemma 17, when $\lambda > 1$, we can show that a^* is a subnormal operator. On the other hand, when $\lambda \leq 1$, the integral in (23) diverges. However, the equations (19) implies that a^* is isometric when $\lambda = 1$. Then, a^* is subnormal even when $\lambda = 1$.

Next, define the unbounded operator $\tilde{A}$ by a linear fractional transform (Möbius transform) of a, as

$$\tilde{A} := -i(a+1)(a-1)^{-1},$$

where the domain $D(\tilde{A})$ of $\tilde{A}$ is defined by $\langle\{|n\rangle_N\}_{n=0}^{\infty}\rangle$[5]. The domain of $\tilde{A}^*$ is dense in $\mathcal{H}_\lambda$, as will be shown in the last part of §5.2. Therefore, $\tilde{A}$ is closable and we can define the operator A by $A := \overline{\tilde{A}} = \tilde{A}^{**}$ (See Reed and Simon [18]). It is shown that $|\zeta\rangle_a \in D(A)$ in the last part of §5.2. Hence we have $A|\zeta\rangle_a = -i\frac{\zeta+1}{\zeta-1}|\zeta\rangle_a$. By defining $|\eta\rangle_A := \left|\frac{\eta-i}{\eta+i}\right\rangle_a$, we can show that $A|\eta\rangle_A = \eta|\eta\rangle_A$ holds. (Formally, the operator a is the Cayley transform of A, with an appropriate discussion on its domain.) From the relations $[a, L_+] = I$, (11), (13) and the definition of a, we can show that the relations

$$\begin{aligned}
2(E_0 - \lambda)(a-1)L_+ &= (L_+ + L_- - \lambda)\big(L_0 - (\lambda - 2) - 2L_+\big) \\
&= (L_0 - L_+ + L_-)(-\lambda + L_0 + 2 + 2L_+) \\
&= -4iE_+(a+1)L_+ \\
(E_0 - \lambda)(a-1)|0\rangle_N &= -(E_0 - \lambda)|0\rangle_N = (L_0 - L_+)|0\rangle_N = -2iE_+|0\rangle_N = -2iE_+(a+1)|0\rangle_N
\end{aligned}$$

hold on $D(\tilde{A})$. Hence, on $D(\tilde{A})$, we have

$$(E_0 - \lambda)(a-1) = -2iE_+(a+1). \tag{24}$$

[5] $\langle X\rangle$ denotes the vector space whose elements are finite linear sums of X.

By using (24), the operator A can be written in another form

$$A = \frac{1}{2} E_+^{-1}(E_0 - \lambda). \tag{25}$$

Now define $v(s,t) := \exp(sE_+)\exp(tE_0)v_0$. Then, from $[A, E_+] = I$, we have

$$A\exp(sE_+) = \exp(sE_+)A + s\exp(sE_+), \quad A\exp(tE_0) = e^{2t}\exp(tE_0)A$$

and hence

$$Av(s,t) = (e^{2t}i + s)v(s,t). \tag{26}$$

From this relation, by letting $\eta = e^{2t}i + s \in H := \{z \in \mathbb{C} | \mathrm{Im}\ z > 0\}$, we have $v(s,t) = e^{i\theta(\eta)}|\eta\rangle_A$ with a real-valued function $\theta(\eta)$ of η. Moreover, from (25), we can show the relation

$$[A, A^*] = -(\lambda - 1)E_+^{-2}$$

formally, and

$$A^*A - AA^* = (\lambda - 1)\left(E_+^{-1}\right)^* E_+^{-1} \quad \begin{cases} \text{on } D(AA^*) & \lambda \geq 1 \\ \text{on } D(A^*A) & 0 < \lambda < 1 \end{cases} \tag{27}$$

in more precise form (The proof of this relation will be given in the last part of §5.2) . Hence it is shown that the point spectrum $\sigma_p(A)$ of A is H, the continuous spectrum $\sigma_c(A)$ is $\mathbb{R}$ and the residual spectrum $\sigma_r(A)$ is the empty set. On the other hand, from Lemma 26, the point spectrum $\sigma_p(A^*)$, the continuous spectrum $\sigma_c(A^*)$ and the residual spectrum $\sigma_r(A^*)$ of A^* are the empty set, $\mathbb{R}$ and H, respectively. Therefore, for $0 < \lambda$, from Lemma 18 and the fact that the operator A has eigenvectors, it is shown that A is not subnormal. When $\lambda < 1$, A^* is not subnormal because the relation (18) shows that the condition $AA^* \geq A^*A$ is not satisfied. Moreover, in a similar manner to the above discussion, the resolution of the identity by the eigenvectors of A

$$(\lambda - 1)\int_{\mathrm{H}} |\eta\rangle_A\, {}_A\langle\eta|\nu_\lambda(d\eta) = I \quad \text{with} \quad \nu(d\eta) := \frac{d^2\eta}{4\pi(\mathrm{Im}\ \eta)^2}$$

holds. Hence, when $\lambda > 1$, we can show that A^* is subnormal from Lemma 17. When $\lambda \leq 1$, the integral in (23) diverges. However, as will be proved in the last part of §5.2, the operator A^* is maximal symmetric when $\lambda = 1$. From Lemma 23, A^* is subnormal even when $\lambda = 1$.

5 Concrete representations of $\mathfrak{su}(1,1)$

5.1 Representation on the space of holomorphic functions

In the following, we will summarize the representations on the space of holomorphic functions in the case where $\lambda > 1$ in general. The corresponding representation of Lie group $SU(1,1)$

cannot necessary be constructed. The case of that it can be constructed i.e. λ is an integer, see Knapp [6]. Define two Hilbert spaces

$$\mathcal{H}(D)_\lambda := \left\{ f : \text{a holomorphic function on } D \,\middle|\, \|f\|^2 := \int_D |f(z)|^2(1-|z|^2)^{\lambda-2}\, d^2z < \infty \right\}$$

$$\mathcal{K}(D)_\lambda := \left\{ f : \text{a complex function on } D \,\middle|\, \|f\|^2 := \int_D |f(z)|^2(1-|z|^2)^{\lambda-2}\, d^2z < \infty \right\}.$$

Define a map $\mathcal{D}_\lambda$ from $\mathcal{H}_\lambda$ to $\mathcal{H}(D)_\lambda$

$$\mathcal{D}_\lambda(f)(z) := \frac{\langle f|z\rangle_a}{(1-|z|)^{\lambda/2}}.$$

From (23), it is easily shown that $\mathcal{D}_\lambda(f)$ belongs to $\mathcal{K}(D)_\lambda$. The holomorphy of $\mathcal{D}_\lambda(f)$ can be confirmed from the modified expression

$$\mathcal{D}_\lambda(f)(z) = \sum_{n=0}^{\infty} \frac{\langle f|(L_+)^n|0\rangle_N}{n!} z^n$$

derived by using (21). It is easily shown that $(\mathcal{D}_\lambda|n\rangle_N)(z) = \sqrt{\frac{\Gamma(\lambda+n)}{n!\Gamma(\lambda)}} z^n$.

From these relations and some considerations, we can show that $\mathcal{D}_\lambda$ is a unitary map from $\mathcal{H}_\lambda$ into $\mathcal{H}(D)_\lambda$. Moreover, by defining $\mathcal{D}_{\lambda,*}(X) = \mathcal{D}_\lambda X \mathcal{D}_\lambda^*$ for the operator X on $\mathcal{H}_\lambda$, the following relations are derived;

$$\begin{aligned}
\mathcal{D}_{\lambda,*}(E_0) &= \lambda z + (z^2-1)\frac{d}{dz}, \quad \mathcal{D}_{\lambda,*}(E_+) = \frac{i}{2}(1-z)\left(\lambda + (z-1)\frac{d}{dz}\right), \\
\mathcal{D}_{\lambda,*}(E_-) &= -\frac{i}{2}(1+z)\left(\lambda + (z+1)\frac{d}{dz}\right), \quad \mathcal{D}_{\lambda,*}(a^*) = z, \\
\mathcal{D}_{\lambda,*}(L_0) &= \lambda + 2z\frac{d}{dz}, \quad \mathcal{D}_{\lambda,*}(L_+) = \lambda z + z^2\frac{d}{dz}, \quad \mathcal{D}_{\lambda,*}(L_-) = -\frac{d}{dz}, \quad \mathcal{D}_{\lambda,*}(N) = z\frac{d}{dz}.
\end{aligned}$$

In this representation, defining the multiplication operators $S_\lambda : f(z) \mapsto zf(z)$ on $\mathcal{K}(D)_\lambda$, we can show that the pair $(S_\lambda, \mathcal{K}(D)_\lambda)$ is a normal extension of $\mathcal{D}_{\lambda,*}(a^*)$.

Next, we consider the case where $\lambda = 1$. Let

$$\mathcal{H}(S)_1 := \left\{ g = \{g_n\}_{n=0}^{\infty} \,\middle|\, \sum_{n=0}^{\infty} |g_n|^2 < \infty \right\}, \quad \mathcal{K}(S)_1 := L^2(S).$$

We can interpret the Hilbert space $\mathcal{H}(S)_1$ as a subspace of the Hilbert space $\mathcal{K}(S)_1$ by the isometric map $V_1 : g \mapsto \frac{1}{2\pi}\sum_{n=0}^{\infty} g_n e^{n\theta i}$. Define a unitary map $\mathcal{D}_1$ from $\mathcal{H}_1$ to $\mathcal{H}(S)_1$ by

$$(\mathcal{D}_1(f))_n := \langle f|n\rangle_N.$$

Moreover, define $\mathcal{D}_{1,*}(X) := V_1\mathcal{D}_1 X \mathcal{D}_1^* V_1^*$ for a operator X on $\mathcal{H}_1$, the following relations are derived;

$$\begin{aligned}
\mathcal{D}_{1,*}(E_0) &= -i(e^{\theta i} - e^{-\theta i})\frac{d}{d\theta}, \quad \mathcal{D}_{1,*}(E_+) = \frac{-1}{2}(1 - e^{\theta i} - e^{-\theta i})\frac{d}{d\theta}, \\
\mathcal{D}_{1,*}(E_-) &= \frac{1}{2}(1 + e^{\theta i} + e^{-\theta i})\frac{d}{d\theta}, \quad \mathcal{D}_{1,*}(a^*) = e^{\theta i}, \quad \mathcal{D}_{1,*}(N) = -i\frac{d}{d\theta} - 1, \\
\mathcal{D}_{1,*}(L_0) &= -i\frac{d}{d\theta}, \quad \mathcal{D}_{1,*}(L_+) = -ie^{\theta i}\frac{d}{d\theta}, \quad \mathcal{D}_{1,*}(L_-) = ie^{-\theta i}\frac{d}{d\theta}.
\end{aligned}$$

In this representation, defining the multiplication operators $S_1 : f(z) \mapsto zf(z)$ on $\mathcal{K}(D)_1$, we can show that the pair $(S_1, \mathcal{K}(D)_1)$ is a normal extension of $\mathcal{D}_{1,*}(a^*)$. Therefore a is subnormal in the case where $\lambda = 1$.

However, these types of construction of the normal extension are not directly corresponding to the construction of physical measurement. We can construct a normal extension of A^* by a similar method, though there arises the same problem that it is not directly corresponding to a physical operation.

5.2 Representation associated with irreducible unitary representation of affine group

Next, we will construct discrete-series-type irreducible unitary representations of the algebra $\mathfrak{su}(1,1)$ from an irreducible unitary representation of the affine group ($ax+b$ group) generated by E_+ and E_0. The representation which will be constructed in this section is closely related to the continuous wavelet transformation [5, 20]. According to Aslaksen and Klauder [2], there is not any irreducible representation of the affine group but the representations equivalent unitarily to the following representation on $L^2(\mathbb{R}^+)$ or $L^2(\mathbb{R}^-)$;

$$E_0 = i(PQ + QP), \quad E_+ = iQ, \tag{28}$$

where E_0 and E_+ are shown to be skew adjoint. In this representation, $\sqrt{\frac{(2\,\mathrm{Im}\,\eta)^{2k+1}}{\Gamma(2k+1)}}x^k e^{i\eta x}$ is called the affine coherent state[6], and it is obtained by operating the affine group on $\sqrt{\frac{2^{2k+1}}{\Gamma(2k+1)}}x^k e^{\mp x}$. In the following, we will construct an irreducible unitary representation of the algebra $\mathfrak{su}(1,1)$ from the above type of unitary representation of the affine group, and will discuss how to interpret the affine coherent states in terms of the unitary representation of the algebra $\mathfrak{su}(1,1)$. Therefore, in addition to the two generators in (28), we should construct the representation of another additional generator E_-. By choosing

$$\tilde{E}_{-,k} := -i(PQP + k^2 Q^{-1}) \quad (k > -1/2) \tag{29}$$

for this additional generator, we can construct an irreducible unitary representation where the triplet E_0, E_+ and E_- satisfies the commutation relations (9). However, we should be careful about the domain of $\tilde{E}_{-,k}$, as follows; First, define the dense subspace $D(\tilde{E}_{-,k})$ of $L^2(\mathbb{R}^+)$ by

$$\begin{aligned} &D(\tilde{E}_{-,k}) \\ := &\left\{ f(x) = x^k f_0(x) \in L^2(\mathbb{R}^+) \cap C^1(\mathbb{R}^+) \,\middle|\, \begin{array}{l} (2k+1)x^k f_0'(x) + x^{k+1} f_0''(x) \in L^2(\mathbb{R}^+), \\ \limsup_{s\to 0} f_0(s) < \infty, \quad x^k f_0(x) \to 0 \text{ as } x \to \infty \end{array} \right\}. \end{aligned}$$

[6]The Fourier transform of this affine coherent state is equivalent to the Cauchy wavelet in signal processing, whose basic wavelet function is $\frac{(\text{const.})}{(\pm i)^{k+1}}$.

Then $\tilde{E}_{-,k}$ is an operator defined on $D(\tilde{E}_{-,k})$. It is confirmed that $i\tilde{E}_{-,k} = PQP + k^2Q^{-1}$ is a symmetric operator on $D(\tilde{E}_{-,k})$, from the fact that the difference

$$\begin{aligned}
&\int_s^t \overline{((PQP+k^2Q^{-1})f)}(x)g(x)\,dx - \int_s^t \overline{f}(x)\left((PQP+k^2Q^{-1})g\right)(x)\,dx \\
=&\ \left[x\overline{f'}(x)g(x) - xg'(x)\overline{f}(x)\right]_s^t \\
=&\ t\overline{f'}(t)g(t) - tg'(t)\overline{f}(t) - \left(sg(s)\left(\overline{f'}(s) - \frac{k}{s}\overline{f}(s)\right) - s\overline{f}(s)\left(g'(s) - \frac{k}{s}g(s)\right)\right) \\
=&\ t\overline{f'}(t)g(t) - tg'(t)\overline{f}(t) - \left(sg(s)s^k\overline{f_0}'(s) - s\overline{f}(s)s^k g_0'(s)\right)
\end{aligned}$$

tends to zero as $s \to 0, t \to \infty$. Moreover, we can show that $D(\tilde{E}_{-,k}^*) \cap C^2(\mathbb{R}^+) = D(\tilde{E}_{-,k})$. Therefore, $\tilde{E}_{-,k}$ is a closable operator and $iE_{-,k} := \overline{i\tilde{E}_{-,k}}$ is a self-adjoint operator. By letting $L_{+,k}, L_{-,k}, L_{0,k}, \tilde{A}_k, N_k$ and $|n\rangle_N^k, |\eta\rangle_A^k$ be $L_+, L_-, L_0, \tilde{A}, N, |n\rangle_N$ and $|\eta\rangle_A$ in this representation, respectively, we have

$$\begin{aligned}
L_{+,k} &= \frac{1}{2}\left(i(PQ+QP) - Q + PQP + k^2Q^{-1}\right), \\
L_{-,k} &= \frac{1}{2}\left(i(PQ+QP) + Q - PQP - k^2Q^{-1}\right), \\
L_{0,k} &= \left(PQP + k^2Q^{-1} + Q\right), \quad \tilde{A}_k = P + ikQ^{-1}, \quad N_k = \frac{1}{2}\left(PQP + k^2Q^{-1} + Q - 1 - 2k\right), \\
|n\rangle_N^k(x) &= \sqrt{\frac{2^{2k+1}n!}{\Gamma(n+2k+1)}}e^{-x}x^k S_n^{2k}(2x), \quad |\eta\rangle_A^k(x) = \sqrt{\frac{(2\,\mathrm{Im}\,\eta)^{2k+1}}{\Gamma(2k+1)}}x^k e^{i\eta x},
\end{aligned}$$

when $S_n^l(x)$ is the Sonine Polynomial (or the associated Laguerre polynomial) defined by[7]

$$S_n^l(x) := \sum_{m=0}^{n} \frac{(-1)^m}{(n-m)!}\frac{\Gamma(n+l+1)x^m}{\Gamma(m+l+1)m!}.$$

Moreover, in this representation, the minimum eigenvalue of $L_{0,k}$ is $\lambda = 2k+1$, and the Casimir operator is $4k^2 - 1$. Therefore, the representations of discrete-series type (defined in §4) in general can be constructed concretely by (28) and (29) on $L^2(\mathbb{R}^+)$.

In the following, we will show the properties of $\tilde{A}_k$ in order to show the properties of $\tilde{A}$ in the representations of discrete-series type. Since the domain of $\tilde{A}_k$ is $\langle\{|n\rangle\}_{n=0}^{\infty}\rangle$ and $\tilde{A}_k = P + ikQ^{-1}$, the following relation (30) is derived, and hence we can show that $D(\tilde{A}_k^*)$ is dense in $L^2(\mathbb{R}^+)$. Thus $\tilde{A}_k$ is shown to be a closable operator. Letting $A_k := \overline{\tilde{A}_k}$, then the following (31) is confirmed. Note that $\bar{X} = X^{**}$ and $X^* = \bar{X}^*$ hold for a densely defined linear operator X.

$$D(\tilde{A}_k^*) \cap C^1(\mathbb{R}^+) = \left\{x^{-k}f(x) \in L^2(\mathbb{R}^+) \cap C^1(\mathbb{R}^+) \,\middle|\, \begin{array}{l} x^{-k}f'(x) \in L^2(\mathbb{R}^+) \\ f(s) \to 0 \text{ as } s \to 0, \end{array}\right\} \quad (30)$$

$$D(A_k) \cap C^1(\mathbb{R}^+) = \left\{x^k f(x) \in L^2(\mathbb{R}^+) \cap C^1(\mathbb{R}^+) \,\middle|\, \begin{array}{l} x^k f'(x) \in L^2(\mathbb{R}^+) \\ \limsup_{s\to 0} f(s) < \infty \end{array}\right\}. \quad (31)$$

[7] Sometimes another definition with $n+l$ instead of l is used.

Note that $\limsup_{x\to\infty} f_0(x) = 0$ for $k > -\frac{1}{2}$, when $x^k f_0(x) \in L^2(\mathbb{R}^+)$. From (31), we can show that $|\zeta\rangle_a \in D(A_k)$. The subspaces $D(A_k^*) \cap C^1(\mathbb{R}^+)$ and $D(A_k) \cap C^1(\mathbb{R}^+)$ are the cores[8] of A_k^* and A_k. In the special case where $\lambda = 1$ (i.e. where $k = 0$), we can show that A_0^* is a symmetric operator. When $-\frac{1}{2} < k < \frac{1}{2}$, the domain of A_k is larger than the domain of A_{-k}^*, though A_k and A_{-k}^* are the same formally, i.e. $A_{-k}^* \subsetneqq A_k$. These relations $D(A_k^*) \subset D(A_k)$ and $D(A_k^*) \subset D(E_+^{-1})$ are shown in the cases where $\lambda < 1$ $(k > 0)$, only the relation $D(A_k^*) \subset D(A_k)$ is shown when $\lambda = 1$ $(k = 0)$, and these relations $D(A_k) \subset D(A_k^*)$ and $D(A_k) \subset D(E_+^{-1})$ are shown when $0 < \lambda < 1$ $(-\frac{1}{2} < k < 0)$. From these discussion and (25), we obtain (27).

5.3 Representation associated with squeezed states

Next, we will discuss the following representation of the algebra $\mathfrak{su}(1,1)$ on the Hilbert space $L^2(\mathbb{R})$; Let

$$E_0 = \frac{i}{2}(PQ + QP), \quad E_+ = \frac{i}{2}Q^2, \quad E_- = -\frac{i}{2}P^2, \tag{32}$$

then we have

$$L_0 = n_b + \frac{1}{2}, \quad L_+ = -\frac{1}{2}(a_b^*)^2, \quad L_- = \frac{1}{2}a_b^2,$$

where $a_b := \sqrt{1/2}(Q + iP)$ and $n_b := 1/2(Q^2 + P^2 - 1) = a_b a_b^*$. In this representation, the Casimir operator is the scalar $-\frac{3}{4}$. From the fact that the Casimir operator is the scalar $\lambda(\lambda-2)$, the solutions are $\lambda = 1/2, 3/2$. Under the representation given in (32), $L^2(\mathbb{R})$ is not irreducible and it is decomposed into two irreducible subspaces as:

$$L^2(\mathbb{R}) = L^2_{\text{even}}(\mathbb{R}) \oplus L^2_{\text{odd}}(\mathbb{R})$$

where $L^2_{\text{even}}(\mathbb{R})$ is the set of square-integrable even functions and $L^2_{\text{odd}}(\mathbb{R})$ is the set of square-integrable odd functions. We have the solution $\lambda = 1/2$ in the subspace $L^2_{\text{even}}(\mathbb{R})$, while we have the solution $\lambda = 3/2$ in the subspace $L^2_{\text{odd}}(\mathbb{R})$. In the subspace $L^2_{\text{even}}(\mathbb{R})$, a, A and N are written in the forms

$$a = (a_b^*)^{-1}a_b, \quad A = Q^{-1}P, \quad N = \frac{1}{2}n_b, \quad |n\rangle_N = (-1)^n|2n\rangle_{n_b}$$

$$D(A) \cap C^1(\mathbb{R}) = \left\{ f \in L^2_{\text{even}}(\mathbb{R}) \cap C^1(\mathbb{R}) \,\middle|\, \frac{1}{x}f'(x) \in L^2(\mathbb{R}) \right\},$$

where $|n\rangle_{n_b}$ denotes the eigenvector in $L^2(\mathbb{R})$ of the boson number operator n_b associated with the eigenvalue n. These are just corresponding to the characteristic equations (1) and (2) of squeezed states explained in §1, and hence it is shown that the eigenvectors of a and A are squeezed states, as:

$$|0;\mu,\nu\rangle = \left| i\frac{\mu+\nu}{\mu-\nu} \right\rangle_A = \left| \frac{\nu}{\mu} \right\rangle_a .$$

[8]The subspace of the domain $D(X)$ of a closed operator X is called a core of the operator X if it is dense in $D(X)$ with respect to the graph norm of the operator X.

Substituting these relations into (22) and (26), we obtain (2) and (1). In the following, the vector $|\zeta\rangle_a$ in $L^2_{\text{even}}(\mathbb{R})$ is denoted by $|\zeta\rangle_{a,\text{even}}$. On the other hand, in $L^2_{\text{odd}}(\mathbb{R})$, a, A and N are written in the forms

$$a = a_b(a_b^*)^{-1}, \quad A = PQ^{-1}, \quad N = \frac{1}{2}(n_b - 1), \quad \text{and} \quad |n\rangle_N = (-1)^n|2n+1\rangle_{n_b}.$$

Next, we will discuss the representation of the algebra $\mathfrak{su}(1,1)$ in the Hilbert space $L^2(\mathbb{R}^n) = \underbrace{L^2(\mathbb{R}) \otimes \cdots \otimes L^2(\mathbb{R})}_{n}$. In this representation,

$$E_0 = \frac{i}{2}\sum_{j=1}^{n}(P_jQ_j + Q_jP_j), \quad E_+ = \frac{i}{2}\sum_{j=1}^{n}Q_j^2, \quad E_- = -\frac{i}{2}\sum_{j=1}^{n}P_j^2, \tag{33}$$

where Q_j and P_j denotes the multiplication operator and the $(-i)$-times differential operator, respectively, with respect to the j-th variable. Let $L^2_{\text{e}}(\mathbb{R}^n)$ be the closure of the linear space generated by $\{|\zeta\rangle^{\otimes n}_{a,\text{even}} := \underbrace{|\zeta\rangle_{a,\text{even}} \otimes \cdots \otimes |\zeta\rangle_{a,\text{even}}}_{n}\}$. Then, $L^2_{\text{e}}(\mathbb{R}^n)$ is irreducible under the representation (33) of the algebra $\mathfrak{su}(1,1)$, and then we have $L^2_{\text{e}}(\mathbb{R}^n) = \{f \in L^2(\mathbb{R}^n) | f \text{ is a function of } \sum_{j=1}^n x_j^2\}$, and then $|\zeta\rangle_a$ in this representation on $L^2_{\text{e}}(\mathbb{R}^n)$ is equivalent to $|\zeta\rangle^{\otimes n}_{a,\text{even}}$.

Letting $A_{n,\text{e}}$ be the operator A in this representation, then we obtain the relation

$$A_{n,\text{e}} = \left(\sum_{j=1}^{n} Q_j^2\right)^{-1}\sum_{j=1}^{n} Q_jP_j = -i\left(\sum_{j=1}^{n}\frac{2x_j}{r}\frac{\partial}{\partial x_j}\right)$$

with $r := 2\sum_{j=1}^n x_j^2$. Now define $U_n : L^2(\mathbb{R}^n) \to L^2(\mathbb{R}^+) \otimes L^2(S^{n-1}) \cong L^2(\mathbb{R}^+ \times S^{n-1})$ by $\Big(U_n(f)\Big)\big(r, (e_1, e_2, \ldots, e_n)\big) = r^{\frac{n-2}{4}} f(\sqrt{\tfrac{r}{2}}e_1, \sqrt{\tfrac{r}{2}}e_2, \ldots, \sqrt{\tfrac{r}{2}}e_n)$, where S^{n-1} denotes the $(n-1)$-dimensional spherical surface and $(e_1, e_2, \ldots, e_n)$ is an element of S^{n-1}. Then, the following relations hold;

$$U_nE_0U_n^* = E_{0,\frac{n-2}{4}} \otimes I, \quad U_nE_+U_n^* = E_{+,\frac{n-2}{4}} \otimes I, \quad U_nE_-U_n^* = E_{-,\frac{n-2}{4}} \otimes I,$$
$$U_nA_{n,\text{e}}U_n^* = \left(P + i\left(\frac{n}{4} - \frac{1}{2}\right)Q^{-1}\right) \otimes I = -i\frac{\partial}{\partial r} + i\left(\frac{n}{4} - \frac{1}{2}\right)\frac{1}{r}, \quad U_nL^2_{\text{e}}(\mathbb{R}^n) = L^2(\mathbb{R}^+) \otimes \psi_n,$$
$$U_nD(A_{n,\text{e}}) \cap \left(C^1(\mathbb{R}^+) \otimes \psi_n\right) = \left\{x^{\frac{n}{4}-\frac{1}{2}}f(x) \in L^2(\mathbb{R}^+) \cap C^1(\mathbb{R}^+) \;\middle|\; \begin{array}{l} x^{\frac{n}{4}-\frac{1}{2}}f'(x) \in L^2(\mathbb{R}^+) \\ f(s) < \infty \text{ as } s \to 0. \end{array}\right\} \otimes \psi_n,$$

where ψ_n denotes the constant function on S^{n-1} such that $\|\psi_n\| = 1$. The compound-system-type normal extension of $A_{n,\text{e}}$ in the above relations is reduced to the discussion of $A_{\frac{n}{4}-\frac{1}{2}}$ which will be treated in §6.1 and §6.2.

6 Construction of compound-system-type normal extension of A^*

6.1 The case where $\lambda = 1$

In this subsection, we will construct an compound-system-type normal extension of A^* when $\lambda = 1$. Let $\{|\uparrow\rangle, |\downarrow\rangle\}$ be a CONS of $\mathbb{C}^2$. From Lemma 23 and the fact that A^* is maximal symmetric, we obtain the following theorem.

Theorem 24 *Define the operator* $T := A \otimes |-\rangle\langle+| + A^* \otimes |+\rangle\langle-|$ *on the domain* $D(T) := D(A) \otimes |+\rangle \oplus D(A^*) \otimes |-\rangle$ *with* $|\pm\rangle := \frac{1}{\sqrt{2}}(|\uparrow\rangle \pm |\downarrow\rangle)$. *$T$ is a self-adjoint operator. Moreover, the triple* $(\mathbb{C}^2, T, |\uparrow\rangle)$ *is a compound-system-type normal extension of* A^*.

Similarly, we can construct a compound-system-type normal extension of a^* according to Lemma 22. The spectrum of the compound-system-type normal extension of A^* for $\lambda = 1$ appears only on the real axis. That of the compound-system-type normal extension of a^* appears only on the unit circle.

6.2 The cases where $\lambda > 1$

In the following, we will discuss the cases when $\lambda > 1$. Let $\{|\uparrow\rangle, |\downarrow\rangle\}$ be a CONS of $\mathbb{C}^2$. We obtain the following theorem.

Theorem 25 *The pair of* $E_+ \otimes I$ *and* $E_0 \otimes I + I \otimes E_0$ *on* $\mathcal{H}_\lambda \otimes \mathcal{H}_{\lambda-1}$ *satisfies the commutation relation of the generators of the affine group. This representation of the affine group is written as follows: There exist a Hilbert space* $\mathcal{H}'$ *and a unitary map* U *from* $\mathcal{H}_\lambda \otimes \mathcal{H}_{\lambda-1}$ *to* $\mathcal{H}' \otimes L^2(\mathbb{R}^+)$ *such that* $U(E_+ \otimes I)U^* = I \otimes E_+, U(E_0 \otimes I + I \otimes E_0)U^* = I \otimes E_0$. *Then, the operator* $U^*I \otimes A_0 U \otimes |-\rangle\langle+| + U^*I \otimes A_0^* U \otimes |+\rangle\langle-|$ *with the domain* $D(U^*I \otimes A_0 U) \otimes |+\rangle \oplus D(U^*I \otimes A_0 U) \otimes |-\rangle$ *is self-adjoint.*

Moreover, the operator $T := U^*I \otimes A_0 U \otimes |-\rangle\langle+| + U^*I \otimes A_0^* U \otimes |+\rangle\langle-| - iE_+^{-1} \otimes E_+ \otimes I$ *with the domain* $D(T) := (D(U^*I \otimes A_0 U) \otimes |+\rangle \oplus D(U^*I \otimes A_0^* U) \otimes |-\rangle) \cap D(E_+^{-1} \otimes E_+) \otimes \mathbb{C}^2$ *is normal.*

Moreover, the triple $(\mathcal{H}'_\lambda := \mathcal{H}_{\lambda-1} \otimes \mathbb{C}^2, T, \psi := |0\rangle_N \otimes |\uparrow\rangle)$ *is a compound-system-type normal extension of* A^*.

Proof It is sufficient to prove them under the representations given in §5.2. Now define the unitary operator U on $L^2(\mathbb{R}^+) \otimes L^2(\mathbb{R}^+)$ by $\big(U(f)\big)(u,v) = \sqrt{v} f(v, uv)$. Then we have $U(E_+ \otimes I)U^* = I \otimes E_+$, $U(E_0 \otimes I + I \otimes E_0)U^* = I \otimes E_0$ and $U(-iE_+^{-1} \otimes E_+)U^* = -E^+ \otimes I$. From the proof of Lemma 23, $A_0 \otimes |-\rangle\langle+| + A_0^* \otimes |+\rangle\langle-|$ is shown to be self-adjoint and its domain is shown to be $D(A^*) \otimes |-\rangle \oplus D(A) \otimes |+\rangle$. In general, for a self-adjoint operator X on $\mathcal{K}_1$ and a skew-adjoint operator Y on $\mathcal{K}_2$, we can show that the operator $X \otimes I + I \otimes Y$ with the domain $D(X) \otimes D(Y) = D(X) \otimes \mathcal{K}_2 \cap \mathcal{K}_1 \otimes D(Y) \subset \mathcal{K}_1 \otimes \mathcal{K}_2$ is normal. Then, the operator $T' := I \otimes (A_0 \otimes |-\rangle\langle+| \oplus A_0^* \otimes |+\rangle\langle-|) - E^+ \otimes I \otimes I$ with the domain given above, is normal. The domain $D(T')$ of T' equals $(D(I \otimes A_0) \otimes |+\rangle \oplus D(I \otimes A_0^*) \otimes |-\rangle) \cap D(E_+) \otimes L^2(\mathbb{R}^+) \otimes \mathbb{C}^2$. Then, we proved that the operator $T(= U^*T'U)$ is normal.

Now, we will prove that the triple $(\mathcal{H}'_\lambda, T, \psi = |0\rangle_N^{k-\frac{1}{2}} \otimes |\uparrow\rangle)$ is a compound-system-type normal extension of A_k^*. Since the set $D(A_k^*) \cap C^1(\mathbb{R}^+)$ is a core of the operator A_k, it is sufficient to show that $(A_k^*\phi) \otimes |0\rangle_N^k \otimes |\uparrow\rangle = T\left(\phi \otimes |0\rangle_N^{k-\frac{1}{2}} \otimes |\uparrow\rangle\right)$ for any $\phi \in D(A_k^*) \cap C^1(\mathbb{R}^+)$.

From the definitions and (30), some calculations result in

$$
\begin{aligned}
&U\left((D(A_k^*) \cap C^1(\mathbb{R}^+)) \otimes |0\rangle_N^{k-\frac{1}{2}}\right)\\
=&\{f(v)u^{k-1/2}e^{-uv} \in L^2(\mathbb{R}^+ \times \mathbb{R}^+) | x^{-k}f'(x) \in L^2(\mathbb{R}^+), f(s) \to 0 \text{ as } s \to 0\}.
\end{aligned}
$$

We can show that a function $u \mapsto u^{k-1/2}e^{-uv}$ is contained by $D(E_+) \subset L^2(\mathbb{R}^+)$ for any $v \in \mathbb{R}^+$. If a function f satisfies the condition $x^{-k}f'(x) \in L^2(\mathbb{R}^+), f(s) \to 0$ as $s \to 0$, then a function $v \mapsto f(v)u^{k-1/2}e^{-uv}$ is contained by $D(A_0^*) \subset L^2(\mathbb{R}^+)$ for any $u \in \mathbb{R}^+$.

Then, the set $U\left((D(A_k^*) \cap C^1(\mathbb{R}^+)) \otimes |0\rangle_N^{k-\frac{1}{2}}\right)$ is included in the set $U\left(D(I \otimes A_0^*) \cap D(E_+ \otimes I) \cap \left(C^1(\mathbb{R}^+) \otimes |0\rangle_N^{k-\frac{1}{2}}\right)\right)$. Hence,

$$
\begin{aligned}
&U\left((D(A_k^*) \cap C^1(\mathbb{R}^+)) \otimes |0\rangle_N^{k-\frac{1}{2}}\right) \otimes |\uparrow\rangle\\
&\subset U \otimes I\left(D(I \otimes A_0^* \otimes I) \cap D(E_+ \otimes I \otimes I) \cap \left(C^1(\mathbb{R}^+) \otimes |0\rangle_N^{k-\frac{1}{2}} \otimes |\uparrow\rangle\right)\right)\\
&\subset U \otimes I\left(D(I \otimes (A_0^* \otimes |+\rangle\langle-| + A_0 \otimes |-\rangle\langle+|) \cap D(E_+ \otimes I \otimes I) \cap \left(C^1(\mathbb{R}^+) \otimes |0\rangle_N^{k-\frac{1}{2}} \otimes |\uparrow\rangle\right)\right)\\
&= U \otimes I\left(D(T) \cap \left(C^1(\mathbb{R}^+) \otimes |0\rangle_N^{k-\frac{1}{2}} \otimes |\uparrow\rangle\right)\right).
\end{aligned}
$$

Thus, for the function $f(x)$ satisfying $\phi(x) = f(x)x^{-k}$, we obtain

$$
\begin{aligned}
T(\phi \otimes |0\rangle_N^{k-\frac{1}{2}} \otimes |\uparrow\rangle) &= -i\frac{d}{dv}\left(f(v)u^{k-\frac{1}{2}}e^{-uv}\right) \otimes (|+\rangle\langle-| + |-\rangle\langle+|)|\uparrow\rangle - if(v)u^{k+\frac{1}{2}}e^{-uv} \otimes |\uparrow\rangle\\
&= -i\frac{df}{dv}(v)u^{k-\frac{1}{2}}e^{-uv} \otimes |\uparrow\rangle\\
&= (A_k\phi) \otimes |0\rangle_N^{k-\frac{1}{2}} \otimes |\uparrow\rangle.
\end{aligned}
$$

The theorem is now immediate. □

In the above discussions, it is sufficient only to choose $\mathcal{H}_{\lambda-1}$ instead of $\mathcal{H}'_\lambda$ in order only to show that the operator T formally satisfies $[T, T^*] = 0$ and formally satisfies (8). However, the above definition of $\mathcal{H}'_\lambda$ is required in order that T may be a normal operator defined in Definition 1.

Since the spectrum of the compound-system-type normal extension of A^* for $\lambda = 1$ appears only in the upper half plane including the real axis, the spectrum of the compound-system-type normal extension of a^* appears only on the unit disc (including the unit circle) if the latter is related to the former by the adjoint of the Cayley transform.

7 Conclusions

We have discussed subnormal operators as a class of generalized observables. The POVM of a subnormal operator defined in Definition 15 has little information about its implementation.

We have defined compound-system-type normal extensions in order to describe not only the probability distributions characterized by the POVMs but also a framework of their implementations. (The heterodyne measurement known in quantum optics is interpreted as a special case of compound-system-type normal extensions.) In these contexts, we have constructed the compound-system-type normal extensions of two subnormal operators a^* and A^* canonically introduced from an irreducible unitary representation of $\mathfrak{su}(1,1)$, when the minimum eigenvalue λ of the generator L_0 is not less than one. The squeezed states are regarded as the coherent states of the algebra $\mathfrak{su}(1,1)$, and have been characterized as the eigenvectors of an operator defined in this mathematical framework. The squeezed states in two-particle or multi-particle systems have been interpreted as the eigenvectors of the adjoints a and A of the subnormal operators a^* and A^*. The coherent states of the affine group have been interpreted in the same framework, as well. The squeezed states in one-particle system have been interpreted as the eigenvectors of the operator a and A, though the operators a^* and A^* are not subnormal and their compound-system-type normal extensions do not exist in this case because λ is less than one in this case.

The information described by a compound-system-type normal extension isn't enough to completely specify the experimental implementation, where the measurement of the normal operator on the compound system is performed by the measurement on each system after some interactions were made between the basic system and the ancillary system. Therefore, the formulation including this specification is one of future problems, where interaction-based normal extensions of subnormal operators, as it were called, will be proposed. As another possibility, since the affine group is closely related to Lorenz group, our results about the affine group may be applicable to the relativistic quantum mechanics.

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Appendix

The following lemma about spectra is well-known. (See Hiai and Yanagi [10].) In Hiai and Yanagi [10], it is proved in the case of bounded operators. But, it can be easily extended to the case of unbounded operators.

Lemma 26 *For a densely defined operator A on $\mathcal{H}$, Let $\sigma_p(A), \sigma_c(A)$ and $\sigma_r(A)$, be the point spectrum, the continuous spectrum and the residual spectrum, respectively. Then we have the following relations:*

- $\lambda \in \sigma_r(A) \Rightarrow \overline{\lambda} \in \sigma_p(A^*)$
- $\lambda \in \sigma_p(A) \Rightarrow \overline{\lambda} \in \sigma_r(A^*) \cup \sigma_p(A^*)$
- $\lambda \in \sigma_c(A) \Rightarrow \overline{\lambda} \in \sigma_c(A^*)$.

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