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Kyoto University
ON THE PRINCIPAL BLOCKS OF FINITE GENERAL LINEAR GROUPS IN NON-DEFINING CHARACTERISTIC

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1 Introduction

Let $k$ be a field of characteristic $\ell > 0$. In this note, we consider the $\ell$-modular representation of a finite general linear group $\mathrm{GL}_n(q)$ with abelian Sylow $\ell$-subgroup of rank 2 where $q$ is a prime power which is not divided by $\ell$. We fix a positive integer $e$ such that $1 < e < \ell$. Let $e(q)$ be the minimal $a > 0$ such that $\ell \mid q^a - 1$. Let $r(q)$ be the maximal $r > 0$ such that $\ell^r \mid q^{e(q)} - 1$. We study the principal block of the group algebra $k\mathrm{GL}_{2e}(q)$ where $e = e(q)$. Note that the Sylow $\ell$-subgroup of $\mathrm{GL}_{2e}(q)$ is isomorphic to $C_{\ell^e} \times C_{\ell^e}$ where $r = r(q)$ and $C_{\ell^e}$ is a cyclic group of order $\ell^e$. On the other hand, the Sylow $\ell$-subgroup of $\mathrm{GL}_{2e-1}(q)$ is isomorphic to $C_{\ell^e}$ and the structure of $k\mathrm{GL}_{2e-1}(q)$ is well-known. Our main result is the following:

Theorem 1.1. Let $q_i$ be a prime power which is not divided by $\ell$ for $i = 1, 2$. Let $B_i$ be the principal block of $k\mathrm{GL}_{2e}(q_i)$ where $e = e(q_1) = e(q_2)$. If $r(q_1) = r(q_2)$, then $B_1$ and $B_2$ are Morita equivalent.

Remark The case $\ell = 3, e = 2, r(q_i) = 1$ is treated in [5]. The proof is essentially same as in [5],[9]. See [5],[9] for the details.

2 Stable equivalence

In this section, we state the outline of the proof of the main theorem. We keep the notation as in §1. First, we define some subgroups.

Definition

$L(q_i) := \left\{ \begin{pmatrix} X & O \\ O & Y \end{pmatrix} \middle| X, Y \in \mathrm{GL}_e(q_i) \right\}$, $H(q_i) := L(q_i)(w_i)$ where $w_i = \begin{pmatrix} O & I \\ I & O \end{pmatrix}$.

Note that $H(q_i)$ is the normalizer of $L(q_i)$ in $\mathrm{GL}_{2e}(q_i)$. By Broué’s theorem ([1]), $B_i$ and the principal block $B_0(kH(q_i))$ of $kH(q_i)$ are stable equivalent of Morita type. Since $B_0(kH(q_1))$ and $B_0(kH(q_2))$ are Morita equivalent, there exists a $(B_1, B_2)$-bimodule $\mathcal{M}$ such that

$- \otimes \mathcal{M} : \text{mod } B_1 \longrightarrow \text{mod } B_2$

induces a stable equivalence.
In order to show that $\mathcal{M}$ induces a Morita equivalence, it suffices to show that $S \otimes_{B_1} \mathcal{M}$ is a simple $B_2$-module for every simple $B_1$-module $S$ by Linckelmann’s theorem [6]. We construct (Corollary 4.3) $B_1$-module $Y$ such that,

1. $Y/\text{rad}
\text{Y}$ and $\text{soc}
\text{Y}$ are isomorphic simple modules.
2. $\text{rad}
\text{Y}/\text{soc}
\text{Y}$ is semisimple.
3. $Y \otimes_{B_1} \mathcal{M}$ satisfies (1) and (2).
4. $T \otimes \mathcal{M}$ is known (and simple) for every composition factor $T$ of $Y$ which is not isomorphic to $S$.
5. The multiplicity of $S$ as a composition factor of $Y$ is one.

Using these properties of $Y$, we can show that $S \otimes \mathcal{M}$ is simple.

3 \ Representation theory of $\mathbf{GL}_n(q)$

In this section, we state some preliminary results on the representation theory of $\mathbf{GL}_n(q)$. First we recall some terminologies on partitions. If $\lambda$ is a partition of $n$, then we write $\lambda \vdash n$.

**Definition** Let $\lambda = (\lambda_1, \lambda_2, \ldots), \mu = (\mu_1, \mu_2, \ldots) \vdash n$.

1. $\lambda > \mu$ if there exists $k$ such that $\lambda_i = \mu_i$ ($i < k$) and $\lambda_k > \mu_k$.
2. $\lambda' \vdash n$ where $(\lambda')_n := \text{Card } \{ j \mid \lambda_j \geq i \}$.
3. By removing e-rim hooks from $\lambda$ as possible, we obtain a partition, which has no hook of length $e$. This partion is uniquely determined by $\lambda$ and $e$, and called the e-core of $\lambda$.
4. (Littelwood-Richardson coefficient $a_{\alpha(1)\lambda}$)

If $\alpha = (\alpha_1, \alpha_2, \ldots) \vdash n - 1$, then $a_{\alpha(1)\lambda} = \begin{cases} 1 & \text{if } \lambda_i = \alpha_i + 1 \text{ for some } i \\ 0 & \text{otherwise.} \end{cases}$

Let $k$ be a field of characteristic $\ell > 0$, $\ell \nmid q$. For each $\lambda \vdash n$, James defines some $k \mathbf{GL}_n(q)$-modules, namely $S(\lambda) := S_k(1, \lambda), D(\lambda) := S(\lambda)/\text{rad} S(\lambda)$ ([3]), and Dipper and James define Young module $X(\lambda) := X(1, \lambda)$ ([2]). For every $\lambda \vdash n$, $D(\lambda)$ is a simple module and every composition factor of $S(\lambda)$ is isomorphic to $D(\mu)$ for some $\mu \vdash n$. We denote the multiplicity of $D(\mu)$ in $S(\lambda)$ as composition factors by $d_{\lambda\mu}$.

Let $U$ be a $k \mathbf{GL}_{n-1}(q)$-module. We may regard $U$ as a module for a parabolic subgroup $P$, where

$$P := \left\{ \begin{pmatrix} X & 0 \\ * & * \end{pmatrix} \in \mathbf{GL}_n(q) \mid X \in \mathbf{GL}_{n-1}(q) \right\}. $$

We define $U \uparrow$ to be the induced module $\text{Ind}_{P}^{\mathbf{GL}_n(q)}(U)$. If $k \mathbf{GL}_n(q)$-module $V$ has the same composition factors as $\bigoplus_{\lambda \vdash n} b_{\lambda} S(\lambda)$, then we write $V \downarrow$ for $\bigoplus_{\lambda \vdash n} a_{\alpha(1)\lambda} \lambda S(\alpha)$.

Let $\Delta_n := (d_{\lambda\mu})_{\lambda,\mu}$, $T_n := (a_{\alpha(1)\lambda})_{\alpha,\lambda}$, $(u_{\alpha\lambda})_{\alpha,\lambda} := \Delta_{n-1}^{1} T_n \Delta_n$. Then the following holds.
Theorem 3.1 (Dipper-James). ([2]) If $\mu \vdash n$, then $X(\mu')$ has the same composition factors as $\bigoplus_{\lambda \vdash n} d_{\lambda\mu}S(\lambda')$.

Theorem 3.2 (James). ([4])

1. If $\lambda \vdash n$, then $X(\lambda') \downarrow$ has the same composition factors as $\bigoplus_{\alpha \vdash n-1} u_{\alpha\lambda}x(\alpha^{;})$.

2. If $\alpha \vdash n-1$, then $D(\alpha) \uparrow$ has the same composition factors as $\bigoplus_{\lambda \vdash n} u_{\alpha\lambda}D(\lambda)$.

4 Inductions of Young modules

Let $B$ be the principal block of $k\mathrm{GL}_{2e}(q)$ where $e = e(q)$, $\text{char} \ k = \ell$, $1 < e < \ell$. In this section, we determine the decomposition matrix $\Delta_{2e}$ and construct the modules mentioned in the last part of §2.

Definition

1. $\lambda := \{\lambda \vdash 2e \mid (e\text{-core of } \lambda) = \emptyset\}$, $\Gamma := \{\alpha \vdash 2e-1 \mid a_{\alpha(1)\lambda} \neq 0 \text{ for some } \lambda \in \Lambda\}$.

2. $\alpha^{-} := \min\{\lambda \in \Lambda \mid a_{\alpha(1)\lambda} \neq 0\}$, $\alpha^{+} := \max\{\lambda \in \Lambda \mid a_{\alpha(1)\lambda} \neq 0\}$ for $\alpha \in \Gamma$.

3. $\lambda_{+} := \max\{\alpha \in \Gamma \mid a_{\alpha(1)\lambda} \neq 0\}$ for $\lambda \in \Lambda$.

Then $\{D(\lambda) \mid \lambda \in \Lambda\}$ is a complete set of isomorphism classes of simple $B$-modules. Using these notation, we can describe Young module $X(\lambda)$ for $\lambda \in \Lambda$.

Theorem 4.1. If $\alpha \in \Gamma$, then $X(\alpha) \uparrow \cdot 1_B \cong X(\alpha^{-})$.

If $\lambda \in \Lambda, \lambda \neq (2e), (e^2)$, then $\lambda = \alpha^{-}$ for some $\alpha \in \Gamma$. Since $\mathrm{GL}_{2e-1}(q)$ has a cyclic Sylow $\ell$-subgroup and the structure of the Young module $X(\alpha) (\alpha \in \Gamma)$ is known, we obtain the decomposition number $d_{\lambda\mu}(\lambda, \mu \in \Lambda)$ by Theorem 3.2. Since $d_{\lambda\mu}(\lambda \not\in \Lambda \text{ or } \mu \not\in \Lambda)$ is well known, we can know all the decomposition numbers.

Corollary 4.2. We can determine $\Delta_{2e}$.

(This means that by [2] we can determine the $\ell$-modular decomposition matrix of $\mathrm{GL}_{2e}(q)$.) Using this result, we have the following result.

Corollary 4.3. Assume that $\lambda \in \Lambda, \lambda \neq (2e), (e^2), (e, 1^e)$. Then the Loewy series of $D(\lambda_{+}) \uparrow \cdot 1_B$ is as follows:

$$D(\lambda_{+}) \uparrow \cdot 1_B = \left( \begin{array}{c} D((\lambda_{+})^{+}) \\ \bigoplus \ C \\ D((\lambda_{+})^{+}) \end{array} \right).$$

Here, $C$ is a direct sum of some $D(\mu)$ where $\mu \in \Lambda, \mu > \lambda$.

Example

1. Let $\lambda = (2e - 1, 1) \in \Lambda$. Then, $\lambda_{+} = (2e - 1), (\lambda_{+})^{+} = (2e)$, and,

$$D(2e - 1) \uparrow \cdot 1_B = \left( \begin{array}{c} D(2e) \\ D((2e - 1, 1) \end{array} \right).$$
2. Let $e = 4$ and $\lambda = (4, 2, 1^2) \in \Lambda$. Then $\lambda_+ = (4, 2, 1), (\lambda_+)^+ = (4, 3, 1)$ and

\[ D(4, 2, 1) \uparrow \cdot 1_B = \left( \begin{array}{ccc} D(4, 3, 1) & D(8) & D(4, 2, 1^2) \\ D(8) & D(4^2) & D(4, 2, 1^2) \\ D(4, 3, 1) & D(4, 2, 1^2) & D(4, 3, 1) \end{array} \right). \]

**Remark** (1) Let $G_n(q)$ be a finite group of Lie type over $\mathbb{F}_q$ whose rank is $n$. Suppose that $e = e(q) = e(q'), r(q) = r(q')$. By Theorem 1.1, the unipotent blocks of $GL_{2e}(q)$ and $GL_{2e}(q')$ are Morita equivalent. We believe that the unipotent blocks of $G_n(q)$ and $G_n(q')$ are Morita equivalent if the types of $G_n(q)$ and $G_n(q')$ are the same. ([10])

(2) After the meeting, we found the paper by M.J.Richards [8]. It seems that some results of this section are contained in his results [8](see also [7, p.126]).

**References**


