On rationality of eigenvalues of the Cartan matrix of a finite group

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$G$ : a finite group
$F$ : an algebraically closed field of characteristic $p > 0$
$B$ : a block of the group algebra $FG$ with defect group $D$ of order $p^d$
$IBr(B)$ : the set of irreducible Brauer characters in $B$
$l(B) := |IBr(B)|$
$C_B = (c_{ij})$ : the Cartan matrix of $B$ i.e. $c_{ij}$ is the multiplicity of an irreducible $FG$-module $S_j$ in a projective cover $P_i$ of $S_i$ as a composition factor, where $S_j$ and $P_i$ belong to $B$
$\rho(B)$ : the Perron-Frobenius eigenvalue of $C_B$ i.e. the largest eigenvalue of $C_B$

It is well known that all $\mathbb{Z}$-elementary divisors of $C_B$ are a power of $p$, the largest one is $p^d = |D|$ and the others are smaller than $p^d$; but eigenvalues need not be an integer. For example, $\rho(B_0) = (7 + \sqrt{33})/2$ for the principal 2-block $B_0$ of the alternating group $A_5$ of degree 5. So the following question is fundamental.

**Question.** When do eigenvalues and elementary divisors of $C_B$ coincide ?

If eigenvalues and elementary divisors coincide, then sure $\rho(B) = |D|$. Does the converse hold ?

We should first consider the case $\rho(B) = |D|$. This case really occurs in the following situation.

**FACTS**

(A) Theorem 4.4 in [K-W]. **If $G$ is $p$-solvable, then $\rho(B) \leq |D|$, and the equality holds if and only if the height of $\varphi = 0$ for all $\varphi \in IBr(B)$**.
(B) Proposition 4.7 in [K-W]. If $D$ is cyclic, then $\frac{|D|}{p} + 1 \leq \rho(B) \leq |D|$, and $\rho(B) = |D|$ if and only if the Brauer tree of $B$ is a star and the exceptional vertex, if it exists, is at the center if and only if $C_B = \begin{pmatrix} m + 1 & m & \cdots & m \\ m & m + 1 & \ddots & \vdots \\ \vdots & \ddots & \ddots & m \\ m & \cdots & m & m + 1 \end{pmatrix}$, where $m$ is the multiplicity, $em + 1 = |D|$ and $e = l(B)$.

(C) (Erdmann) If $B$ is tame i.e. $p = 2$ and $D$ is dihedral, generalized quaternion or semi dihedral, then $l(B) = 1, 2$ or $3$ and in this case $\rho(B) = |D|$ if and only if the following hold.

(i) $l(B) = 1$.

(ii) $D \simeq E_4$ (elementary abelian group of order 4) and $C_B = \begin{pmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{pmatrix}$, $\rho(B) = 4 = |D|$. Example. $G = PSL(2, q), q \equiv 3 \pmod{8}, B = B_0$.

(iii) $D \simeq Q_8$ (the quaternion group of order 8) and $C_B = \begin{pmatrix} 4 & 2 & 2 \\ 2 & 4 & 2 \\ 2 & 2 & 4 \end{pmatrix}$, $\rho(B) = 8 = |D|$. Example. $G = SL(2, q), q \equiv 3 \pmod{8}, B = B_0$.

(D) Proposition 4.3 in [K-W]. If $D \triangleleft G$, then $\rho(B) = |D|$.

So we should next consider when eigenvalues and elementary divisors coincide in the examples above.

**Proposition 1.** Let $G$ be a $p$-solvable group and $B$ a block of $FG$. Then the following are equivalent.

1. eigenvalues and elementary divisors of $C_B$ coincide.
2. $\rho(B) = |D|$.
3. the height of $\varphi = 0$ for all $\varphi \in IBr(B)$.
4. $f = (f_1, \cdots, f_{\iota(B)})$ is an eigenvector of $C_B$, where $f_i = \varphi_i(1)$ for $\varphi_i \in IBr(B)$.

**Proof.** We have already proved equivalence on (2), (3) and (4) in [K-W]. So we should prove (3) $\rightarrow$ (1).

(a) (Isaacs [I], Rukolaïne) Let $G$ be a $p$-solvable group and $\eta_G$ the character afforded by the projective cover of the trivial $FG$-module. Then $\eta_G(x)$ is a power of $p$ for all $p$-regular
element $x \in G$.

(b) Let $G$ be a $p$-solvable group, $B$ a block of $FG$ of full defect. If the height of $\varphi = 0$ for all $\varphi \in \text{IBr}(B)$, then eigenvalues and elementary divisors of $C_B$ coincide.

**proof of (b).** Let $P_G$ be a projective cover of the trivial $FG$-module and $S_i, P_i$ be a simple $B$-module, its projective cover, respectively. Then, since $\dim S_i$ is not divided by $p$, $P_G \otimes S_i \simeq P_i$ for all $1 \leq i \leq l(B)$. Let $\Phi_B$ be an $l(B) \times l(B)$ matrix $(\varphi_i(x_j))$, where $\{x_1, \ldots, x_{l(B)}\}$ is a set of representatives of $p$-regular classes associated with $B$. Then the above statement means that $C_B\Phi_B = \Phi_B \text{diag}\{\eta_G(x_1), \ldots, \eta_G(x_{l(B)})\}$. This implies that $\eta_G(x_i)$ is an eigenvalue of $C_B$ and $^t(\varphi_1(x_i), \ldots, \varphi_{l(B)}(x_i))$ is its eigenvector for $1 \leq i \leq l(B)$. Since $\eta_G(x_i)$ is a power of $p$ by (a), and $\Phi_B$ is a unimodular matrix over a complete discrete valuation ring $R$, $\eta_G(x_1), \ldots, \eta_G(x_{l(B)})$ are also elementary divisors of $C_B$.

(c) We have a conclusion by Fong reduction.

**Proposition 2.** Let $B$ be a block of $FG$ with cyclic defect group $D$. Then the following are equivalent.

1. eigenvalues and elementary divisors of $C_B$ coincide.
2. $\rho(B) = |D|$.

If one of the conditions above holds, then the set of eigenvalues of $C_B$ is $\{ |D|, 1, \cdots, 1 \}$ and $1^t(1, \cdots, 1)$ is an eigenvector of $C_B$ for $\rho(B)$.

**Proof.** If (2) holds, then $C_B = \begin{pmatrix} m + 1 & m & \cdots & m \\ m & m + 1 & \cdots & \vdots \\ \vdots & \vdots & \ddots & m \\ m & \cdots & m & m + 1 \end{pmatrix}$, where $m$ is the multiplicity, $em + 1 = |D|$ and $e = l(B)$. It is clear that the set of eigenvalues and elementary divisors of $C_B$ are equal to $\{ |D|, 1, \cdots, 1 \}$. $1$ is an eigenvector for $\rho(B)$. Let $x_i := ^t(1, 0, \cdots, -1, \cdots, 0)$ (i.e. the column vector whose first entry is $1$ and the $i$ th entry is $-1$, and others are $0$), then $C_Bx_i = x_i$ for $2 \leq i \leq l(B)$.

**Proposition 3.** If $B$ is a block of $FG$ with a normal defect group $D$, then the set of eigenvalues and elementary divisors of $C_B$ are equal to $\{ |C_D(x_1)|, \cdots, |C_D(x_{l(B)})| \}$, where $\{x_1, \cdots, x_{l(B)}\}$ is a set of representatives of $p$-regular classes of $G$ associated with $B$. In this case, $f = ^t(f_1, \cdots, f_{l(B)})$ is an eigenvector for $|D|$.

**Proof.** Let $M := \Phi_B \text{diag}\{ |C_D(x_1)|, \cdots, |C_D(x_{l(B)})| \} \Phi_B^{-1}$. Then by [A-C-S] $C_B = C_B M$, where $\overline{B}$ is a homomorphic image of $B$ of an algebra epimorphism $\tau : FG \rightarrow FG$. 
for $\overline{G} = G/D$. Now since $D$ is a defect group of $B$, if $\overline{B} = \overline{B}_1 + \cdots + \overline{B}_r$, each $\overline{B}_i$ is a block of $F\overline{G}$ of defect 0. So $C_{\overline{B}}$ is the identity $l(B) \times l(B)$ matrix. This means $C_B = \Phi_B \text{diag}([C_D(x_1)], \cdots, [C_D(x_{l(B)})]) \Phi_B^{-1}$. Since $\Phi_B$ is a unimodular matrix over $R$, we have a conclusion (see Proposition 4.3 in [K-W]).

Proposition 4. Let $B$ be a tame block of $FG$. Then the following are equivalent.

1. eigenvalues and elementary divisors of $C_B$ coincide.
2. $\rho(B) = |D|$.

If one of the conditions above holds, then $1 = ^t(1, \cdots, 1)$ ia an eigenvector of $C_B$ for $|D|$.

Proof. In [E] Erdmann has classified all tame blocks of finite group algebras via Morita equivalence. There are 6 types of Cartan matrices of blocks with dihedral defect group, 5 types with generalized quaternion defect group and 10 types with semi dihedral defect group. We can check any types of Cartan matrices have no rational largest eigenvalues without two cases. For example, there is the case that $D$ is semi dihedral, $l(B) = 3$, and

$$C_B = \begin{pmatrix} 8 & 4 & 4 \\ 4 & s + 2 & 2 \\ 4 & 2 & 3 \end{pmatrix},$$

where $s = |D|/4$. Now $|D| \geq 16$, then $s \geq 4$. If $s = 4$, then a minimal and a maximal row sum of $C_B$ is 9 and 16, respectively and so $9 < \rho(B) < 16$. If $\rho(B)$ is an integer, then it must be a power of $2 \leq |D|$ by Corollary 4.6 in [K-W]. This is a contradiction. Then $s \geq 8$. We have $s + 2 = c_{22} < \rho(B) < \text{maximal row sum} = s + 8$. So if $\rho(B) \in \mathbb{Z}$, then $\rho(B) = 2s$ or $4s$. If $\rho(B) = 2s$, then $2 < s < 8$. This contradicts to $s \geq 8$. Similarly, if $\rho(B) = 4s$, then $s = 2$ and this is also a contradiction. We can also prove that $\rho(B)$ is not an integer for other cases.

There remain two types of Cartan matrices. One of them is the case that $D$ is dihedral, $l(B) = 3$ and $C_B = \begin{pmatrix} 2 & 1 & 1 \\ 1 & m + 1 & m \\ 1 & m & m + 1 \end{pmatrix}$, where $m = |D|/4$. In this case, if $\rho(B) \in \mathbb{Z}$, then $m = 1$ and so $\rho(B) = |D| = 4$. This can actually occur, for example, when $B = B_0(PSL(2, q)), q \equiv 3(\text{mod} 8)$. The other case is that $D$ is generalized quaternion, $l(B) = 3$ and $C_B = \begin{pmatrix} 4 & 2 & 2 \\ 2 & k + 2 & k \\ 2 & k & k + 2 \end{pmatrix}$, where $k = |D|/4$. In this case, if $\rho(B) \in \mathbb{Z}$, then $k = 2$ and $\rho(B) = |D| = 8$. This can also actually occur, for example, when $B = B_0(SL(2, q)), q \equiv 3(\text{mod} 8)$.
From these results the following questions seem natural.

**Question 1.** For any finite group \( G \), and any block \( B \) of \( FG \), are the following equivalent?

1. eigenvalues and elementary divisors of \( C_{B} \) coincide.
2. \( \rho(B) = |D| \).

Furthermore,

**Question 2.** For any finite group \( G \), and any block \( B \) of \( FG \), does \( \rho(B) \in \mathbb{Z} \) mean \( \rho(B) = |D| \)?

We have an affirmative answer for Question 2 if \( D \) is cyclic or \( B \) is tame as is shown later.

**Further examples for \( \rho(B) = |D| \).**

If two algebras \( A, B \) are Morita equivalent, then the Cartan matrix of \( A \) and \( B \) coincide, and then \( \rho(A) = \rho(B) \). On the other hand, if a defect group \( D \) of a block \( B \) of \( FG \) is normal in \( G \), then \( \rho(B) = |D| \). Suppose a block \( B \) of \( FG \) is Morita equivalent to its Brauer correspondent \( b \) of \( FN_{G}(D) \). Then we have \( \rho(B) = \rho(b) = |D| \). There are several such examples, in verifying Broué conjecture.

**Broué conjecture.** Let \( B \) be a block of \( FG \) with abelian defect group \( D \). Then \( B \) and its Brauer correspondent \( b \) are derived equivalent.

If two algebras \( A, B \) are Morita equivalent, then they are derived equivalent. Several authors prove that Morita equivalence between \( B \) and \( b \) actually occur in the following groups.

1. \( p = 2, D \simeq E_{4} \) (tame), \( D \) is a Sylow \( p \)-subgroup of \( G \).
   - (Erdmann) \( G = PSL(2, q), q \equiv 3(\text{mod } 8) \implies B_{0}(G) \sim B_{0}(N_{G}(D)) \) (Morita equivalent).

2. \( p = 3, D \simeq E_{9} \), \( D \) is a Sylow \( p \)-subgroup of \( G \).
   - (Koshitani-Kunugi) \( G = PSU(3, q^{2}), 3||q + 1 \implies B_{0}(G) \sim B_{0}(N_{G}(D)) \) (Morita equivalent).
   - (Miyachi) \( G = GL(5, q), 3||q + 1 \implies B_{0}(G) \sim B_{0}(N_{G}(D)) \) (Morita equivalent).

3. Non abelian case, \( p = 2, D \simeq Q_{8} \) (tame), \( D \) is a Sylow \( p \)-subgroup of \( G \).
   - (Erdmann) \( G = SL(2, q), q \equiv 3(\text{mod } 8) \implies B_{0}(G) \sim B_{0}(N_{G}(D)) \) (Morita equivalent).
Several authors have shown that $B$ and $b$ are derived equivalent but not Morita equivalent. Such Cartan matrices of $B$ and $b$ are sure different and further $\rho(B)$ and $\rho(b)$ are different. So the following question might have an affirmative answer.

**Question 3.** Let $B$ be a block of $FG$ and $b$ a Brauer correspondent of $B$ i.e. $b$ is a block of $FN_G(D)$ with $b^G = B$. Then are the following equivalent?

1. $\rho(B) \in \mathbb{Z}$.
2. $B \sim b$ (Morita equivalent).

Question 3 is affirmative in the case that $D$ is cyclic or $B$ is tame. Suppose $D$ is cyclic. If $\rho(B) \in \mathbb{Z}$, then $\rho(B) = |D|$ from the first inequality in case(B). Then the second inequality means that the Brauer tree of $B$ is a star with an exceptional vertex, if it exists, is at the center. This is the same shape of Brauer tree for $b$. The Brauer tree of $B$ determines every shape of projective indecomposable $FG$-modules and star means that every projective indecomposable $FG$-module is uniserial, and then any indecomposable $FG$-module is uniquely determined as a homomorphic image of a projective indecomposable $FG$-module. Thus $B$ and $b$ are Morita equivalent.

Suppose that $B$ is tame. Among 21 types there are just two types of Cartan matrices of tame blocks which satisfy $\rho(B) \in \mathbb{Z}$ by [E]. Blocks belonged to any one of these two types are Morita equivalent by [E].

As assertion (2) in Question 3 is very strong, we might as well adopt (2)' in stead of (2)

(2)' there are a finite group $\tilde{H}$ and a block $\tilde{b}$ with $D(\tilde{b}) \triangleleft \tilde{H}, D(\tilde{b}) \simeq D(B)$ such that $B \sim \tilde{b}$ (Morita equivalent).

If $G$ is a $p$-solvable group with $p$-length 1, then in Question 3 $\rho(B) = |D|$ means (2)'. We do not have an affirmative answer to Question 3 even if $G$ is a $p$-solvable group. If $D$ is abelian, then recently, it has been shown $B \sim b$ (Morita equivalent) in [H-L].

In known cases (A),(B),(C),(D), the Cartan matrices satisfying $\rho(B) = |D|$ have $f = (f_1, \cdots, f_{\ell(B)})$ or $1 = (1, \cdots, 1)$ as its eigenvector. Except the case that $G$ is $p$-solvable, all vectors above are dimension vectors for simple modules in $b$. So it seems natural to ask Question 3 in this sense. In particular, if $G$ is $p$-solvable with abelian defect group, then [H-L] implies that dimension vectors $f_B$ of $B$ and $f_b$ of $b$ are proportional and so $f_B = cf_b$ for some $c \in \mathbb{N}$. 

References


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