<table>
<thead>
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<th>Title</th>
<th>A multiplication on the twisted tensor product (Cohomology theory of finite groups)</th>
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<tbody>
<tr>
<td>Author(s)</td>
<td>Kuribayashi, Katsuhiko; Mimura, Mamoru; Tezuka, Michishige</td>
</tr>
<tr>
<td>Citation</td>
<td>数理解析研究所講究録 (2000), 1140: 52-60</td>
</tr>
<tr>
<td>Issue Date</td>
<td>2000-04</td>
</tr>
<tr>
<td>URL</td>
<td><a href="http://hdl.handle.net/2433/63854">http://hdl.handle.net/2433/63854</a></td>
</tr>
<tr>
<td>Type</td>
<td>Departmental Bulletin Paper</td>
</tr>
<tr>
<td>Textversion</td>
<td>publisher</td>
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Kyoto University
A multiplication on the twisted tensor product

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1 Introduction

Let $G$ be a connected topological group. We define the right adjoint action $ad : G \times G \to G$ by $ad(g, h) = h^{-1}gh$. Then the cohomology $H^*(G; \mathbb{Z}/l)$ is regarded as a right $H^*(G; \mathbb{Z}/l)$-comodule under the coaction induced by the adjoint action. The comodule is denoted by $H^*(G; \mathbb{Z}/l)_c$ below. In this note, the algebra structure of

$$E := \text{Cotor}_{H^*(G; \mathbb{Z}/l)}(H^*(G; \mathbb{Z}/l)_c, \mathbb{Z}/l)$$

is considered from the viewpoint of the differential graded algebra structure of the twisted tensor product due to Brown [1]. The existence of the following three spectral sequences motivates the consideration of the algebra structure of $E$.

(1) Let $G(\mathbb{F}_q)$ be a finite Chevalley group of Lie type over the finite field $\mathbb{F}_q$ of $q$ elements and $l$ a prime number. By applying the Deligne spectral sequence in the case where the characteristic of $\mathbb{F}_q$ is prime to $l$, Tezuka [7] has constructed a spectral sequence converging to $H^*(BG(\mathbb{F}_q); \mathbb{Z}/l)$. In particular if $q - 1 \equiv 0$ modulo $l$, then the $E_2$-term of the spectral sequence is isomorphic to $E$ as an algebra for many cases.

(2) Let $BLG$ be the classifying space of the loop group $LG$ consisting of all continuous maps from the circle to $G$. Then there exists the
Eilenberg-Moore spectral sequence, whose $E_2$-term is isomorphic to $E$ as an algebra, converging to $H^*(BLG;\mathbb{Z}/l)$.

(3) Let $X$ be a simply connected finite CW-complex. Following Milnor’s description of universal bundles over a space, we can regard the loop space $\Omega X$, which is the subspace of the free loop space $LX$ consisting of based loops, as a topological group $G$. Therefore we have the Eilenberg-Moore spectral sequence converging to $H^*(LX;\mathbb{Z}/l)$ with $E_2 \cong E$ as an algebra.

One will know that it is important to clarify the algebra structure of $E$ as the first step in computing those spectral sequences.

Let $G$ be a connected complex Lie group with the same Lie type as that of a finite Chevalley group $G(\mathbb{F}_q)$. As for the cohomology algebras of $BG(\mathbb{F}_q)$ and $BLG$, Tezuka [15] has proposed a problem whether the cohomologies $H^*(BG(\mathbb{F}_q);\mathbb{Z}/l)$ and $H^*(BLG;\mathbb{Z}/l)$ are isomorphic as an algebra in the case where $l$ is odd and divides $q−1$ but does not divide $q$ or $l = 2$ and 4 divides $q−1$. As mentioned in [15], the answer is affirmative if the integral cohomology of $G$ has no $l$-torsion. The main theorem in [6] and the explicit calculation of $H^*(BG(\mathbb{F}_q);\mathbb{Z}/l)$ due to Kleinerman [3] guarantee the result. To shed light on left part of the problem, we will consider the structure of $E$ for the case where $H^*(G;\mathbb{Z})$ has $l$-torsion.

2 Results

Before stating our results, we recall a construction of the twisted tensor product due to Brown (see [1], [14] or [4]). Let $A$ be a coalgebra over $\mathbb{Z}/l$ with coproduct $\phi_A$ and augmentation $\varepsilon$. Let $L$ be a $\mathbb{Z}/lp$-subspace of $A$, $\iota : L \to A$ the inclusion and $\theta : A \to L$ a map such that $\theta \circ \iota = id_L$. We define the map $\bar{\theta} : A \to sL$ by $\bar{\theta} = s \circ \theta$ and $\bar{\iota} : sL \to A$ by $\bar{\iota} = \iota \circ s^{-1}$, where $s : L \to sL$ is a suspension. Construct the tensor product $X = T(sL)$ and denote by $\psi$ the product in $T(sL)$. The map $\bar{\theta}$ induces a map $A \to T(sL)$ which is again denoted by $\bar{\theta}$. Let $I$ be the ideal of $T(sL)$ generated by $(\psi \circ (\bar{\theta} \otimes \bar{\theta}) \circ \phi_A) (\ker \bar{\theta})$. The twisted tensor product $(W, d)$ with respect to $\bar{\theta}$ is defined as follows; we put...
$W = A \otimes X/I = A \otimes \bar{X}$ and define the differential operator $d_W$ by

$$d_W = 1 \otimes d_X + (1 \otimes \psi) \circ (1 \otimes \bar{\theta} \otimes 1) \circ (\phi_A \otimes 1),$$

where

$$d_X = -\psi \circ (\bar{\theta} \otimes \bar{\theta}) \circ \phi_A \circ \bar{\iota}.$$

We may denote the twisted tensor product $W$ with respect to $\bar{\theta}: A \rightarrow sL$ by $A \otimes_{\theta} \bar{X}$.

Let $G$ be a compact, simply connected, simple exceptional Lie group. Then it is known [9] that a suitable choice of a subspace $L$ of $H^*(G; \mathbb{Z}/l)$ makes the twisted tensor product into an injective resolution $0 \rightarrow \mathbb{Z}/l \rightarrow H^*(G; \mathbb{Z}/l) \otimes_{\theta} \bar{X}$ over the coalgebra $A$. Moreover the algebra structure of $\bar{X}$ induces that of the complex

$$(\mathbb{Z}/l \square_{H^*(G; \mathbb{Z}/l)}(H^*(G; \mathbb{Z}/l) \otimes_{\theta} \bar{X}), 1 \square d_W) \cong (\bar{X}, d_{\bar{X}})$$

Consequently we have

$$\text{Cotor}_{H^*(G; \mathbb{Z}/l)}(\mathbb{Z}/l, \mathbb{Z}/l) \cong H(\bar{X}, d_{\bar{X}})$$

as an algebra.

In this note, we consider a multiplication $m_W$ on the twisted tensor product $A \otimes_{\theta} \bar{X}$ for a Hopf algebra $A$, in the sense of Milnor and Moore [8], such that the differential $d_W$ is derivative under the multiplication. In order to define a multiplication $m_W$ explicitly, we will assume that the $\mathbb{Z}/l$-subspace $L$ of $A$ satisfies the following condition.

(1) There exist the set $Q$ of indecomposable elements of $A$ and a basis $\{x_i\}$ of $L$ such that $\{x_i\} \subset Q \cup Q^2$, where $Q^2 = \{\alpha^2|\alpha \in Q \cap \text{Prim} A\}$ and, as an algebra,

$$A \cong \bigotimes_{x_s \in S} \mathbb{Z}/p[x_s]/(x_s^{p^m}) \otimes \bigotimes_{x_t \in T} \Lambda(x_j),$$

where $S \cup T = Q \cap \{x_i\}$ and $S \cap T = \phi$. Moreover, we also assume that

(II) $(\psi \circ (\tilde{\theta} \otimes \bar{\theta}) \circ \phi_A)(\ker \bar{\theta}) = \mathbb{Z}/l\{(\psi \circ (\bar{\theta} \otimes \bar{\theta}) \circ \phi_A)(x_ix_j)|x_i, x_j \in \{x_i\}, i \neq j\}$,

(III) for any $a \in Q$, $\bar{\theta}(ya_i'') = 0$ for any $y \in \bar{A}$, where $\phi_A(a) = \Sigma_i a_i' \otimes a_i'' + a \otimes 1 + 1 \otimes a$ and that

(IV) for any $x$ and $y \in \{x_i\}$, $\bar{\theta}(xy) \neq 0$ if and only if $x = y$ and
$x^2 \in Q^2$.

We mention here that the conditions (I), (II) (III) and (IV) hold in the cases $(PU(3), 3), (F_4, 3), (E_8, 3), (E_6, p), (E_7, p)$ for $l = 2$ and 3 which have been studied by Kono, Mimura, Sambe and Shimada ([4],[5], [10], [11]).

The following is one of the our main theorem.

**Theorem 2.1.** Let $A$ be a Hopf algebra over $\mathbb{Z}/l$. For any elements $a \otimes \theta x$ and $b \otimes \theta y$ of $A \otimes_{\theta} \tilde{X}$, define $m_W : A \otimes_{\theta} \tilde{X} \otimes A \otimes_{\theta} \tilde{X} \rightarrow A \otimes_{\theta} \tilde{X}$ by

$$m_W(a \otimes \theta x \otimes b \otimes \theta y) = a \otimes \theta x \cdot b \otimes \theta y = \sum_i (-1)^{|\theta x||b_i^l|} a b_i^l \otimes \theta(x b_i^r) \theta y,$$

and

$$(\theta x_1 \cdots \theta x_s) \cdot a = (\theta x_1(\theta x_2(\cdots(\theta x_s \cdot a))\cdots),$$

where $\phi_A(b) = \sum_i b_i^l \otimes b_i^r$. If $m_W$ is well-defined, then $(A \otimes_{\theta} \tilde{X}, d_W, m_W)$ is a differential graded algebra.

By comparing the differential algebra structure of the cobar resolution [13, 7.A, 1.2] of the left $A$-comodule $\mathbb{Z}/l$ and that of the twisted tensor product mentioned above, we can prove Theorem 1.

**Theorem 2.2.** If $l = 2$ or 3 and the condition (I), (II), (III) and (IV) hold, then the multiplication $m_W$ is well-defined.

In the case where $A = H^*(E_8; \mathbb{Z}/5)$, explicit calculation for the differential $d_W$ and the multiplication $m_W$ on $A \otimes_{\theta} \tilde{X}$ allow us to obtain the following theorem.

**Theorem 2.3.** Let $A \otimes_{\theta} \tilde{X}$ be the twisted tensor product of $H^*(E_8; \mathbb{Z}/5)$ constructed in [12]. Then $(A \otimes_{\theta} \tilde{X}, d_W, m_W)$ is a well-defined differential graded algebra.

In the case where $A = H^*(E_8; \mathbb{Z}/2)$, indecomposable elements $x$ on $A$ can be chosen so that $\tilde{\Delta}(x)$ is in $P \otimes P$, where $P$ is the $\mathbb{Z}/2$-subspace of $A$ consisting of primitive elements. Thanks to this fact, we can easily verify that the multiplication $m_W$ is well-defined.
**Theorem 2.4.** Let $A \otimes_{\theta} \overline{X}$ be the twisted tensor product of $H^*(E_8; \mathbb{Z}/2)$ constructed in [9]. Then $(A \otimes_{\theta} \overline{X}, d_W, m_W)$ is a well-defined differential graded algebra.

In order to prove that the multiplication $m_W$ induces the algebra structure on $\text{Cotor}_A(A, \mathbb{Z}/p)$, it suffices to prove

**Proposition 2.5.** Let $p$ be a prime number and $\mu : A \otimes A \to A$ the multiplication of $A$. Then the map $m_W : A \otimes_{\theta} \overline{X} \otimes A \otimes_{\theta} \overline{X} \to A \otimes_{\theta} \overline{X}$ is a $\mu$-morphism if $m_W$ is well-defined, that is, the following diagram is commutative:

\[
\begin{array}{ccc}
A \otimes_{\theta} \overline{X} \otimes A \otimes_{\theta} \overline{X} & \xrightarrow{\psi_1} & (A \otimes A) \otimes A \otimes_{\theta} \overline{X} \otimes A \otimes_{\theta} \overline{X} \\
\downarrow m_W & & \downarrow \mu \otimes m_W \\
A \otimes_{\theta} \overline{X} & \xrightarrow{\psi_2} & A \otimes A \otimes_{\theta} \overline{X} ,
\end{array}
\]

where $\psi_1$ and $\psi_2$ are the comodule structures of $A \otimes_{\theta} \overline{X} \otimes A \otimes_{\theta} \overline{X}$ and $A \otimes_{\theta} \overline{X}$ respectively.

Let $A$ denote the mod $l$ cohomology $H^*(G; \mathbb{Z}/p)$. Since $\text{ad}^* \otimes 1 : A \otimes \overline{X} \to A \square_A (A \otimes \overline{X})$ is the isomorphism with the inverse $1 \otimes \epsilon \otimes 1$, we can define a differential on $A \otimes \overline{X}$ by the compositions

\[
A \otimes \overline{X} \xrightarrow{\text{ad}^* \otimes 1} A \square_A (A \otimes \overline{X}) \xrightarrow{\text{inc}} A \otimes (A \otimes \overline{X}) \xrightarrow{1 \otimes d_W} A \otimes (A \otimes \overline{X}) \xrightarrow{1 \otimes \epsilon \otimes 1} A \otimes \overline{X}.
\]

A straightforward calculation for the differential $d : A \otimes \overline{X} \to A \otimes \overline{X}$ enables us to obtain the following explicit formula for $d$.

**Lemma 2.6.** We write as $\Delta_A(x) = x \otimes 1 + 1 \otimes x + \sum i x'_i \otimes x''_i$ for $x \in A$. If $x'_i$ is primitive for any $i$, then

\[
dx = - \sum (-1)^{|x''_i|(|x'_i|+1)} x''_i \otimes \theta x'_i + \sum (-1)^{|x'_i|} x'_i \otimes \theta x''_i.
\]

The multiplication $m_W$ on the twisted tensor product $A \otimes_{\theta} \overline{X}$ induces a multiplication $m$ on $A \otimes \overline{X}$ defined by

\[
A \otimes \overline{X} \otimes A \otimes \overline{X} \xrightarrow{\text{ad}^* \otimes 1 \otimes \text{ad}^* \otimes 1} A \square_A (A \otimes \overline{X}) \otimes A \square_A (A \otimes \overline{X}) \xrightarrow{\text{inc}} A \otimes (A \otimes \overline{X}) \otimes A \otimes (A \otimes \overline{X}) \xrightarrow{m_A \otimes m_W} A \otimes (A \otimes \overline{X}) \xrightarrow{1 \otimes \epsilon \otimes 1} A \otimes \overline{X}.
\]
We can obtain an explicit formula for the multiplication $m$ on $A \otimes \overline{X}$.

**Lemma 2.7.** We write as $\Delta_A(a) = a \otimes 1 + 1 \otimes a + \sum_i a'_i \otimes a''_i$ for $a \in A$. If $a'_i$ is primitive for any $i$, then

$$\theta x \cdot a = (-1)^{\|\theta x\|}a \otimes \theta x - \sum_i (-1)^{|a''_i|+|\theta x||a''_i|}a''_i \otimes \theta(xa'_i)$$

$$+ \sum_i (-1)^{|a'_i|\|\theta x\|}a'_i \otimes \theta(xa''_i).$$

Thus we can obtain a differential graded algebra $(A \otimes \overline{X}, d, m)$. From the construction of this differential graded algebra, we have

**Theorem 2.8.** For the case where $A = H^*(G; \mathbb{Z}/l)$, if the twisted tensor product $(A \otimes_{\theta} \overline{X}, d_W, m_W)$ is a well-defined differential graded algebra, then, as an algebra,

$$\text{Cotor}_{H^*(G; \mathbb{Z}/l)}(H^*(G; \mathbb{Z}/l)_c, \mathbb{Z}/l) \cong H(A \otimes \overline{X}, d, m).$$

The proofs of theorems and propositions in this note will be given in a further article [7].

This note will be concluded with some examples of the differential graded algebras $A \square_A (A \otimes_{\theta} \overline{X})$ for computing the algebras $\text{Cotor}_A(A, \mathbb{Z}/l)$.

The case $(G, p) = (PU(3), 3)$.

$W' = A \square_A (A \otimes_{\theta} \overline{X}) = \mathbb{Z}/3[x_2]/(x_2^3) \otimes \Lambda(x_1, x_3) \otimes \mathbb{Z}/3\{a_2, a_3, c_5, b_4\}/I$,

$$db_4 = -a_2a_3, \quad dc_5 = a_3^2, \quad d(x_3) = x_2 \otimes a_2 + x_1 \otimes a_3,$$

$$a_3 \cdot x_3 = -x_3 \otimes a_3 + x_1 \otimes c_5.$$

Therefore, we have, as a $\text{Cotor}_{H^*(PU(3); \mathbb{Z}/3)}(\mathbb{Z}/3, \mathbb{Z}/3)$-module,

$$\text{Cotor}_{H^*(PU(3); \mathbb{Z}/3)}(H^*(PU(3); \mathbb{Z}/3), \mathbb{Z}/3) \cong \{\mathbb{Z}/3[x_2]/(x_2^3) \otimes \Lambda(x_1) \otimes \mathbb{Z}/3\{y_2, y_3, y_7, y_8, y_{12}\}/(y_2y_3, y_3^2, y_2y_7, y_7^2, y_2y_8 + y_3y_7)
\oplus x_3 \cdot (x_1x_2, x_1y_7, x_1y_8 + x_2y_7, x_2y_2, y_3)}/(x_2y_2 + x_1y_3).$$
The case $(G, p) = (F_4, 3)$.

\[ W' = \bigodot_A(A \otimes_\theta X) = \mathbb{Z}/3[x_8]/(x_8^3) \otimes \Lambda(x_3, x_7, x_{11}, x_{15}) \otimes \mathbb{Z}/3\{a_4, a_8, a_9, b_{12}, b_{16}, c_{17}\}/I, \]

\[ d(x_j) = x_8 \otimes a_{j-8+1} + x_{j-8} \otimes a_9 \quad (j = 11, 15), \]

\[ d|_{\mathbb{Z}/3}\{\} / I = \text{the ordinary differential on } \mathbb{Z}/3\{\} / I, \]

\[ a_9 \cdot x_j = -x_j \otimes a_9 + x_{j-8} \otimes c_{17} \quad (j = 11, 15). \]

The case $(G, p) = (E_6, 3)$.

\[ W' = \bigodot_A(A \otimes_\theta X) = \mathbb{Z}/3[x_8]/(x_8^3) \otimes \Lambda(x_3, x_7, x_9, x_{11}, x_{15}, x_{17}) \otimes \mathbb{Z}/3\{a_4, a_8, a_9, a_{10}, b_{12}, b_{16}, b_{18}, c_{17}\}/I, \]

\[ d(x_j) = x_8 \otimes a_{j-8+1} + x_{j-8} \otimes a_9 \quad (j = 11, 15, 17), \]

\[ d|_{\mathbb{Z}/3}\{\} / I = \text{the ordinary differential on } \mathbb{Z}/3\{\} / I, \]

\[ a_9 \cdot x_j = -x_j \otimes a_9 + x_{j-8} \otimes c_{17} \quad (j = 11, 15, 17). \]

The case $(E_7, 3)$.

\[ W' = \bigodot_A(A \otimes_\theta X) = \mathbb{Z}/3[x_8, x_{20}]/(x_8^3, x_{20}^3) \otimes \Lambda(x_3, x_7, x_{15}, x_{19}, x_{27}, x_{35}) \otimes \mathbb{Z}/3\{a_4, a_8, a_9, a_{20}, a_{21}, b_{16}, b_{28}, c_{17}, e_{36}\}/I, \]

\[ d(x_j) = x_8 \otimes a_{j-8+1} + x_{j-8} \otimes a_9 \quad (j = 11, 15, 27), \]

\[ d(x_{35}) = x_8 \otimes b_{28} + x_{27} \otimes a_9 - x_8^2 \otimes a_{20} + x_{19} \otimes c_{17}, \]

\[ d|_{\mathbb{Z}/3}\{\} / I = \text{the ordinary differential on } \mathbb{Z}/3\{\} / I, \]

\[ a_9 \cdot x_j = -x_j \otimes a_9 + x_{j-8} \otimes c_{17} \quad (j = 11, 15, 27, 35). \]

The case $(E_8, 3)$.

\[ W' = \bigodot_A(A \otimes_\theta X) = \mathbb{Z}/3[x_8, x_{20}]/(x_8^3, x_{20}^3) \otimes \Lambda(x_3, x_7, x_{15}, x_{19}, x_{27}, x_{35}, x_{39}, x_{47}) \otimes \mathbb{Z}/3\{a_4, a_8, a_9, a_{20}, a_{21}, c_{17}, c_{41}, b_{16}, b_{40}, d_{28}, e_{36}, e_{48}\}/I, \]

\[ d(x_{15}) = x_8 \otimes a_8 + x_7 \otimes a_9, \quad d(x_{39}) = x_{20} \otimes a_{20} + x_{19} \otimes a_{21}, \]

\[ d(x_{27}) = x_8 \otimes a_{20} + x_{19} \otimes a_9 + x_{20} \otimes a_8 + x_7 \otimes a_{21}, \]
\[ d(x_{35}) = x_{8} \otimes d_{28} + x_{27} \otimes a_{9} - x_{8}^{2} \otimes a_{20} + x_{19} \otimes c_{17} + x_{20} \otimes b_{16} \\
+ x_{15} \otimes a_{21} + x_{20} x_{8} \otimes a_{8}, \\
\]
\[ d(x_{47}) = x_{8} \otimes b_{40} + x_{39} \otimes a_{8} + x_{20} \otimes d_{28} + x_{27} \otimes a_{21} + x_{7} \otimes c_{41} \\
- x_{20}^{2} \otimes a_{8} + x_{20} x_{8} \otimes a_{20}, \\
\]
\[ d|_{\mathbb{Z}/3\{} I = \text{the ordinary differential on } \mathbb{Z}/3\{} I/\]
\[ a_{9} \cdot x_{15} = -x_{15} \otimes a_{9} + x_{7} \otimes c_{17}, \quad a_{21} \cdot x_{39} = -x_{39} \otimes a_{21} + x_{19} \otimes c_{41}, \\
\]
\[ a_{9} \cdot x_{27} = -x_{27} \otimes a_{9} + x_{19} \otimes c_{17}, \quad a_{21} \cdot x_{27} = -x_{27} \otimes a_{21} + x_{7} \otimes c_{41}, \\
\]
\[ a_{9} \cdot x_{35} = -x_{35} \otimes a_{9} + x_{27} \otimes c_{17}, \quad a_{21} \cdot x_{35} = -x_{35} \otimes a_{21} + x_{15} \otimes c_{41}, \\
\]
\[ a_{9} \cdot x_{47} = -x_{47} \otimes a_{9} + x_{39} \otimes c_{17}, \quad a_{21} \cdot x_{47} = -x_{47} \otimes a_{21} + x_{27} \otimes c_{41}. \\
\]
The differential operator \( d \) and the bracket \([ , ]\) are trivial on the generators if they are not indicated above.

References


