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$p$-good groups and $P$-good modules

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This is a preliminary report of my joint work with Professors T.Wada, M.Murai and A.Hanaki. Let us fix our notation.

$G$ = a finite group
$k$ = an algebraically closed field of characteristic $p > 0$
$P_G(S)$ = the projective cover of a simple $kG$-module $S$
$1_G$ = the trivial $kG$-module
$P(G) = P_G(1_G)$, the projective cover of $1_G$

**Definition 1.** A finite group $G$ is called $p$-good if for every simple $kG$-module $S$ the following condition holds;

$$P(G) \otimes S \cong m_S P_G(S)$$

with some positive integer $m_S$.

We are interested in group structures of $p$-good groups. If $G$ is a $p$-group or a $p'$-group, then $G$ is clearly $p$-good. $G$ is $p$-good if and only if $G/O_p(G)$ is $p$-good. $S_4$, symmetric group of degree 4, is 2-good but not 3-good. We have two fundamental questions on group structures of $p$-good groups.

**Question 1.** If $G$ is $p$-good, then is it true that $G$ is $p$-solvable?

**Question 2.** Let $G$ be $p$-solvable and $p$-good. Is it true that the $p$-length of $G$ is bounded?

Now we will define good $kG$-modules.

**Definition 2.** Let $M$ be a $kG$-module and $P$ a projective $kG$-module. We call $M$ $P$-good if a simple $kG$-module $S$ is a composition factor of $M$ whenever $P_G(S)$ is a direct factor of $P \otimes M$. $M$ is called good if $M$ is $P(G)$-good.
By definition, $M$ is $P_1 \oplus P_2$-good if and only if $M$ is $P_i$-good ($i = 1, 2$). Note that the following statements are equivalent to each other:

1. $G$ is $p$-good,
2. $M$ is good for every (finitely generated) $kG$-module $M$,
3. $S$ is good for every simple $kG$-module $S$.

The following lemma is well-known.

**Lemma 1.** Let $H \triangleleft G$, $S$ be a simple $kG$-module and $X$ a simple $kH$-module. Then $X$ is a composition factor of $S_H$ if and only if $S$ is a composition factor of $X^G$. Here $S_H$ is the restriction of $S$ to $H$, and $X^G$ is the induction of $X$ to $G$.

Using Lemma 1 we can prove the following propositions.

**Proposition 1.** Let $H \triangleleft G$, $M$ be a $kG$-module and $P$ a projective $kG$-module. If $M$ is $P$-good, then $M_H$ is $P_H$-good.

**Proposition 2.** Let $H \triangleleft G$, $N$ be a $kH$-module and $Q$ a projective $kH$-module. If $N$ is $Q^x$-good for all $x \in G$, then $N^G$ is $Q^G$-good.

Since $P(H)$ is a direct summand of $P(G)_H$ and $P(G)$ is a direct summand of $P(H)^G$, we have the following

**Corollary 3.** Let $H \triangleleft G$, $M$ be a $kG$-module and $N$ a $kH$-module.

1. If $M$ is good, then $M_H$ is good.
2. If $N$ is good, then $N^G$ is good.

When the index $|G : H|$ is a power of $p$, the converse of Proposition 1 also holds.

**Proposition 4.** Let $H$ be a normal subgroup of $G$ with $p$-power index. Let $M$ be a $kG$-module and $P$ a projective $kG$-module. Then $M$ is $P$-good, if and only if $M_H$ is $P_H$-good.

**Corollary 5.** Let $H$ be a normal subgroup of $G$ with $p$-power index. Let $M$ be a $kG$-module. Then $M$ is good, if and only if $M_H$ is good.

Corollary 5 yields the following two results on $p$-good groups.

**Corollary 6.** Let $H$ be a normal subgroup of $G$ with $p$-power index. If $H$ is $p$-good then $G$ is $p$-good.
Corollary 7. If $G$ is a $p$-solvable group with $G = O_{p,p',p}(G)$, then $G$ is $p$-good.

Question 3. Does the converse of Corollary 6 hold?

Question 4. When $G$ is $p$-good and $p$-solvable with $p$-length 1, describe the group structure of $G$. Recall that $S_4$ is 3-solvable with 3-length 1, but not 3-good.

For $p$-solvable groups $G$, we have the following criterion for a simple $kG$-module $S$ to be good.

Proposition 8. Let $G$ be a $p$-solvable group with a $p$-complement $L$. Let $S$ be a simple $kG$-module. Then the following inequality holds;

$$\dim_k \text{End}_k (S_L) \leq (\dim_k S)_p.$$ 

Moreover, the following statements are equivalent;

(1) $S$ is good,
(2) $\dim_k \text{End}_k (S_L) = (\dim_k S)_p$,
(3) $\text{Hom}_k (S_L, T_L) = 0$ for every simple $kG$-module $T$ with $T \not\cong S$.

Corollary 9. Let $G$ be a $p$-solvable group. Then the following statements are equivalent;

(1) $G$ is $p$-good,
(2) $\dim_k \text{End}_k (S_L) = (\dim_k S)_p$ for every simple $kG$-module $S$,
(3) $\text{Hom}_k (S_L, T_L) = 0$ for all simple $kG$-modules $S, T$ with $S \not\cong T$. 