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ランダム行列理論と金属・非金属転移の問題

(Random Matrix Theories and the Problem of Metal-Insulator Transition)

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内容のアウトライン

「金属・非金属転移」の問題(より狭い視点では「アンダーソン局在」の問題)は固体物理の領域の一つの中心課題として研究が盛んであるが、90年代に入って特に「量子準位統計」の角度からの研究が進展している。アンダーソン局在とは金属-非金属転移において発生する電子の局在化現象であり、準位統計の立場でみれば、それはWigner-Dyson統計-Poisson統計の間の転移と考えることができるので、一般的な「中間準位統計」のなかでも相転移の問題がからんだ一つの重要なテーマと考えられるようになっている。この分野での最近10年間における発展で見逃すことが出来ないのはB.L.Al'tshulerとその協力者らが行った仕事であろう。その一つの到達点はmobility edgeにおける「準位圧縮率の結果」である。それは長さSのエネルギー区間に含まれる 固有値の「数分散」に関する性質で、情報理論・統計力学から見ても興味深いものがあるが、その視点から十分満足できる理論を提示するのがわれわれの目的である。(われわれの目標は、局在性」を示す準位相関のexponential decayがどのように導かれるかを見極めることであるが、今回、そこまで示すことはできなかったので、上のようなタイトルを採用した。) (1999年5月31日)

Part I

Information Theoretical Basis of Random Matrix Distributions (to appear in *Journal of Mathematical Physics*)

Part II

Long-Range Level Statistics Characterizing Metal-Insulator Transition (by H. Hasegawa, B. Hu, B. Li, and J-Z. Ma at Hong Kong Baptist University: preprint HKBU-CNS-9815).

Part III

Supplement to HKBU-CNS-9815(Long-Range Level Statistics... (private communications to colleagues at HKBU)

Part I

INFORMATION THEORETICAL BASIS OF RANDOM MATRIX DISTRIBUTIONS

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Abstract. A general expression of N joint level distribution used in random matrix theory,

$$P(x_1, x_2, .., x_N) = C_{N,\beta}\mathrm{exp}\left[-\beta\left(\sum_{j<k}\phi(x_j - x_k) + \sum_j V(x_j)\right)\right] \qquad \beta = 1, 2 \text{ and } 4,$$

is examined along Balian's axiomatic strategy, namely, (A) $P(\{x_j\})\prod_{j=1}^N dx_j = invariant$ under a specified class of unitary transformations on the basis of metric on matrix spaces, and (B) $P(\{x_j\})$ satisfies a maximum entropy principle under two sorts of constraint, i.e. a *geometric constraint* and a *level-density constraint*. An analogy to constructing a canonical equilibrium state is employed for the so-called Hamiltonian level-dynamical system. In this way, it is shown that the most general joint distribution must be of the above form with a possible pair-potential function ϕ in a 2-dimensional space:

$$\phi(\mathrm{r}) = \frac{1}{4}\mathrm{log}\left(1 + 2\left(\frac{a}{r}\right)^2 \mathrm{cos}2\theta + \left(\frac{a}{r}\right)^4\right), \text{ parametrized by } a > 0 \text{ and } \theta;\ 0 \leq \theta < \pi/2.$$

It excludes the possibility of many body interaction higher than the pair. A physical significance of this description is discussed with an application to metal-insulator transition in mind.

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1. Introduction

The standard form of N joint level distribution for the so-called Gaussian matrix ensembles[1](β=1 for GOE, 2 for GUE, and 4 for GSE) is expressed as follows:

$$P_G(H)dH = Ce^{-\frac{1}{2\sigma^2}\mathrm{Tr}H^2}\prod_{\alpha,\nu} dH_\alpha^{(\nu)} \ \alpha = (m \leq n) \text{ and } \nu \leq \beta (\text{to specify } \beta \text{ fold degeneracy}). \tag{1.1}$$

The quantum level statistics that uses distribution (1.1) will be called Wigner-Dyson statistics, and it is characterized by the short range repulsion in the pair-potential

$$\phi_{WD}(r) = -\log r \qquad (e^{-\beta\phi(r)} = r^{\beta}). \tag{1.2}$$

A simple logic to deduce (1.1) and (1.2) is provided by a maximum entropy principle stated as follows. Let any $N \times N$ hermitian matrix be expressed as a linear combination of matrix units (e_{mn}) $H = \sum_{m,n} H_{mn} e_{mn}$ so that a distribution P over $N \times N$ hermitians may be specified by $P(\{H_{mn}\})$. Then,

among all possible distributions $P(\{H_{mn}\})$ possessing 1st and 2nd moments(this set of P being denoted by $\mathcal{E}$), distribution P_G (1.1) is the unique one that satisfies

A. *unitary invariance* $P(\{(U^*HU)_{mn}\}) = P(\{H_{mn}\})(U \in \text{invariant unitarygroup}\, G_{\beta})$.

B. *maximum entropy principle* $\max_{P\in\mathcal{E}} h(P) = h(P_G)$
under constraint

$$\langle H_{mn}\rangle_P = 0 \qquad \langle H_{nn}^2\rangle_P = 2\langle |H_{mn}|^2\rangle_P = \sigma^2, \tag{1.3a}$$

$$\langle H_{mn}H_{rs}\rangle_P = 0 \ (mn) \neq (rs), \tag{1.3b}$$

where $\langle\,\cdot\,\rangle_P$ *denotes an average over distribution* P, *and*

$$h(P) \equiv \int -P(\{H_{mn}\})\log P(\{H_{mn}\}) \prod dH_{mn} (= \langle -\log P\rangle_P) \tag{1.4}$$

(entropy functional of P).

Once distribution (1.1) is so constructed, the repulsion (1.2) can be seen to arise from a change of variables $(H_{mn}) \to (x_j)$ (N eigenvalues of H) and the other cyclic variables not entering the Gaussian exponent of (1.1) so that

$$\prod_{m\leq n} dH_{mn} \propto \prod_{j<k} |x_j - x_k|^{\beta}. \tag{1.5}$$

It is remarkable that the special constraint (1.3b) expresses *statistical independence* between any different matrix units, implying that a correlation between different eigenvalues arises totally from the repulsion facter (1.5), i.e. from a purely geometrical origin.

Balian's paper in 1968[2], aiming to extract the above geometrical aspect of random matrices, proposed summarizing postulates (A) and (B) as two guiding prescriptions for construction of a more general form of distributions:

(A) $ds^2 = \mathrm{Tr}(dMdM^*)$ (metric between two matrices M and $M + dM$) that ensures the unitary invariance

(B) for a hermitian $M = H$, $\quad I\{P[H]\} \equiv \int d[H]P[H]\log P[H](= -h(P))$, and $\min_{P\in\mathcal{E}} I\{P[H]\}$ under constraint $\langle f_x\rangle_P \equiv \int d[H]P[H]f_x[H] = C_x$

(typically, $f_x[H] = \mathrm{Tr}\delta(x - H)$ for a given level density $C_x = \rho(x)$)

to get P_m so that $\min_{P\in\mathcal{E}} I\{P[H]\} = I\{P_m[H]\}$.

In the present paper, we aim to find out a most general form of P_m by performing the above program, in particular, by specifying more detailed conditions on the Riemannian geometry of matrix spaces, following the recent work by Petz[3], to clarify the actual context of (A).

2. Possible Riemannian Metrics and Gaussian Distributions on Random Matrix Spaces

2.1. Unitary Covariant Bilinear Form

We introduce a Riemannian metric into the space of matrices according to Balian's postulate (A) concerning the distance between two infinitesimally separated matrices. A Riemannian metric tensor $(g_{\mu\nu})$ can then be defined as the coefficient tensor of the distance ds^2 with respect to a quadratic form of an infinitesimal parameter set, or of a velocity vector called *tangent vector*. Let us denote, following Petz[3], the space of $N \times N$ complex matrices by $\mathcal{M}_N$ on which a sesqui-linear form $\mathrm{K}(B, A)$(linear with respect to A and anti-linear to B; $A, B \in \mathcal{M}_N$) is defined. The Hilbert-Schmidt inner product defined by $\mathrm{K}_{H-S}(B, A) \equiv \mathrm{Tr} B^* A$ gives a simple example that satisfies the unitary invariance, namely

$$\mathrm{K}(U^* BU, U^* AU) = \mathrm{K}(B, A). \tag{2.1}$$

Here, we seek a more general class of sesqui-linear form K, not satisfying the unitary invariance, but still yields a useful tool for our purpose: we need a Gaussian distribution on $\mathcal{M}_N$ whose quadratic variables in the exponential play a role of heat reservoir(called a reservoir variable) against the system we are interested in(called an object variable), and after disposing the reservoir variables by integrating them out the result may recover the desired strict invariance(for a detail, see [4]). We shall show that such a situation may arise for a class of those $\mathrm{K}'s$ which depend on another hermitian matrix H representing the system of interest, and which satisfy the property of *unitary covariance* (the unitary invariance of $A, B, \mathrm{and} H$ all together). It is desirable to classify such inner products under a system of axioms. Denoting the set of all hermitian matrices in $\mathcal{M}_N$ by $\mathcal{M}_N^s$, we list up the properties of the expected K-form as follows.

(a)**symmetry** $\mathrm{K}_H(A^*, B^*) = \mathrm{K}_H(B, A)$; $\quad H \in \mathcal{M}_N^s, \quad A, B \in \mathcal{M}_N$. When A and B are restricted to hermitians, the form K becomes real and symmetric, and hence it is a bilinear form. [1]

(b)**positive definiteness** $\mathrm{K}_H(A, A) \geq 0$, $\quad$ and the equality holds only when $\quad A = 0$.

(c)**continuity of the map** $H \mapsto \mathrm{K}_H$: the continuity holds for every A in $\mathrm{K}_H(A, A)$.

(d′)**unitary covariance** $\mathrm{K}_{U^*HU}(U^* BU, U^* AU) = \mathrm{K}_H(B, A)$: this relaxes the condition of unitary invariance in the strict sense to the same condition but with an inclusion of the subsideary matrix H, and hence the bilinear form K_H belongs to much wider class than the Hilbert-Schmidt inner product.

This last condition (d′) is essential in the present context, and actually is weaker than the condition (d) below of *monotonicity* which Petz proposed, setting it up for a density matrix D that is more restricted than just a hermitian H. (A density matrix D in $\mathcal{M}_N$ is a special hermitian matrix, positive and $\mathrm{Tr} D = 1$.)

(d)**monotonicity** $\mathrm{K}_{T(D)}(T(A), T(A)) \leq \mathrm{K}_D(A, A)$, where T, a super-operator(a linear map) $\mathcal{M}_n \mapsto \mathcal{M}_m$, in which a positive matrix is mapped to a positive matrix(called *stochastic map*).

1 The K-form with this symmetry is equivalent to Petz's K': $\mathrm{K}'(A, B) = \frac{1}{2}(\mathrm{K}(A, B) + \mathrm{K}(B^*, A^*))$ [3].

An intuitive understanding of the monotonicity of T is that by any coarse-graining of the pertaining matrices in K_D, i.e. both A and D, the metric represented by K_D must be a non-increasing quantity. When T is a unitary map, the above monotonicity inequality becomes the equality, because now T can be an invertible super-operator from $\mathcal{M}_N$ onto itself. Therefore, condition (d) includes (d')((d) is more stringent than (d'): if (d) is valid for a form K, (d') is also valid for the same form, but the converse is not necessarily true).

Condition (d') enables one to take the representation of the pertinent matrices where H is diagonal, and to exhibit the form of K in terms of the matrix elements A_{jk} with $H = \mathrm{diag}(\lambda_1, \lambda_2, .., \lambda_N)$

$$\mathrm{K}_H(A, A) = \sum_{j \leq k} c(\lambda_j, \lambda_k)|A_{jk}|^2 \qquad A \in \mathcal{M}_N^s. \tag{2.2}$$

Petz[3] showed that, under more stringent condition (d) than (d') on $\mathrm{K}_D(A, A)$ with D diagonalized, the real function $c(\lambda, \mu)$ above satisfies that

$$c(\lambda, \mu) = c(\mu, \lambda), \quad c(\lambda, \lambda) = 1/\lambda, \quad c(t\lambda, t\mu) = t^{-1}c(\lambda, \mu). \tag{2.3}$$

Thus, only a single, continuous function $c(x)$ is enough to represent a monotone metric on a matrix space, as far as the dimensionality is finite, which is related to an operator-monotone function [3] to characterize a quantum mechanical Fisher metric[5]. We will seek the same kind of representation of $\mathrm{K}_H(A, A)$ under condition (d'). For this purpose, let us adopt another condition (d''):

(d'')translational invariance with respect to H $\quad \mathrm{K}_{H+aI}(B, A) = \mathrm{K}_H(B, A)$.

It is straightforward to show that, under conditions (d') and (d'') with $H = \mathrm{diag}(\lambda_1, .., \lambda_n)$ of $\mathrm{K}_H(A, A)$ in (2.2), the real function $c(\lambda, \mu)$ satisfies that

$$c(\lambda, \mu) = c(\lambda - \mu) > 0 \quad \lambda \neq \mu \quad \text{and} \quad c(\lambda, \lambda) = (\text{independent of} \lambda) \geq 0. \tag{2.4}$$

We have just obtained a Riemannian metric form $g_{\mu\nu}v^\mu v^\nu$ with metric tensor $g_{\mu\nu}$ and a tangent vector v^μ on a matrix space $\mathcal{M}_N$ under conditions (a) $\sim$ (d') and (d''), where the quadratic quantity $|A_{jk}|^2$ indexed by $\frac{1}{2}N(N+1)$ pairs $(j, k)(\equiv \mu)$ represents the square of a tangent vector component.

Remark1. The above formulation of the metric form with complex tangent vector applies directly to the unitary ensemble(UE) with 2 degrees of freedom for each pair (j, k). It also applies to the orthogonal ensemble(OE) by restricting each vector to a real quantity with 1 degree of freedom for each pair, and to the symplectic ensemble(SE) by restricting each vector to a quaternion real [2] with 4 degrees of freedom for each pair. It is also remarked that the metric tensor $g_{\mu\nu}$ here is a diagonal tensor that stems from our choice of *H-diagonalized representation* under the unitary covariance.

2 An $(N \times N)$ quaternion-real matrix Q is defined by the one whose every matrix element is of the form $q = q_0 + \mathbf{q}\tau$ with 3-component quaternion τ and real coefficients $q_i; i = 0, 1, 2, 3$ so that it satisfies the time-reversal symmetry for a symplectic system conditioned by $Q^R = Q^\dagger$ [6] (cf.[1]).

2.2. Complexitized Riemannian Metrics

Here, we discuss a generalization of the above formulation of the real metric by means of complexitizing the c-function: this is because, if we ask ourselves whether the expression (2.2) yields the most general form of physically meaningful, unitary covariant metrics, the answer must be no, since the restriction to a hermitian tangent vector $A \in \mathcal{M}_N^s$ enforces the c-function to be real by virtue of symmetry (a).

If we allow a general vector $A \in \mathcal{M}_N$ under conditions (a) $\sim$ (d′) and (d″) for $\mathrm{K}_H(A, A)$, expression (2.2) should read, with a generally complex function $c(\lambda - \mu)$,

$$\mathrm{K}_H(A, A) = \sum_{j \leq k} \left(c(\lambda_j - \lambda_k) A_{jk} A_{jk}^T + \bar{c}(\lambda_j - \lambda_k) \bar{A}_{jk} A_{jk}^\dagger \right), \tag{2.2'}$$

(A^T and $A^\dagger$ denote the transpose and the hermitian conjugate of A, respectively)

and the positive-definiteness condition (b) requires

$$Re\ c(\lambda - \mu) > 0. \tag{2.4'}$$

The argument applies in its form to UE, also to SE by pairing two components of the four arising from a product of the two quaternions in a given site (j, k) where the reality of the components is removed, leading us to 2-sets of independent expressions of the form(2.2′). For OE, we do not use (2.2′) directly, but discard one of the two terms there, and by rewriting $c(\cdot) = |c(\cdot)|e^{i\psi}$, we absorb the factor $e^{i\psi}$ into the tangent vector component, which replaces the c-function by its absolute magnitude.

2.3. Maximizing the Entropy for a Gaussian Distribution under Geometric Constraint

A Gaussian distribution in probability theory has a power of information property that the covariance of its variables prescribed tells us that the muximmum of entropies of all probability distributions with a fixed covariance is attaind by that Gaussian distribution[7]. Thus, we may regard a given covariance tensor as the constraint for the maximization problem associated to a (multi-dimensional) Gaussian distribution P_G, and call this a geometric constraint for the present problem.

We aim at a Gaussian-reservoir distribution on the matrix space $\mathcal{M}_N$ by means of the so obtained metric with a $d(=\frac{1}{2}N(N+1))$-dimensional complex tangent vector typically for UE. We adopt a new notation Y_{jk} for a reservoir(r-) variable in a Gaussian exponent, and x_j for an object(o-) variable that replaces λ_j, an eigenvalue of H, and that only enters the metric tensor of the Gaussian exponent. We identify the r-variables (Y_{jk}) to be a *cotangent vector* rather than the tangent, as defined by

$$Y_{j,k} \equiv c(x_j - x_k) A_{jk} \quad j \neq k; \quad Y_{j,j} \equiv 0 \quad (c(0) = 0 \text{ assumed}). \tag{2.5}$$

Then,

$$\mathrm{K}_H(A(Y), A(Y)) = \sum_{j<k} \frac{1}{c(x_j - x_k)} |Y_{j,k}|^2, \tag{2.6}$$

or, more generally,

$$\mathrm{K}_H(A(Y), A(Y)) = \sum_{j<k} \left(\frac{1}{\bar{c}(x_j - x_k)} Y_{jk} Y_{jk}^T + \frac{1}{c(x_j - x_k)} \bar{Y}_{jk} Y_{jk}^\dagger \right), \tag{2.6'}$$

which is put in an exponential for a Gaussian distribution to write

$$P_G(x,Y) = \frac{1}{Z}\exp[-\frac{1}{2}\mathrm{K}_H(A(Y),A(Y))] \qquad Z = \int_{R^{2d}} e^{-\mathrm{K}_H(A(Y),A(Y))/2}dY, \tag{2.7}$$

yielding, in general,

$$\mathrm{mean}(Y) = 0, \quad \mathrm{Cov}(Y,Y) = \mathrm{diag}(..,\bar{c}(x_j - x_k), c(x_j - x_k),..) \tag{2.8}$$

i.e. $\langle Y_{j,k}Y^T_{m,n}\rangle = \bar{c}(x_j - x_k)$, and $c(x_j - x_k)$ for $(j,k) = (m,n)$; $= 0$ for $(j,k) \neq (m,n)$.

Then, on the basis of the maximum entropy principle under constraint (2.8), the resulting Gaussian distribution (2.7) expresses the following properties.

(i) statistical independence of different matrix units

$$\mathrm{for}(j,k) \neq (m,n), \quad P(Y_{j,k}, Y_{m,n}) = P(Y_{j,k}) \cdot P(Y_{m,n}). \tag{2.9}$$

(ii) identical distribution for all the matrix units with off-diagonal type

$\mathrm{Cov}(Y_{j,k}Y_{j,k})$ depends on the pair (j,k) only through $x_j - x_k$ in a common function $c(.)$. (2.10)

2.4. Reduced Probability Distribution

Consequently, the normalization integral Z in (2.7) is simply the product of all the variances $c(x_j - x_k)$, and we can get the reduced probability distribution for the object eigenvalue system in a form

$$P(x_1, x_2, .., x_N) = C_N \prod_{j<k} |c(x_j - x_k)|^{\beta/2}, \tag{2.11}$$

where,

$$C_N = [\int_D \prod_{j<k} |c(x_j - x_k)|^{\beta/2} dx_1..dx_n]^{-1} \qquad \beta = 1,\ 2 \quad \text{and } 4, \tag{2.12}$$

the integer β being the multiplicity of the components of each cotangent vector $Y_{jk}(j \neq k)$ i. e. $\beta = 1$ for OE, $\beta = 2$ for UE, and $\beta = 4$ for SE. Also, by regarding this index β as a continuous parameter of *inverse temperature*, and apart from the pure numerical factor $\log(2\pi e)^{d\beta/2}$ to change merely the normalization factor, we can write the distribution of N joint eigenvalue distribution in terms of the sum of pair potentials as follows.

$$P(x_1, x_2, .., x_N) = C_{N\beta} \prod_{j<k} \exp\left[-\beta\left(\sum_{j<k}\phi(x_j - x_k)\right)\right], \tag{2.13}$$

where

$$\phi(r) = \frac{1}{2}\log|c(r)| = \frac{1}{2}Re\ \log c(r) \quad \text{if } c(r) \text{ is complex.} \tag{2.14}$$

This shows that level interactions are limited to a sum of pair potentials under our axioms (a),(b),(c),(d′) and (d″). At present, we assume an analogy to hold to statistical mechanics of gases, postponing a detailed specification of the potential function $\phi(r)$ to Sec.3.

2.5. Maximizing the Entropy for the Eigenvalue Distribution under Level-Density Constraint

An important application which Balian clarified to establish in the 1968 paper[2] was to find a scheme of obtaining a matrix eigenvalue distribution so as to satisfy an agreement of the single-level density deduced from it with a given, or observed level density by means of maximizing entropy, where the identification between the deduced and observed densities is expressed as a constraint. His treatment, which was specialized to the standard form of the geometric factor (1.5) of Wigner-Dyson, is entirely applicable to the foregoing geometry of more general type, which is presented here.

A prototype scheme of maximum entropy principle in classical statistics[5] is summarized: *Let $C_1, C_2, .., C_n$ be a set of observables of our object system ($C_i = C_i(\{\xi\})$; a function of o-variables), and a (repeated) measurement of them is supposed to show, with a probability measure μ multiplied by a hypothetized distribution $P, \int P d\mu = 1$,*

$$\langle C_i \rangle_P = \eta_i \qquad i = 1, 2, .., n. \tag{2.15}$$

A maximizing the entropy $\langle -\log P \rangle_P$ of the distribution P under constraint(2.15) yields the most unbiased distribution called exponential family given by

$$P = \exp[\theta_i C_i - \psi(\{\theta_i\})] \qquad \psi(\{\theta_i\}) = \log \int \exp[\theta_i C_i] d\mu \tag{2.16}$$

in terms of the Lagrange multiplier θ_i's.

There exists one-to-one correspondence between parameter set $\{\eta_i\}$ and $\{\theta_i\}$, and under the satisfaction of so-called *potential condition* $\frac{\partial \theta_i}{\partial \eta_j} = \frac{\partial \theta_j}{\partial \eta_i}$, a covariance to express fluctuations of the measurement(2.15) is expressed as

$$\langle (C_i - \langle C_i \rangle_P)(C_j - \langle C_j \rangle_P) \rangle_P = \frac{\partial \eta_i}{\partial \theta_j} \left(= \frac{\partial^2 \psi}{\partial \theta_i \partial \theta_j}\right) \tag{2.17}$$

that is called Fisher metric associated to the measurement whose outcome is (2.15). This is shown to yield the minimum of all covariances for any observables $\{\hat{C}_i\}$ satisfying $\langle \hat{C}_i \rangle_P = \eta_i$ (the so called Cramér-Rao bound[5]).

The above stated scheme is now applied to the eigenvalue distribution presented in (2.13) by associating the set of observables $\{C_i\}$ to the level-density observable $\rho(x)$:

$$\rho(x) = \sum_i \delta(x - x_i) = \mathrm{Tr}\delta(x - H), \tag{2.18}$$

where the free continuous parameter x plays the role of index i in(2.15) that is assumed to be discrete there. The corresponding Lagrange multiplier is denoted by $V(x)$ so that the exponential family may be written as

$$\exp[-\beta \int V(x)\rho(x)dx + \beta\psi(V)] = \exp\left[-\beta\left(\mathrm{Tr}V(H) - \psi(V)\right)\right] \quad \text{(satisfying invariance)}$$

which is multiplied by (2.13) as the coefficient of the starting measure μ to get

$$P(x_1, .., x_N) = C_{N,\beta}\exp[\left[-\beta\left(\sum_{j<k}\phi(x_j - x_k) + \sum_j V(x_j)\right)\right]. \tag{2.19}$$

A usefulness of the argument is that it provides a concise basis, from a viewpoint of statistics (parameter estimation theory), of *functional derivative method* developed by Beenakker[8] and used frequently for discussions of 2-point correlation functions for nuclei, mesoscopic systems and quntum transport, quantum chaos and so on[9]. Namely, the Fisher metric (2.17), when applied to the level-density function $\rho(x)$ (2.18), represents just the 2-point density correlation function in random matrix theories so that expression (2.17) offers Beenakker's basic functional derivative

$$\frac{\delta\langle\rho(x)\rangle}{\delta V(x')}\left(=\frac{\delta\langle\rho(x')\rangle}{\delta V(x)}\right)= -\beta\left(\langle\rho(x)\rho(x')\rangle-\langle\rho(x)\rangle\langle\rho(x')\rangle\right). \tag{2.20}$$

We shall come back to an issue about 2-point correlation functions in Section 4, after establishing the precise form of the pair potetial in (2.19).

3. Canonical Equilibrium States of Hamiltonian Level Dynamical Systems

In a previous paper[10], we have treated two types of Hamiltonian level dynamics, generalized Calogero-Moser and generalized Calogero-Sutherland systems. Here, we only use the former system whose Hamiltonian is given by

$$\mathcal{H}_{gCM}=\frac{1}{2}\sum_j p_j^2+\frac{1}{2}\sum_{j\neq k}\frac{\|f_{jk}\|^2}{(x_j-x_k)^2} \tag{3.1}$$

in terms of N-canonical conjugate variables $(x_j,p_j)_{j=1}^N$ and $d\beta(d=N(N-1)/2,\beta=1,2\text{and}4)$ multi-dimensional angular-momentum variables $(f_{j<k})$: these satisfy the following three sets of Poisson bracket relations. Namely,

$$\{x_j,p_k\}=\delta_{jk};\qquad \{x_j,x_k\}=\{p_j,p_k\}=0, \tag{3.2a}$$

$$\{f_{jk}^{(\mu)},f_{rs}^{(\nu)}\}=-\sum_{pqc}c_{jk\mu,rs\nu}^{pq\lambda}f_{pq}^{(\lambda)}, \tag{3.2b}$$

(c's represent structure constants of the underlying Lie algebra) [3]

and

$$\{x_j,f_{rs}\}=\{p_j,f_{rs}\}=0\qquad \text{(separation of } o \text{ and } r \text{ variables)}. \tag{3.2c}$$

Superscript $\mu,\nu,\lambda.$ denotes the 2-components of a complex number i.e. *real* and *imaginary* part for UE and the 4-components of a quaternion for SE, respectively, and

$$\|f_{jk}\|^2=\sum_{\nu=1}^{\beta}|f_{jk}^{(\nu)}|^2. \tag{3.2d}$$

These angular momentum variables, present in the Hamiltonian (3.1), are essential ingredient playing the role of the Gaussian-reservoir variables in Sec.2. It is well known in mechanics that an angular moentum vector arises as the conjugate variable to an angular velocity vector, and that is a cotangent vector versus the latter tangent vector as regards

3 For OE where the β-fine structure is absent, the relation is given explicitly by $\{f_{jk},f_{rs}\}=(1/2)(\delta_{js}f_{rk}+\delta_{jr}f_{ks}+\delta_{ks}f_{jr}+\delta_{kr}f_{sj})$. The relations for UE and SE are discussed in [11].

the pertinent Riemannian metric form that corresponds to (2.6), or more generally to (2.6′).

We have used in [10] a canonical equilibrium distribution of the g-CM system with Hamiltonian (3.1) to write a Gaussian distribution of the form

$$P_G = \frac{1}{Z_{N,\beta}} \exp[-\beta \mathcal{H}_{gCM} - \gamma \mathcal{Q}], \tag{3.3}$$

where

$$\mathcal{Q} \equiv \frac{1}{2} \sum_{j<k} ||f_{jk}||^2 \quad \text{square of angular momentum vector,} \tag{3.4}$$

and β and γ are real constants(β here is different from the one usd for the 3-symmetry class). Then, the form in the exponential, $\beta\mathcal{H}_{gCM} + \gamma\mathcal{Q}$, provides a typical metric form (2.6) in terms of the two cotangent vectors, (p_j) and (f_{jk}) with a real c-function. We may remark that the choice of the linear combination of $\mathcal{H}_{gCM}$ and $\mathcal{Q}$ is necessitated because these provide the only two constants of motion of the gCM system written in the metric form of the angular momentum vector[12]. However, the choice of two coëfficients, β and γ to be real and positive, appears to be too restrictive: more precisely, a real, positive β is necessitated for the reason of the variance relation

$$\langle p_j^2 \rangle_{P_G} = \beta^{-1}, \tag{3.5}$$

but another positivity of the variance relation involving γ must be different from the positivity of γ. Hence, let us allow the constant γ a generally complex number to write a possible variance function $c(r)$ to be put in (2.8). This can be written in accordance with Sec.2.2 as

$$c(r) = (1 + \frac{\hat{a}^2}{r^2})^{-1} \qquad \hat{a}^2 \equiv \frac{\beta}{\gamma}, \qquad Re\, c(r) \geq 0 \text{ ensured by} \beta > 0. \tag{3.6}$$

(A non-zero complex constant is absorbed to the normalization factor, $Z_{N,\beta}$).

Writing $\gamma = |\gamma| e^{i2\theta}$, we are now led to the most general form of the potential function in (2.13), $\phi(r)(= \phi(r; a, \theta)) = \frac{1}{2}\log|c(r)|$ parametrized by a and θ:

$$\phi(r) = \frac{1}{4}\log\left(1 + 2\left(\frac{a}{r}\right)^2 \cos 2\theta + \left(\frac{a}{r}\right)^4\right) \quad \hat{a} = a^{-i\theta}, a > 0, \text{ and } 0 \leq \theta < \pi/2. \tag{3.7}$$

The specification of the pair potential (3.7) in the Gibbs type distribution (2.18) now provides us with a concrete framework of equilibrium statistical mechanics to treat quantum level statistics. Here, we show some feature of the potential function $\phi(r)$.

(1)short- and long range properties. For $0 < r \ll a$, the inverse quartic term in logarithm dominates to yield $\phi(r) \to \phi_{WD}(r) = -\log r + \text{const.}(1.2)$ irrespective of θ, whereas for $r \to \infty$, $\phi(r) \to \frac{1}{2r^2}a^2\cos 2\theta$, the universal inverse square decay, but from positive or negative side depending on θ.

(2)long-range attractiveness for $\pi/4 < \theta < \pi/2$. Under this circumstace, the potential function $\phi(r)$ has a unique minimum in a positive finite range of r at $r_m = a/\sqrt{-\cos 2\theta}$, and the attractive range is specified by

$$\frac{1}{\sqrt{2}} r_m < r < \infty. \tag{3.8}$$

(3)Fourier transform of $\phi(r)$(*see*Appendix).– regularity and stability of $\phi(r)$.

$$\mathcal{F}_\phi(k) \equiv \int_{-\infty}^{\infty} \phi(r)e^{ikr}dr \text{ exists } = \frac{\pi(1-e^{-a|k|\cos\theta}\cos(kr\sin\theta))}{|k|} > 0 \quad -\infty < k < \infty. \tag{3.9}$$

This together with property1 shows that $\int_0^\infty |1-e^{-\beta\phi(r)}|dr < \infty$ (regularity), and that $\sum_{j,k}\phi(x_j - x_k) \geq -nB, B \geq 0$ for any n variables $x_1, .., x_n$ (stability)[13].(The positivity of $\mathcal{F}_\phi$ ensures that $\phi(r)$ can be represented as a sum of a positive function and a function of positive type– the Fourier transform of a bounded positive function, which admits the latter inequality.) These two properties provide an analytic method of treating the present level gas, in particular, the assurance of *thermodynamic limit*[13].

4. On 2-Point Correlation Functions for Level Statistics

The present work has been motivated by several recent papers [14],[15](and references therein) which seem to converge to an idea that in a metallic state a pair of energy levels, repelling to each other by Wigner-Dyson repulsion (1.2) when short-ranged, are in fact subject to a long range attractive force that is evidenced by studies of a pertinent 2-point density correlation function. As a last topic of the present paper, we argue this point rather briefly leaving our detailed report elsewhere.

Let us denote the quantity $\langle\rho(x)\rho(x')\rangle - \langle\rho(x)\rangle\langle\rho(x')\rangle$ in (2.20) by $K(r)$, where the function K is supposed to depend on the single variable $r \equiv x - x'$. This supposition can be regarded as legitimate, when the one level potential $V(x)$ in (2.19) is weak for a given density $\rho(x)$ so that Beenakker's functional derivative is treated by perturbation:

$$\rho(x) = -\frac{1}{\beta}\int_{-\infty}^{\infty} K(x,x')V(x')dx', \qquad K(x,x') = K(x-x') \text{ independent of } V. \tag{4.1}$$

On the other hand, the relation between the one level potential V and the one level density ρ via an integral kernel was an important subject in early random matrix theories: for the case of Wigner-Dyson repulsion (1.2) it has been expressed as

$$V(x) = -\int_D \log|x-x'|\rho(x')dx' + \text{const.} \quad (D \text{ represents support of } \rho) \tag{4.2}$$

which can be verified in the limit $N \to \infty$ (for the standard Gaussian statistics (1.1) with the parabollic $V(x)$ and the Wigner semicircle $\rho(x)$, a discussion is given at length in [1]). This led Beenakker to suppose that the validity of the relation (4.2) to hold for any pair potential($\phi(x-x')$ for our case), and to propose a universal relationship between the kernel $K(x-x')$ and the inverse of the potential kernal so that [9]

$$K(r) = \frac{1}{\beta}\phi^{inv}(r), \qquad \text{or, } \mathcal{F}_K(k) = \frac{1}{\beta\mathcal{F}_\phi(k)}. \tag{4.3}$$

Remark 2. There exists another definition of 2-point density correlation function (denoted by $R(x-x')$) used first by Dyson[1]: $K(x-x')$ to denote the variance of $\rho(x)$ in (2.20) includes the self correlation $\delta(x-x')$. Hence, both are related by $K(r) = \delta(r) + R(r) - 1 = \delta(r) - Y(r)$ ($Y(r)$ is called the cluster function).

However, an explicit investigation of the exact spectral form factor (Fourier transform of the 2-point cluster function $Y(r)$), first obtained by Gaudin for UE in case of $\theta = 0$ (see [10]), indicates that Beenakker's identity(4.3) is not generally valid, but limited to a vicinity of the Wigner-Dyson form(2.1). In other words, within this limitation we may have a good approximate formula for the 2-point correlation function by using (3.9). Namely, for $a \gg 1$

$$\mathcal{F}_K(k) = \frac{|k|}{\beta\pi(1 - e^{-a|k|\cos\theta}\cos(kr\sin\theta))} \quad |k|a \leq 2\pi; \; = 0 \; |k|a > 2\pi. \tag{4.4}$$

The usefulness of this formula in contrast to those presented in the literature ([14],[15]) should be emphasized from the standpoint of equilibrium statistical mechanics, which will be demonstrated shortly.

Appendix. Fourier transform of the potential function $\phi(r)$ (3.9)

$$\int_{-\infty}^{\infty} \frac{1}{4} Re[\log(1 + \frac{a^2}{r^2} e^{-i2\theta})] e^{ikr} dr = \frac{\pi}{|k|} \left(1 - e^{-|k|a\cos\theta}\cos(ka\sin\theta)\right), \; 0 \leq \theta < \frac{\pi}{2}. \tag{A1}$$

<u>**derivation**</u> We set $\hat{a} \equiv ae^{-i\theta}$, and show that

$$I = \frac{1}{2}\int_{-\infty}^{\infty} \log(1 + \frac{\hat{a}^2}{r^2}) e^{ikr} dr = \frac{1}{2}\int_{-\infty}^{\infty} \log(1 + \frac{\hat{a}^2}{r^2})\cos(kr) dr = \frac{\pi}{|k|}(1 - e^{-|k|\hat{a}}). \tag{A2}$$

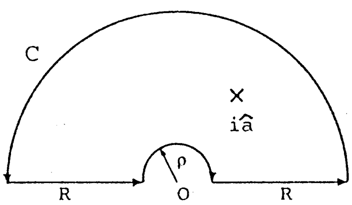
Then, the real part of I yields the desired result $(A1)$. The proof of $(A2)$ is as follows. By an integration by part, we can write

$$I = \int_{-\infty}^{\infty} \frac{\hat{a}^2}{\hat{a}^2 + r^2} \frac{e^{ikr}}{ikr} dr, \tag{A3}$$

which we can perform by means of a contour integration on the complex $r(= z)$-plane:

$$I_C \equiv \frac{1}{2\pi i}\int_C \frac{\hat{a}^2}{\hat{a}^2 + z^2} \frac{e^{ikz}}{kz} dz \qquad (I = 2\pi I_R \text{ in the sense of } \textit{pricipal value}), \tag{A4}$$

where the contour C comprises a large and a small semicircle and two segments on the real axis: $I_C = I_R + I_{\text{semicircle}} + I_{\text{Semicircle}}$ whose radius of the Semicircle and the semicircle are denoted by R and ρ, respectively. Since the only singularity of the complex analytic function of the integrand in $(A4)$ inside C is the simple pole at $z = i\hat{a}$,



$$I_C = Res[z = i\hat{a}](Im[i\hat{a}] > 0) = -\frac{1}{2}\frac{e^{-|k|\hat{a}}}{|k|}, \text{ and} \tag{A5}$$

$$I_R = I_C - I_{\text{semicircle}} - I_{\text{Semicircle}} \longrightarrow \quad I_C + \frac{1}{2}Res[z = 0], \quad \text{as}$$

$$I_{\text{Semicircle}} \longrightarrow 0, \text{ and } I_{\text{semicircle}} \longrightarrow -\frac{1}{2}Res[z = 0] = \frac{1}{2|k|}, \tag{A6}$$

when $R \to \infty$ and $\rho \to 0$, respectively. Multyplying $(A5)$ and $(A6)$ by a factor 2π and adding them up, we get the desired result $(A2)$ and so $(A1)$.

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Part II

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Long-Range Level Statistics Characterizing Metal- Insulator Transition

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Abstract

We study two-level correlation function $X_2(r)$ and spectral number variance $\Sigma^2(L)$ by means of *Gaussian matrix ensemble with preferential basis* (GMEPB) to see its effectiveness on level statistics involving metal-insulator transition. The generalized scheme of GMEPB admits an attractive as well as a repulsive potential between distant pair of levels. The attractiveness is responsible for an "overshoot" of $X_2(r)$ above unity and a non-monotone increase of the $\Sigma^2(L)$ curve that conform to the prediction by another type of correlation function for matrix dynamics $H = H_0 + \lambda H_1$. In contrast, the equilibrium nature of GMEPB captures an *intermediate compressibility* of the level gas, which ensures a static crossover between the metallic and the insulating phases.

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In recent years, there have been considerable efforts in condensed matter physics and random matrix theories (RMT) to formalize the metal-insulator transition phenomena as regards the pertinent electron energy level statistics. These efforts seek a powerful and unified method to generalize the standard Gaussian ensembles initiated by Wigner, Dyson and Mehta (see a comprehensive review on the recent development [1]). Indeed, literatures tell us that a framework exists for computing the two-level correlation function, as a function of $r = x - x'$, of the level density $\rho(x)$:

$$X_2(r) \equiv -\delta(r) + \langle \rho(x)\rho(x') \rangle, \tag{1}$$

$$(\lim_{r \to \infty} X_2(r) = 1, \text{and } \langle \rho \rangle = 1 \text{ assumed}) \tag{2}$$

which depends on an external parameter λ such that $X_2(r; \lambda)$ represents a correlation for a pair of eigenvalues x and x' of a perturbed $N \times N$ hermitian,

$$H = H_0 + \lambda H_1. \tag{3}$$

Here, H_0 and H_1 are assumed to belong to Poisson and Gaussian (typically, unitary) ensemble, respectively. One thus expects the resulting $X_2(r; \lambda)$ to describe properly a transition from the uncorrelated eigenvalue sequence($\lambda = 0$) to that of the full correlation with Wigner-Dyson repulsion ($\lambda = \infty$) continuously. The study was initiated by Leyvraz and Seligman [2] who treated expression (3) as a perturbation of the pure uncorrelated sequence by the weak λ part, and later developed by Guhr [3] for the whole range of this parameter by means of supersymmetry. A characteristic feature of the X_2 function obtained was the so called "overshoot" implying that $X_2(r; \lambda)$, normalized as unity at ∞ as in (2), goes beyond unity peaking in a finite range, the feature already noticed in the perturbation treatment [2]. The latest two papers [4] and [5] have clarified more detailed aspect of this effect on *long-range level statistics* manifest in the number variance curve $\Sigma^2(L)$ (the variance of the number of levels lying in an interval of length L, see [6]) that

(A) this curve exhibits a change of its 2nd derivative from minus to plus at a point denoted by a_0, slightly smaller than λ, that may be called the transition point, and

(B) its asymptote for $L \to \infty$(i.e. $a_0 \ll L$) is a straight line but with coefficient unity corresponding to the Poisson line having a large, positive intersection on the L-axis.

According to a statement by Kunz and Shapiro [4], these two characteristics may be expressed as: **(A)** the inter-level interaction, when represented as a pair potential (denoted by $\phi(r)$ here), must be attractive around the overshooting point a_0 and $a_0 < r \to \infty$, and **(B)** the total area surrounded by the cluster function $Y_2(r)(= 1 - X_2(r))$ [6] on abscissa vanishes due to the precise cancellation of the positive (repulsive) and negative (attractive) parts of the cluster function i.e. $\int_{-\infty}^{\infty} Y_2(r)dr = 0$, which also allows one to express it in terms of the spectral form factor (the Fourier transform of the cluster function) that

$$B(0) = 0, \quad \text{where} \quad B(t) \equiv \int_{-\infty}^{\infty} Y_2(r) e^{i2\pi tr} dr. \tag{4}$$

Another paper by Frahm *et al* [5], in agreement with [4] by their numerical computation of $\Sigma^2(L)$, argued that these features of the curve could be regarded as the characteristics of level statistics in metallic states that undergoes a transition to insulating states accompanied by localization (or, at least, 'weak localization'), discussed first by Al'tshuler and Shklovskii [7] who expected and aimed to clarify an intermediate nature of the long-range level statistics [8]. Al'tshuler *et. al.*'s studies were inherited by successors [9], and finally provided a conclusion that in an intermediate situation between metallic and insulating states, called *mobility edge*, the asymptote line of $\Sigma^2(L)$ must be expressed as a straight line χL with coefficient χ generally $0 < \chi < 1$ [10]. We shall call this an *intermediate compressibility*, because χ can be expressed, when the assembly of electron levels in a metal is treated as (1-dimensional) gas as a statistical mechanical object, in a form of the density-pressure relation for the gas [11]:

$$\chi = \frac{1}{\beta}\left(\frac{\partial \rho}{\partial p}\right)_{\beta}, \tag{5}$$

where β is the number in RMT to specify the three symmetry classes. Although the above two author's view [4,5] on the long range attractiveness of the level gas **(A)** would be correct and new, the feature **(B)** contradicts with the conclusion of intermediate compressibility, and we attribute it to the "dynamical" nature of their approach expressed in (3) (here, by "dynamical" we mean that one pursues a statistical quantity as a function of "time" λ). In this Letter, we wish to present a counter description of the long range level statistics based on an analog to equilibrium statistical mechanics that conforms to the static nature, or better to say "isothermal" nature as implied in Eq.(5), of the subject matter.

We employ the concept of *Gaussian matrix ensemble with preferential basis*(GMEPB) proposed by Pichard and Shapiro [12] for the above purpose. Let us consider an ensemble of $N \times N$ hermitian matrices and take one of them H for representing every one in the H-diagonal representation. We suppose all matrix elements of any (another) H to be Gaussian distributed but its H-diagonal elements biasedly weighted such that

$$W(\{H_{jk}\}) \propto \exp\left[-\frac{1}{2}\sum_{j=1}^{N} H_{jj}^2 - (1+\mu)\sum_{j<k}|H_{jk}|^2\right], \tag{6}$$

where μ is presently an arbitrary real positive parameter. Upon changing the distribution variables to $\{E_\alpha\}$ and $\{U_{j\alpha}\}$, where E_α is an eigenvalue of H and $U_{j\alpha}$ is a unitary matrix element of connecting the original basis to the new diagonalizing basis, the distribution becomes $W(\{E_\alpha, U_{\alpha,j}\}) \propto \exp[-\frac{1}{2}\sum_{\alpha=1} E_\alpha^2 - \mu\sum_{\alpha,\alpha'}(E_\alpha - E'_\alpha)^2 \sum_j U_{\alpha j}^2 U_{\alpha' j}^{*2}] \prod_{\alpha<\alpha'}(E_\alpha - E'_\alpha)^2$.

By linearizing the quartic part in the exponential as $U = 1 + A$ (an infinitesimal anti-hermitian), we get $W(\{E_\alpha, U_{\alpha j}\}) \propto \exp[-\frac{1}{2}\sum_{\alpha=1} E_\alpha^2 - \mu\sum_{\alpha,\alpha'}(E_\alpha - E'_\alpha)^2 |A_{\alpha,\alpha'}|^2] \prod_{\alpha<\alpha'}(E_\alpha - E'_\alpha)^2$ that is a Gaussian distribution on $\{A_{\alpha,\alpha'}\}$. A maximum entropy principle under the constraints

$$\langle \mathrm{tr} H^2\rangle = C_1,\ \langle \sum_{\alpha,\alpha'} |A_{\alpha,\alpha'}|^2\rangle = C_2,$$

$$\text{and} \qquad \langle \sum_{\alpha,\alpha'}(E_\alpha - E'_\alpha)^2 \sum_{\alpha,\alpha'}(E_\alpha - E'_\alpha)^2 |A_{\alpha,\alpha'}|^2\rangle = C_3, \tag{7}$$

then yields a solution that satisfies

$$\langle \sum_{\alpha,\alpha'}[1 + \mu(E_\alpha - E'_\alpha)^2]|A_{\alpha,\alpha'}|^2\rangle = C_2 + \mu C_3. \tag{8}$$

Although the three constants $C_i (i = 1, 2, 3)$ must be positive, the constraint condition (8) does not require the parameter μ to be a positive quantity, but it does require that $C_2 + \mu C_3 > 0$, to ensure integrability of $W(\{A_{\alpha,\alpha'}\})$.

An integration of the distribution $W(\{E_\alpha\},\{A_{\alpha,\alpha'}\})$ over the auxiliary variables $A_{\alpha,\alpha'}$ yields the N-joint level distribution of the form $P(x_1,\cdots,x_N) = C_{N,\beta}\exp\left[-\beta\sum_{j<k}\phi(x_j-x_k)\right], x_j \equiv E_j$, where the pair potential for $x_j - x_k \equiv r$ is given by

$$\phi(r) = \frac{1}{2}\log\left|1+\frac{1}{\mu r^2}\right|. \tag{9}$$

For the reason stated above, the real parameter μ could be negative as far as the inside of logarithm is positive, which may provide an attractive potential for the range $r_0 \equiv 1/\sqrt{2|\mu|} < r < \infty$, as shown in Fig.1(inset). But it has a logarithmic singularity at $r_c = 1/\sqrt{|\mu|}$. If we adopt an *ad hoc* postulate that the parameter μ may be complex- valued by an analogy to Breit-Wigner width in a line-shape function, then we can remove this logarithmic singularity to write

$$\begin{aligned}&\phi(r) = \frac{1}{2}Re\log(1+\frac{1}{\mu r^2}) = \frac{1}{4}\log(1+2\frac{a^2}{r^2}\cos2\theta+\frac{a^4}{r^4}),\\ &\text{where}\quad 1/\mu \equiv a^2e^{-i2\theta}, a>0;\ 0\le\theta<\pi/2.\end{aligned} \tag{10}$$

We can show that the *ad hoc* postulate of this complex parametrization is justified, if the GMEPB is properly generalized (See [13]). The potential function $\phi(r)$ is plotted in Fig.1 for three cases, namely,

(a)**attractive region** : $\pi/4 < \theta < \pi/2$, and on the positive r axis, $r_0 \equiv r_m/\sqrt{2} < r < \infty$, where $r_m = a/\sqrt{-\cos2\theta}$ is the unique potential minimum there.

(b)**repulsive regeion** : $\phi(r)$ is always repulsive(≥ 0) for $0 \le r \ll a$ (Wigner-Dyson repulsive region), but for $0 \le \theta < \pi/4$, there is no potential minimum, and it is always repulsive.

(c)**boundary between the two regions** : $\theta = \pi/4$ ($\cos2\theta = 0$), for which $r_m = \infty$.

The three cases in Fig.1 represent our view on the spectral statistics of solid states, namely (a) the metallic states, (b) non-metallic(including the insulating) states, and (c) the boundary between a metal and an insulator, i.e. the mobility edge situation. It may be remarked that in both situations (a) and (b) the long range tail of the potential as well as of the lowest-order approximate correlation function Eq.(11) retains the r^{-2} universality, though in the opposite direction to each other as regards (a) vs. (b). It should be pointed out that the Gibbs type distribution $P(x_1, \cdots, x_N)$ with pair potential so specified has its physical origin of the canonical equilibrium state of the Hamiltonian system (so called "g-CM system" [11]) whose trajectories are identified with (3).

In order to see the difference between the metallic and the non-metallic phases in a measurable quantity, we have computed the resulting number variance curves for two regimes of the transition parameter a. In small a regime, the correlation function and the number variance is provided by the 1st order virial expansion of the distribution $P(x_1, x_2, \cdots, x_N)$ i.e.

$$X_2(r; a, \theta) = \frac{r^2}{\sqrt{a^4 + 2a^2r^2 \cos 2\theta + r^4}}, \tag{11}$$

$$\Sigma^2(L; a, \theta) = L - L^2 + 2\int_0^L \frac{(L-r)r^2}{\sqrt{a^4 + 2a^2r^2\cos 2\theta + r^4}} dr. \tag{12}$$

For large parameter a regime, they can be derived via Beenakker's relation [14] between the Fourier transform of the potential ϕ(see [15]) and the spectral form factor $B(k) = 1 - (\beta\mathcal{F}_\phi(k))^{-1}$, hence

$$X_2(r; a, \theta) = 1 - \int_{-\infty}^{\infty} B(t) \cos(2\pi rt) dt$$
$$= 1 - \int_{-1}^{1} (1 - \frac{|t|}{1 - e^{-2\pi|t|a\cos\theta}\cos(2\pi|t|a\sin\theta)}) \cos(2\pi rt) dt, \tag{13}$$

$$\Sigma^2(L; a, \theta) = L - \int_{-1}^{1} (1 - \frac{|t|}{1 - e^{-2\pi|t|a\cos\theta}\cos(2\pi|t|a\sin\theta)}) \left(\frac{\sin(\pi tL)}{\pi t}\right)^2 dt. \tag{14}$$

The asymptotic evaluation of the integral in Eq.(14) for $L \to \infty$ where $(\cdot)^2 dt$ becomes $L \times \delta(x)dx$ yields

$$\chi = 1 - B(0) = \frac{1}{\beta\pi a\cos\theta}, \quad (\beta = 2; \mathrm{GU case}). \tag{15}$$

We draw $X_2(r)$ for two different values of a at a fixed $\theta = \pi/2.8$ (in metallic regime) in Fig.2. The "overshoot" is clearly seen at small $a = 0.22$: this is similar to that obtained by Guhr[3] with $\lambda = 0.1$ (see Fig.1 in [3]). The "overshoot" at large $a = 5$ is also demonstrated by magnifying the figure around $X_2 = 1$ (shown in the inset).

Very interesting things are shown in the curves of number variance $\Sigma^2(L)$. As can be seen from Fig.3, a specific behavior, we call it *non-monotone character*, is common for all the parameter values, although the overshoot becomes obscure in Fig.2 quickly as a increases. The asymptotic form of these curves in the normal plot gives us the compressibility χ, namely, $\Sigma^2(L) = \chi_0 + \chi L$. Indeed the linear asymptote of $\Sigma^2(L)$ at finite a is clearly shown in Fig. 3, where the three curves of $a = 0.22, 5$ and 10 in the large L regime are parallel and having slopes almost identical to unity. The best fit in normal scale gives rise to: 1) $a = 0.22, \chi_0 = 0.46$, $\chi = 0.55$; 2) $a = 5, \chi_0 = -6.83 \times 10^{-2}, \chi = 7.13 \times 10^{-2}$; 3) $a = 10, \chi_0 = 8.88 \times 10^{-2}, \chi = 3.45 \times 10^{-2}$. The latter two numbrs of χ are consistent with that from Eq.(15)($\chi = 7.32 \times 10^{-2}$ for $a = 5$ and $\chi = 3.66 \times 10^{-2}$ for $a = 10$). As a goes to infinity, Eqs. (13) and (14) become the respective form of GUE, thus χ goes to zero smoothly in the metallic limit.

In summary, we have derived expressions for the two-level correlation function $X_2(r)$ and the spectral number variance $\Sigma^2(L)$ that have the same physical origin of dynamics as that in previous version (Eq.(3) Refs. [2–5]), but via a different context. Here, first presenting the dynamics in the Hamiltonian form, we put it in an orthodox equilibrium statistical mechanics to compute every statistical quantity along the same line as the treatment in [11]. Therefore, it is not strange that the outcomes of some quantity by the both approaches, of which *compressibility* of the level gas is a typical one, sharply differ.

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FIGURES

FIG. 1. The ϕ function in Eq.(10) in different regimes. (a) $\theta = \pi/2.1$, (b) $\theta = 0.01$, and (c) $\theta = \pi/4$, which correspond to the metallic states, non-metallic states and the mobility edge situation, respectively. The inset is for $\theta = \pi/2(\mu =$ real negative; the singular case).

FIG. 2. The two-level correlation function $X_2(r)$ Eq.(11) for small a, and Eq.(13) for large a to simulate Guhr's $X_2(r,\lambda)$: 1) $a = 0.22$ (for $\lambda = 0.1$); 2) $a = 5$ (for large λ's) $\theta = \pi/2.8$ for both cases in the unfolded scale of abscissa. The inset is a magnification of the curve inside box.

FIG. 3. The Number variance $\Sigma^2(L)$ for different values of the transition parameter a from Eq.(12) for small a (=0.22) and Eq.(14) for large a (=5, 10)$\theta = \pi/2.8$ for all three cases in the unfolded scale. Note that the compressibility χ can be estimated from the intersection of each asymptote line and the abscissa for $\Sigma^2(L) = 1$.

Fig.1

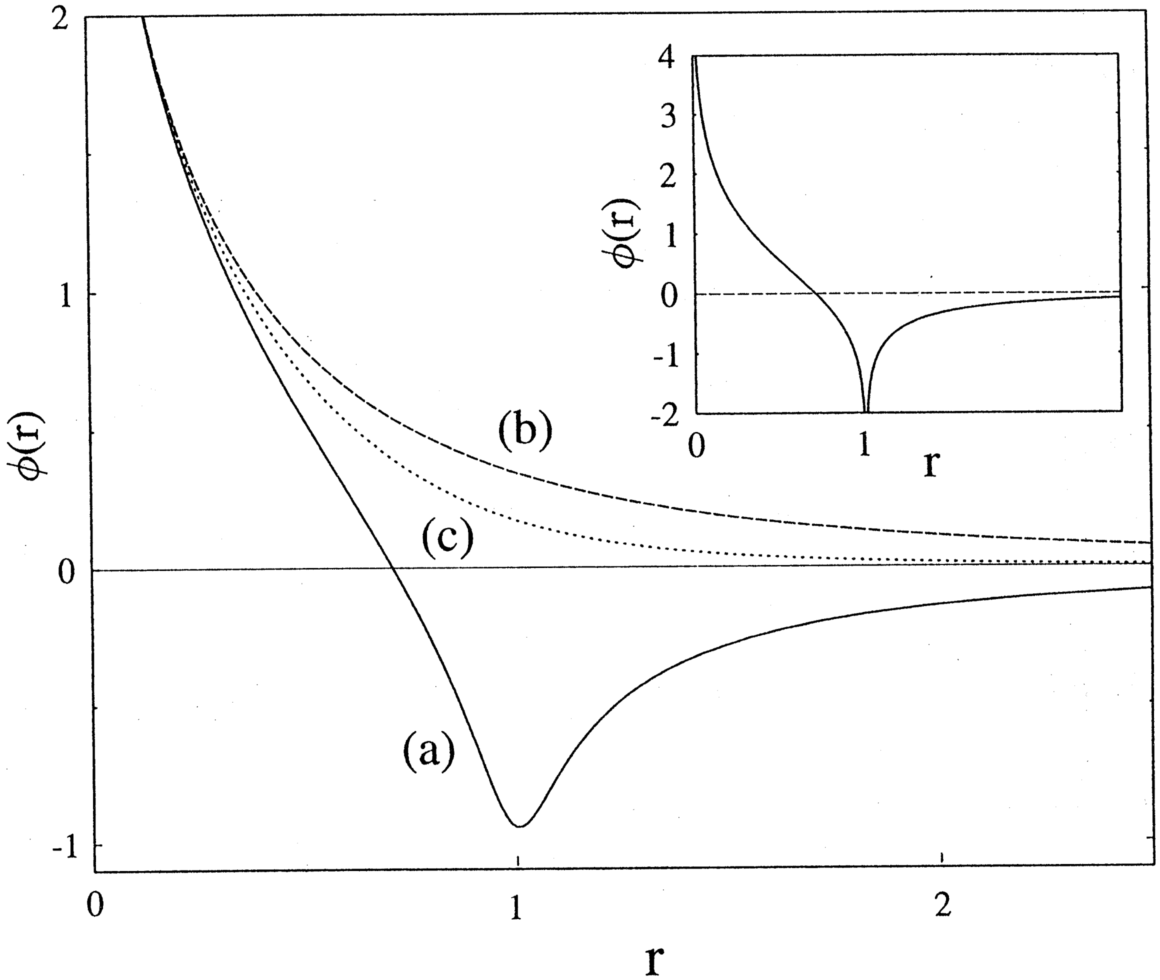
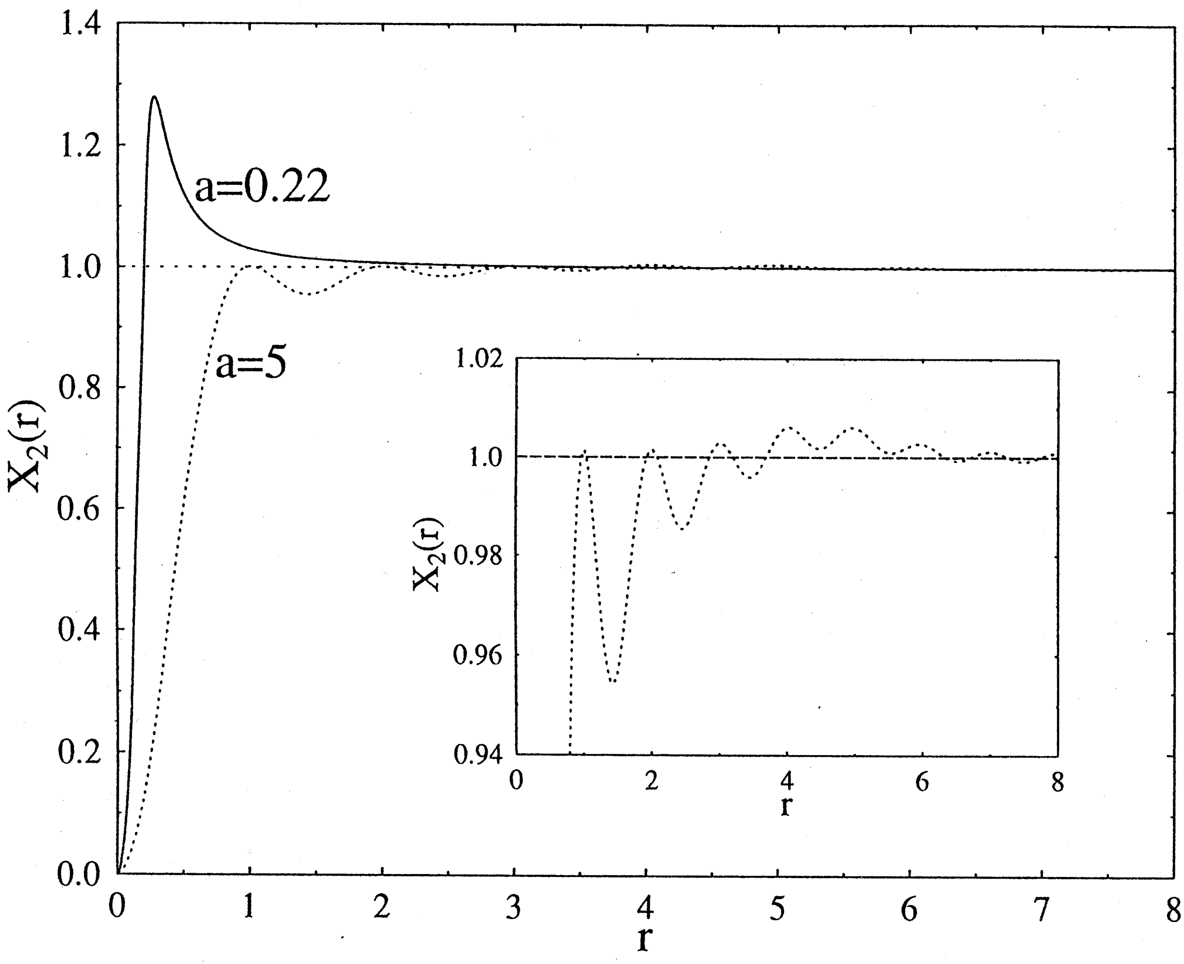
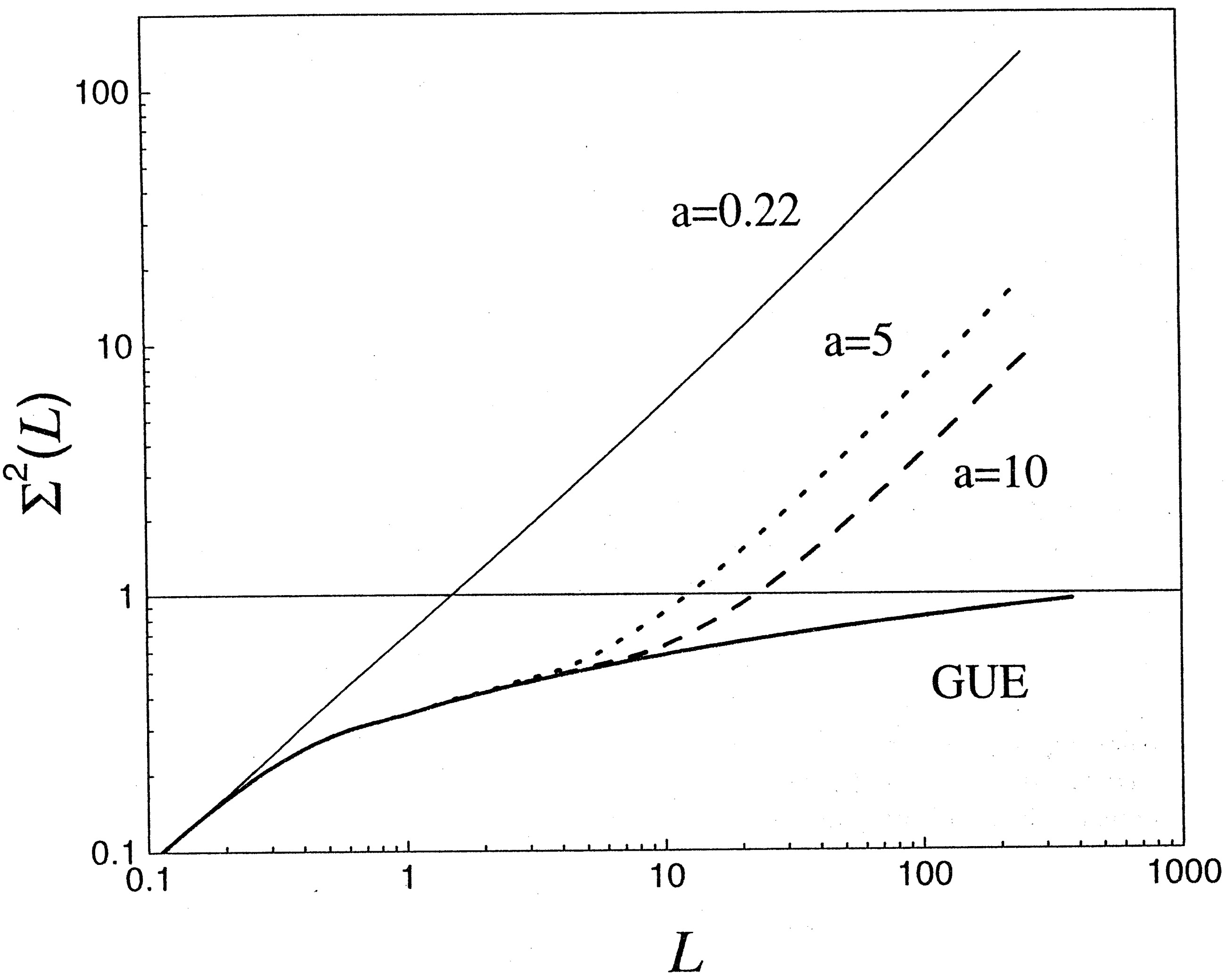


Fig.2



1.4
1.2
1.0
0.8
0.6
0.4
0.2
0.0
X2(r)
a=0.22
a=5
0
1
2
3
4
5
6
7
8
r
1.02
1.0
0.98
0.96
0.94
X2(r)
0
2
4
6
8
r

Fig 3



Part III

SUPPLEMENT TO HKBU-CNS-9815(Long-Range Level Statistics...) 1.*Perturbation Theory* (H. Hasegawa Oct. 1998, revised May 1999)

$$P_G(\tilde{A}) \propto \prod_{j<k} \exp\left[-\frac{1}{2f(x_j-x_k)}|\tilde{A}_{jk}|^2\right], \quad \text{with } f(r) = |\frac{\mu r^2}{1+\mu r^2}| \text{ (hermitian case)},$$

and hence

$$\phi(r) = \frac{1}{2}\log|1+\frac{1}{\mu r^2}|. \tag{9}$$

(r stands for $x_j - x_k$ with any pair (j,k)).

We use the notation a for the inverse square-root μ: $a \equiv 1/\sqrt{\mu}(\mu > 0)$. Then, the variance and the potential function of the Gaussian distribution are rewritten as

$$f(r) = \frac{r^2}{a^2+r^2}, \qquad \phi(r) = \frac{1}{2}\log(1+\frac{a^2}{r^2}), \tag{$S1$}$$

that is identical to the linear gas model of Gaudin[1]. As noted by him, the variance function $f(r)$ above has a meaning of the lowest-order (virial expansion of) correlation function for the interacting gas, and hence the corresponding cluster function(in the RMTsense) $Y_2(r) = 1 - f(r)$ can be written simply as

$$Y_2(r) = \frac{a^2}{a^2+r^2} > 0, \tag{$S2$}$$

(showing no overshoot of the correlation function $f(r)$). Here, we discuss the modified Gaudin model (with an imaginary parameter $ia(a > 0)$ for which the pair potential becomes *attractive*)in some detail:

$$\phi(r) = \frac{1}{2}\log(\frac{a^2}{r^2}-1) \quad |r| < a; \qquad \frac{1}{2}\log(1-\frac{a^2}{r^2}) \quad |r| > a, \tag{$S3$}$$

$$\text{with the attractive range} \qquad \frac{a}{\sqrt{2}} < |r| < \infty. \tag{$S4$}$$

It is quite easy to write the corresponding (low density) correlation function as

$$f(r) = |\frac{r^2}{a^2-r^2}| = \frac{r^2}{a^2-r^2}, \quad |r| < a; \qquad = \frac{r^2}{r^2-a^2} \quad |r| > a, \tag{$S5$}$$

and the cluster function, $Y_2(r) = 1 - f(r)$ as

$$Y_2(r) = \frac{a^2-2r^2}{a^2-r^2} \quad |r| < a; \qquad = -\frac{a^2}{r^2-a^2} \quad |r| > a. \tag{$S6$}$$

The overshoot of $f(r)$ (the negativeness of $Y_2(r)$) on the same range as ($S4$) can be seen readily from these expressions. Note that the figures exhibit a strong divergence reflecting the logarithmic divergence of the potential function ($S3$) that may be regarded as unphysical. Accordingly, we will discuss a treatment of eliminating this divergence by means of introducing a Breit-Wigner type broadening factor in the next page.

Let us recall Forrester's paper [2], where a useful representation of the N-level distribution $(x_1, x_2, .., x_N)$ is given by means of the Cauchy double alternant identity:

$$\det\left[\frac{i}{x_j - x_k + i(\beta/\gamma)^{1/2}}\right]_{j,k=1,\dots,N} = (\beta/\gamma)^{-N}\prod_{j<k}\frac{(x_j - x_k)^2}{[(x_j - x_k)^2 + \beta/\gamma]}. \qquad (S7)$$

Forrester assumed the positiveness of the parameter β/γ throughout, and we want to generalize his treatment by replacing $i(\beta/\gamma)^{1/2}$ by a complex parameter, $\alpha + i\delta(\alpha$ real; δ real and positive) so that

$$\det\left[\frac{1}{x_j - x_k + \alpha + i\delta}\right]_{j,k=1,\dots,N} = (\alpha + i\delta)^{-N}\prod_{j<k}\frac{(x_j - x_k)^2}{[(x_j - x_k)^2 - (\alpha + i\delta)^2]}.$$

Defining

$$\delta + i\alpha \equiv ae^{i\theta}, \qquad (S8)$$

where

$$a = \sqrt{\alpha^2 + \delta^2}, \qquad \theta = \mathrm{Arctan}\,(\alpha/\delta), \qquad (S9)$$

and taking the absolute magnitude of the right hand side of the above equality to provide it with *positivity* for probability, we can write

$$\prod_{j<k}\frac{(x_j - x_k)^2}{[(x_j - x_k)^4 + 2a^2(x_j - x_k)^2\cos 2\theta + a^4]^{1/2}} = a^N|\det\left[\frac{1}{x_j - x_k + iae^{-i\theta}}\right]_{j,k=1,\dots,N}|. \qquad (S10)$$

We can see that the left hand expression defines the distribution of an interacting level gas with a pair potential

$$\phi(r) = \frac{1}{4}\log(1 + 2\frac{a^2}{r^2}\cos 2\theta + \frac{a^4}{r^4}), \qquad (S11)$$

and that it is repulsive or partially attractive, respectively, according to the condition

$$0 \le 2\theta < \pi/2\,(\text{repulsive}); \qquad \pi/2 < 2\theta < \pi\,(\text{partially attractive}) \qquad (S12)$$

or,

$$\delta > \alpha\,(\text{dissipation dominates}); \quad \delta < \alpha\,(\text{Thouless energy dominates}). \qquad (S12')$$

In the latter case the unique maximum of the variance function $f(r)$ in a finite range of r(the potential minimum) exists at

$$r_m = a/\sqrt{-\cos 2\theta} \qquad (S13)$$

that is located in the attractive range, $\frac{1}{\sqrt{2}}r_m < r < \infty$(cf.$(S4)$), where

$$f(r) = \frac{r^2}{\sqrt{r^4 + 2a^2r^2\cos 2\theta + a^4}} \ge 1 \quad (\text{Guhr's overshoot}). \qquad (S14)$$

It should be noted that the last statement is under the restriction of lowest-order Mayer expansion theory for which more exact analysis is required by means of performing the grand canonical series of Forrester(SUPPLEMENT2).

Number variance curve and Compressibility by the perturbation theory

$$\Sigma^2(L) = L - 2\int_0^L (L-r)Y_2(r)dr, \quad Y_2(r) = \frac{a^2-2r^2}{a^2-r^2}, 0 \le r < a\ ; = -\frac{a^2}{r^2-a^2}, a \le r. \qquad (S15)$$

To avoid the singularity of $Y_2(r)$ at $r = \pm a$, it is assumed that $Y_2(r) =$ constant for $r \in (-\epsilon + \pm a, \pm a + \epsilon)$ with a small ϵ, and the constant is determined by the condition

$$\int_0^\infty Y_2(r)dr = 0. \qquad (S15a)$$

(i) Σ^2 curve inside the critical point $a : L < a$

$$L - 2\int_0^L (L-r)(2 - \frac{a^2}{a^2-r^2})dr = L - 2L^2 + 2\int_0^L \frac{(L-r)a^2}{a^2-r^2}dr, \qquad 0 \le L < a \ll 1.$$

Hence,

$$\Sigma^2(L) = L - 2L^2 - (a-L)a\log\left(\frac{a+L}{a-L}\right) + 2a^2\log\left(\frac{a+L}{a}\right). \qquad (S16)$$

This satisfies in the limit $\epsilon \to 0$

$$\frac{d}{dL}\Sigma^2(L) = 1 - 4\ L + a\log\left(\frac{a+L}{a-L}\right). \quad \frac{d^2}{dL^2}\Sigma^2(L) = -2Y_2(L) \quad L < a. \quad (S17)$$

(ii) Σ^2 curve outside the critical point $a : a < L$

$$L - 2\int_0^L (L-r)Y_2(r)dr = L - 2\left(\int_0^a + \int_a^L\right)(L-r)Y_2(r)dr$$

$$= \Sigma^2(a) + (L-a)[1 - 2\int_0^a Y_2(r)dr] + 2\int_a^L (L-r)\frac{a^2}{r^2-a^2}dr$$

$(-2\int_0^a Y_2(r)dr = 2\int_a^\infty Y_2(r)dr$ in the [ ] above by (S15a)), hence

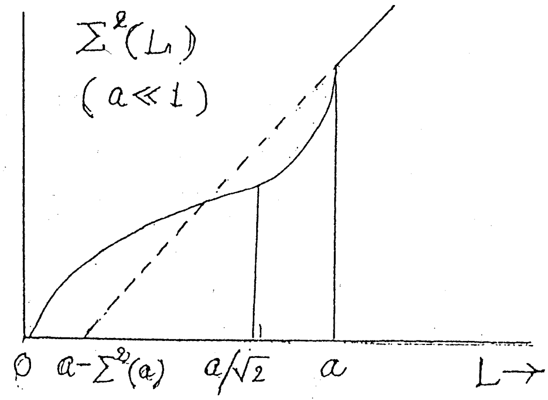
$$\Sigma^2(L) = \Sigma^2(a) + (L-a)\left[1 + a\log\left(\frac{L-a}{L+a}\right)\right] - 2a^2\log\left(\frac{L+a}{2a}\right) \quad a \le L, \ (S18)$$

where from $(S16)$ $\quad \Sigma^2(a) = a - 2a^2(1-\log 2)$. $\qquad (S19)$

This satisfies, under the condition $(S15a)$,

$$\frac{d}{dL}\Sigma^2(L) = 1 + a\log\left(\frac{L-a}{L+a}\right), \quad \frac{d^2}{dL^2}\Sigma^2(L) = -2Y_2(L) \quad a < L. \qquad (S20)$$

Remark The function $\Sigma^2(L)$ is continuous everywhere on the L axis(even for $\epsilon = 0$), but its first derivative is not at the critical point $S = a$, where $(d/dL)\Sigma^2(L)$ is logarithmically divergent as $\epsilon \to 0$.
The function $\Sigma^2(L)$ is illustrated:



Note that the point $a/\sqrt{2}$ is an inflection point of $\Sigma^2(L)$ from negative to positive, and, at the same time, at the singular point a the $\Sigma^2(L)$ changes its slope from steep to slow so that an s-shape behavior can be seen.

SUPPLEMENT TO HKBU-CNS-9815(Long-Range Level Statistics...) 2.*Grand Canonical Formulation* (H. Hasegawa May.1999)

I. Forrester's formulas for computing the two-point correlation functions

$$\Xi(a) = \sum_{N=0}^{\infty} \frac{\zeta^N}{N!} \left(\prod_{l=1}^{N} \int_{-L/2}^{L/2} dx_l a(x_l) \right) W_{N^2}, \qquad W_{N^2} = \det \left[\frac{i}{x_j - x_k + \alpha + i\delta} \right]_{j,k=1,..,N}$$

$$\text{(with a complex parameter } \alpha + i\delta) \qquad (S2.1)$$

It is related to the Fredholm determinant $\det(\mathbf{1} + \zeta\mathbf{K})$ of the integral operater $\mathbf{K}$ with kernel $K(x,y) = i/(x - y + \alpha + i\delta)$. The n-point correlation function is defined by

$$\rho(x_1,..,x_n) = \frac{1}{\Xi(1)} \left[\sum_{N=n}^{\infty} \frac{\zeta^N}{(N-n)!} \left(\prod_{l=n+1}^{N} \int_{-L/2}^{L/2} dx_l \right) W_{N^2} \right], \qquad (S2.2)$$

and can be computed from

$$\rho(x_1,..,x_n) = \det[G(x_j,x_k)]_{j,k=1,..,n}, \quad G(x,y) \equiv G(x-y), \qquad (S2.3)$$

where

$$\mathbf{G} = \frac{\zeta\mathbf{K}}{1+\zeta\mathbf{K}} \quad (\zeta : \text{ to be replaced by } 2\pi\zeta \text{ after } (S2.5)). \qquad (S2.4)$$

Remark. Forrester's treatment is restricted to the case $\alpha = 0$ for which the integral operator $\mathbf{K}$ is self-adjoint($K(y,x) = K(x,y)^*$) on the function space of a finite domain $(-L/2, L/2)$. The Fredholm theory is known to be valid for non-self adjoint integral operators, and this fact is essential here for the treatment of case $\alpha \neq 0$.

Accordingly, Forrester's solution of the Green's function $G(x)$ for $\mathbf{G}$ can be extended to the present case by a simple replacement of his parameter by a general complex parameter, i. e. $i(\beta/\gamma)^{1/2} \to \alpha + i\delta$, namely, under a thermodynamic-limit situation

$$G(r) = \int_0^{\infty} dt \frac{2\pi\zeta \exp[-2\pi(\delta - i\alpha)t]}{1 + 2\pi\zeta \exp[-2\pi(\delta - i\alpha)t]} \exp(2\pi irt), \qquad \delta > 0. \qquad (S2.5)$$

In particular, for $n = 1$ in $(S2.3)$

$$G(0) = \frac{\log(1+2\pi\zeta)}{2\pi(\delta - i\alpha)} = \frac{e^{i\theta}}{2\pi a} \log(1+2\pi\zeta). \quad (\delta - i\alpha \equiv ae^{-i\theta}, \quad a > 0.) \qquad (S2.6)$$

Eqs.(S2.5,6) are our starting formulas in the present analysis, and we will obtain the two-point correlation function in sevral situations. Because of the above complex parametrization, the Green function $G(r)$ does not satisfy the hermiticity condition unless $\alpha = 0$, i. e.

$$G(-r) = G(r)^* \qquad \text{only when } \alpha = 0. \qquad (S2.7)$$

Consequently, the basic quantity for statistics, $\rho(x_1,..,x_n)$ in $(S2.3)$, is in general complex-valued, and it is necessary to take its absolute magnitude in order to assign it as a probability distribution. From $(S2.6)$, the fugacity ζ can be real-valued.

We may write the level density and the 2-point correlation function as follows.

$$G(0) \equiv \rho e^{i\theta}, \qquad \rho = \frac{1}{2\pi a}\log(1+2\pi\zeta), \tag{S2.8}$$

or,

$$2\pi\zeta = e^{2\pi\rho a} - 1, \tag{S2.9}$$

and for the normalized 2-point correlation function as

$$X_2(r) = \left|1 - \frac{G(r)G(-r)}{G(0)^2}\right| = \sqrt{1 - 2Re\left[\frac{G(r)G(-r)}{G(0)^2}\right] + \frac{|G(r)G(-r)|^2}{|G(0)|^4}}. \tag{S2.10}$$

Note that, if the hermiticity condition ($S2.7$) holds, then the phase angle $\theta = 0$, and $X_2(r)$ reduces to

$$1 - |G(r)/\rho|^2, \tag{S2.11}$$

implying that there is no overshoot of the 2-level correlation function(the positive cluster function in the RMT sense).

From ($S2.9$), we can see that the fugacity ζ is represented by a function of the single product ρa =(ratio of the characteristic length a(absolute magnitude of the introduced broadning factor) vs the mean level-spacing), and hence that the situations of low, or high density can be represented by smallness, or largeness, respectively, of the parameter a while the density ρ is kept at a fixed value. Accordingly, we can classify the situation into 3 different regimes: **A.** $\rho a \ll 1$, **B.** $\rho a \gg 1$, **C.** $\rho a \simeq 1$.

II. Outline of Results

A. low fugacity limit $\zeta \ll 1(\rho a \ll 1)$.

In eq.(S2.5), ζ term in the denominator of the integrand is ignored so that we obtain

$$G(r) = \int_0^\infty dt 2\pi\zeta \exp[-2\pi(\delta - i\alpha - ir)t] = \frac{a\rho}{\delta - i\alpha - ir}, \quad \zeta \simeq a\rho, \tag{S2.12}$$

and

$$G(-r) = \frac{a\rho}{\delta - i\alpha + ir}.$$

Hence,

$$1 - \frac{G(r)G(-r)}{G(0)^2} = \frac{r^2}{a^2e^{-2i\theta} + r^2}; \; X_2(r) = \frac{r^2}{\sqrt{r^4 + 2a^2r^2cos2\theta + a^4}}, \text{(independent of}\rho) \tag{S2.13}$$

in agreement with the previous Supplement1, ($S14$), with an overshoot of $X_2(r)$ above unity for $\frac{\pi}{2} < 2\theta < \pi$. It satisfies

short-range property $X_2(r) \sim r^2/a^2 \quad r \ll a$ (Wigner-Dyson repulsion)

long-range property $X_2(r) \sim 1 - a^2\cos 2\theta/r^2 \; r \gg a$ (inverse-square universality).

Note that the latter also results from the approximation $X_2(r) \simeq 1 - ReG(r)G(-r)/G(0)^2$.

B. high fugacity case $\zeta \gg 1(\rho a \gg 1)$.

The spectral form factor associated with the 2-point cluster function $Y_2(r)$ is defined as the Fourier transform of $Y_2(r)$, given by(cf. Gaudin(4.18): here $e^{i2\pi tr}$ is in place of e^{ikr} there)

$$b(t) = \rho \int_{-\infty}^{\infty} Y_2(r) e^{i2\pi tr} dr, \quad \text{and then } Y_2(r) = \frac{1}{\rho} \int_{-\infty}^{\infty} b(t) e^{-i2\pi tr} dt. \qquad (S2.15)$$

We assume that the cluster function $Y_2(r)$ is an even function of r so that

$$Y_2(-r) = Y_2(r) \quad \text{then, } b(t) = \rho \int_0^{\infty} 2Y_2(r) \cos(2\pi tr) dr = b(-t). \qquad (S2.15')$$

We may utilize a simple formula of the variance function for the Fourier component, namely $2\int_0^L (L-r)\cos(2\pi tr)dr = \left(\frac{\sin(\pi tL)}{\pi t}\right)^2$. Hence, in terms of the form factor $B(t)(= B(-t)) = \frac{1}{\rho} b(t)$,

$$\Sigma^2(L) = L - \int_{-\infty}^{\infty} B(t) \left(\frac{\sin(\pi tL)}{\pi t}\right)^2 dt \ (\text{satisfying } \frac{d^2}{dL^2}\Sigma^2(L) = -2Y_2(L)). \quad (S2.16)$$

The asymptotic evaluation of this function for $L \gg \rho^{-1}$ can be obtained by the property $(\sin\pi tL/\pi t)^2 \to \delta(t)L$ as $L \to \infty$ so that

$$\Sigma^2(L \gg \rho^{-1}) = (1 - B(0))L \qquad (B(0) = \lim_{t\to 0} B(t)). \qquad (S2.17)$$

In the original Gaudin model, the form factor $B(t;a)$ is explicitly given by

$$B(t;a) = \frac{1}{1-e^{-2\pi|t|a}} \left(\frac{1}{2\pi a\rho} \log[1 + e^{-2\pi|t|a}(e^{2\pi a\rho} - 1)] - e^{-2\pi|t|a}\right), = 1 - \frac{1-e^{-2\pi\rho a}}{2\pi a\rho}$$

for $t = 0$. (S2.18)

Accordingly, the compressibility is given by $\chi = 1 - \int_0^{\infty} 2Y_2(r)\rho dr = (1-e^{-2\pi\rho a})/2\pi a\rho$. (S2.18′)

A simplification of $B(t;a)$ for $a \gg 1$ can be made by taking the dominant part of log[ ]above so that $B(t;\infty) = 1 - |t|/\rho(a=\infty), |t| \le \rho$; 0 otherwise, providing the $B_{\rm GUE}(t)$. Up to the next dominant part, thus

$$B(t; a \gg 1) = 1 - \frac{1}{\rho}\frac{|t|}{1-e^{-2\pi|t|a}} \quad |t| \le \rho, \text{ then } \chi(= 1 - B(0)) = \frac{1}{2\pi a\rho}. \quad (S2.19)$$

Remark $1 - B(t; a \gg 1)$ above, setting $\rho = 1$, is identical to the inverse of the Fourier transform of the potential function $\beta\phi(r)(\beta = 2)$, indicating that Jalabert,Pichard, Beenakker's way[3] of constructing the potential function from the form factor can be validated here, but only for the large parameter regime $a \gg 1$.

The above remark suggests that in order to draw the variance curve for $a \gg 1$ with an attractive potential$(\pi/4 < \theta < \pi/2)$, we may use the generalized form factor $B(t : a \gg 1, \theta)$ that can be derived easily from the potential function $\phi(r;\theta)(S11)$ with the result: $B(t; a \gg 1, \theta) = 1 - |t|/[1 - e^{-2\pi|t|a\cos\theta}\cos(2\pi|t|a\sin\theta)]$, (S2.20) under the unfolded scale $\rho = 1$, then

$$\Sigma^2(L) = L - \int_{-1}^{1} (1 - \frac{|t|}{1 - e^{-2\pi|t|a\cos\theta}\cos(2\pi|t|a\sin\theta)}) \left(\frac{\sin(\pi tL)}{\pi t}\right)^2 dt, \text{and } \chi = \frac{1}{2\pi a\cos\theta}.$$

(S2.21)

C. an approximate $X_2(r)$ for a general value of fugacity
In order to investigate the asymptotic behavior of $X_2(r)$, in paricular, to get the number variance curve $\Sigma^2(L)$ and the compressibility χ for an arbitrary fugacity, we make use of the following approximation for $(S2.10)$:

$$X_2(r) = 1 - Re\left[\frac{G(r)G(-r)}{G(0)^2}\right] \qquad (S2.22)$$

which is valid for $a \ll r$ (because it is correct up to $O(r^{-2})$ in the inverse-power expansion of $X_2(r)$). This is equivalent to an approximation $Y_2(r) = Re\left[\frac{G(r)G(-r)}{G(0)^2}\right]$, and enables us to apply the treatment of foregoing **B.** by means of the spectral form factor for this (approximate) cluster function: here, the above form of $\left[\frac{G(r)G(-r)}{G(0)^2}\right]$ is generally complex, but still it is a symmetric function of r taking the value unity at $r = 0$ so that eqs.$(S2.15, 15')$ are applicable.
We first show an exact result:

$$\int_{-\infty}^{\infty} Re\left[\frac{G(r)G(-r)}{G(0)^2}\right]\rho dr = \cos\theta\left(1 - \frac{1-e^{-2\pi a\cos\theta}}{2\pi a\cos\theta}\right). \qquad (S2.23)$$

It implies that the unfolding scale for a non-zero θ(but still $\cos\theta > 0$) to be chosen is $\rho = \cos\theta$ by which $(S2.23)$ becomes

$$\int_{-\infty}^{\infty} Re\left[\frac{G(r)G(-r)}{G(0)^2}\right] dr = 1 - \frac{1-e^{-2\pi a\cos\theta}}{2\pi a\cos\theta}. \qquad (S2.24)$$

This yields the expression for compressibility χ for an arbitrary transition parameter a, namely

$$\chi = \frac{1-e^{-2\pi a\cos\theta}}{2\pi a\cos\theta}. \qquad (S2.25)$$

A simple understanding of Eq.$(S2.25)$ is the previous result by HaMa[4] for $\theta = 0$ that it is obtainable by replacing the transition parameter a there is simply replaced by $a\cos\theta$, and for large parameter regime $2\pi a \gg 1$ it reduces to $(S2.21)$.
Similarly, on a less rigorous basis we may have

$$B(t;a,\theta) = \int_{-\infty}^{\infty} Re\left[\frac{G(r)G(-r)}{G(0)^2}\right]\cos(2\pi tr)dr = 1 - \frac{1}{1-e^{-2\pi|t|a\cos\theta}\cos(2\pi|t|a\sin\theta)}\times$$
$$\left(1 - \frac{1}{2\pi a\cos\theta} Re\log[1 + (e^{2\pi a\cos\theta} - 1)e^{-2\pi|t|\hat{a}}]\right). \qquad (S2.26)$$

This expression is shown to be valid under a legitimate inequality

$$Re(1-e^{-2\pi|t|\hat{a}}) \gg |Im(1-e^{-2\pi|t|\hat{a}})| = e^{-2\pi|t|a\cos\theta}|\sin(2\pi|t|a\sin\theta)| \hat{a} = ae^{-i\theta}. \quad (S2.27)$$

For $a \gg 1$, Eq.$(S2.26)$ can be shown to reduce to Beenakker's relation

$$B(t; a \gg 1, \theta) = 1 - \frac{|t|}{1-e^{-2\pi|t|a\cos\theta}\cos(2\pi|t|a\sin\theta)} \quad |t| \le 1; \ = 0 \ |t| > 1. \quad (S2.28)$$

derivation of Eq.($S2.23$).

From ($S2.6$), we have two expressions concerning the Green function $G(r)$ i.e.

$$G(0)=\int_0^\infty \frac{dt}{1+(2\pi\zeta)^{-1}e^{2\pi\hat{a}t}} \quad \text{and} \int_{-\infty}^{\infty} G(r)G(-r)dr=\int_0^\infty \frac{dt}{(1+(2\pi\zeta)^{-1}e^{2\pi\hat{a}t})^2}, \tag{S2.29}$$

for which we have exact identities(derivable in an elementary manner). Namely,

$$(1)\ \int_0^\infty \frac{dt}{1+(2\pi\zeta)^{-1}e^{2\pi\hat{a}t}}=\frac{1}{2\pi\hat{a}}\log(1+2\pi\zeta) \qquad \hat{a}=ae^{-i\theta} \tag{S2.30}$$

$$(2)\ \int_0^\infty \frac{dt}{(1+(2\pi\zeta)^{-1}e^{2\pi\hat{a}t})^2}=\frac{1}{2\pi\hat{a}}\left(\log(1+2\pi\zeta)-\frac{2\pi\zeta}{1+2\pi\zeta}\right). \tag{S2.31}$$

Eq.($S2.30$) gives rise to the fugacity-density relation ($S2.8$) by the assignment $G(0)=\rho e^{-i\theta}$. With this relation we may write

$$\int_{-\infty}^{\infty} \frac{G(r)G(-r)}{\{G(0)\}^2}\rho dr=e^{-i\theta}\left(1-\frac{1-e^{-2\pi a\rho}}{2\pi a\rho}\right), \tag{S2.32}$$

and, since $(\cdot)$ is a real quntity, taking the real part of both sides yields Eq.($S2.23$).

derivation of Eq.($S2.26$).

Consider the Fourier transform of the quantity $\frac{G(r)G(-r)}{\{G(0)\}^2}$: $\int_{-\infty}^{\infty}\frac{G(r)G(-r)}{\{G(0)\}^2}\cos(2\pi at)\rho dr$ (which must be nessesarily a cosine transform). The explicit form of $G(r)$Eq.($S2.5$) yields (after some manipulations) an exact result:

$$\int_{-\infty}^{\infty} \frac{G(r)G(-r)}{\{G(0)\}^2}\cos(2\pi at)\rho dr=e^{-i\theta}\times$$
$$\left[1-\frac{1}{1-e^{-2\pi\hat{a}|t|}}\left(1-\frac{1}{2\pi a\rho}\log[1+(e^{2\pi a\rho}-1)e^{-2\pi\hat{a}|t|}]\right)\right]. \tag{S2.33}$$

The quantity on the right hand side contains three imaginary factors $e^{-i\theta}$, but except the first they are accompanied by the variable $2\pi|t|a$, and for small values of $|t|a$ the imaginary part of $|t|\hat{a}$ is small compared to unity, also for large values of $|t|a$ it can be neglected since it appears in the exponentially decaying part. This leads us to ($S2.28$) and to neglect of the imaginary part of log[ ]. After taking the real part of both side ofEq.($S2.33$) and the unfolding $\rho=\cos\theta$, we arrive at Eq.($S2.26$).

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