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Some properties of Laver forcing

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1 Introduction

The notion of Laver forcing **LT** was first introduced by Laver [3] in order to construct a generic model in which Borel conjecture holds. Many properties which Laver forcing satisfies have been known (see [1]). One of fundamental properties of Laver forcing is the Laver property: For any increasing $h : \omega \rightarrow \omega$, it holds that

$$\Vdash_{\mathbf{LT}} \ \forall f \in \prod_{n<\omega} h(n) \ \exists S \in (\prod_{n<\omega} [h(n)]^{\leq n})^{\mathbf{V}} \ \forall^{\infty} n < \omega \ (\ f(n) \in S(n) \ ).$$

In this paper, we first discuss this property more closely by introducing the notion of simple conditions. After that, we give two applications of it. One is that Laver forcing satisfies the skip splitting property (for the definition, see section 3). Another one is a direct proof of the following known result.

Theorem (CH) Let P be the ω_2-stage countable support iteration by Laver forcing. Then, it holds that

$$\mathbf{V}^{P} \models \text{"the splitting number of } [\omega]^{\omega} = \omega_1\text{"}.$$

In the next section, we introduce the notion of simple conditions and show that, under some assumption, the set of simple conditions is $\leq_0^*$-dense in Laver forcing. The first application will be given in section 3 and the next in section 4.

2 Laver forcing

For each $s \in \omega^{<\omega}$, $[s]$ denotes the set $\{ t \in \omega^{<\omega} \mid s \subset t \}$. Let $q \subset \omega^{<\omega}$ be a tree. For each $s \in q$, $\mathrm{succ}_q(s)$ denote the set $\{ s^\frown\langle i \rangle \in q \mid i < \omega \}$. $s \in q$ is called a splitting node, if $|\mathrm{succ}_q(s)| > 1$. The first spliiting node of q is

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denoted by stem(q). q is called a Laver tree, if it holds that

$\forall t \in q \cap [\text{stem}(q)]$ ($\text{succ}_q(t)$ is infinite).

Laver forcing **LT** is the forcing notion which is defined by

$\mathbf{LT} = \{ q \subset \omega^{<\omega} \mid q$ is a Laver tree $\}$, and

$q' \leq q$ if and only if $q' \subset q$, for any $q,\ q' \in \mathbf{LT}$.

Let $q \in \mathbf{LT}$. For each $s \in q$, $q[s]$ denotes the condition $\{ t \in q \mid s \subset t \text{ or } t \subset s \}$. For each $S \subset q$, $\text{Succ}_q(S)$ denotes the set $\bigcup_{s \in S} \text{succ}_q(s)$.

2.1 simple conditions

Define the relation $\leq_0^*$ on **LT** by

$q' \leq_0^* q$ if and only if $q' \leq q$ and $\text{stem}(q) = \text{stem}(q')$.

The following fact is well-known (see e.g. [1]).

Fact 1 *Let $m < \omega$, $\dot{a}$ be a* **LT***-name, and $q \in \mathbf{LT}$. If $q \Vdash \dot{a} < m$, then there exist $q' \leq_0^* q$ and $i < m$ such that $q' \Vdash \dot{a} = i$.*

Let $h \in \omega^\omega$. We denote by $\mathcal{F}_h$ the set $\{ x \in \omega^\omega \mid \forall j < \omega\ (x(j) \leq h(j)) \}$. Let $\dot{f}$ be a **LT**-name such that $\Vdash_{\mathbf{LT}} \dot{f} \in \mathcal{F}_h$.

For each $q \in \mathbf{LT}$, define $H(q) = H_{\dot{f}}(q)$ by

$H(q) = \{ \delta \in \omega^{<\omega} \mid \exists q' \leq_0^* q\ (q' \Vdash \delta \subset \dot{f}) \}$.

For each $f,\ g \in \omega^{\leq\omega}$, we denote by $\Delta(f,g)$ the least $n \in \text{dom}(f) \cap \text{dom}(g)$ such that $f(n) \neq g(n)$ if such n exists, otherwise undefined. For each tree $H \subset \omega^{<\omega}$, we denote by $\text{Lim}(H)$ the set $\{ x \in \omega^\omega \mid \forall j < \omega\ (x \restriction j \in H) \}$.

The next fact can be easily verfied by using Fact 1. We left a proof to the reader.

Fact 2 *For any $q \in \mathbf{LT}$, it holds that*

(1) *$\delta \in H(q)$ if and only if $\exists^\infty s \in \text{succ}_q(\text{stem}(q))$ ($\delta \in H(q[s]$), for any $\delta \in \omega^{<\omega}$.*

(2) *$\langle\rangle \in H(q)$, and $H(q)$ is a tree, and $H(q)$ does not have a maximal node.*

(3) *$\text{Lim}(H(q)) \neq \phi$ and $\text{Lim}(H(q)) \subset \mathcal{F}_h$.*

(4) *If $q' \in \mathbf{LT}$ and $q' \leq_0^* q$ then $H(q') \subset H(q)$.*

We say that a condition q is $\dot{f}$-simple, if $|\text{Lim}(H(q[s]))| = 1$, for all $s \in q \cap [\text{stem}(q)]$.

Lemma 2.1 *For any $q \in \mathbf{LT}$, there exists $q' \leq_0^* q$ such that $|\text{Lim}(H(q'))| = 1$.*

Proof Let $q \in \mathbf{LT}$.

Case 1 $\text{Lim}(H(q))$ is finite.

Take $g \in \mathrm{Lim}(H(q))$. Choose $n < \omega$ such that

$g \restriction n \neq g' \restriction n$, for any $g' \in \mathrm{Lim}(H(q)) \setminus \{ g \}$.

Take $q' \leq_0^* q$ such that $q' \Vdash g \restriction n \subset \dot{f}$. Since $H(q') \subset H(q)$, we have that $\mathrm{Lim}(H(q')) = \{ g \}$.

Case 2 $\mathrm{Lim}(H(q))$ is infinite.

Take $g \in \mathcal{F}_h$ such that

$\sup\{ \Delta(g, g') \mid g' \in \mathrm{Lim}(H(q)) \setminus \{ g \} \} = \omega$.

Take $g_i \in \mathrm{Lim}(H(q))$ (for $i < \omega$) such that

$\Delta(g, g_i) < \Delta(g, g_{i+1})$, for all $i < \omega$.

For each $i < \omega$, take $s_i \in \mathrm{succ}_q(\mathrm{stem}(q))$ and $q_i \leq_0^* q[s_i]$ such that

$s_i \neq s_j$, for all $j < i$ and $q_i \Vdash g_i \restriction (\Delta(g, g_i) + 1) \subset \dot{f}$.

Set $q' = \bigcup_{i<\omega} q_i$. Then, it holds that $q' \leq_0^* q$ and $\mathrm{Lim}(H(q')) = \{ g \}$. □

Corollary 2.2 *For any $q \in \mathbf{LT}$, there exists $q' \leq_0^* q$ such that q' is $\dot{f}$-simple.* □

For each $\dot{f}$-simple condition q, let $\dot{f}[q] \in \mathcal{F}_h$ denote the function such that $\mathrm{Lim}(H(q)) = \{ \dot{f}[q] \}$.

Lemma 2.3 *Let q be an $\dot{f}$-simple condition. Then, for any $s \in q \cap [\mathrm{stem}(q)]$, the following* (a) <u>*or*</u> (b) *hold.*

(a) $\exists^\infty t \in \mathrm{succ}_q(s)\ (\ \dot{f}[q[s]] = \dot{f}[q[t]]\ )$.

(b) $\sup\{ \Delta(\dot{f}[q[s]], \dot{f}[q[t]]) \mid t \in \mathrm{succ}_q(s) \} = \omega$.

Proof Easy. □

Corollary 2.4 *For any $q \in \mathbf{LT}$, there exists $q' \leq_0^* q$ such that*

(1) *q' is $\dot{f}$-simple, and,*

(2) *for any $s \in q' \cap [\mathrm{stem}(q')]$, the following* (2.a) *or* (2.b) *hold.*

(2.a) $\forall t \in \mathrm{succ}_{q'}(s)\ (\ \dot{f}[q'[s]] = \dot{f}[q'[t]]\ )$.

(2.b) *For any $t, t' \in \mathrm{succ}_{q'}(s)$,*

if $t(|s|) < t'(|s|)$ then $\Delta(\dot{f}[q'[s]], \dot{f}[q'[t]]) < \Delta(\dot{f}[q'[s]], \dot{f}[q'[t']])$.

(3) *For any $s \in q' \cap [\mathrm{stem}(q')]$, any $t \in q' \cap [s]$, and any $u \in q' \cap [t]$,*

if $\dot{f}[q'[s]] \neq \dot{f}[q'[t]]$ and $\dot{f}[q'[t]] \neq \dot{f}[q'[u]]$ then $\Delta(\dot{f}[q'[s]], \dot{f}[q'[t]]) < \Delta(\dot{f}[q'[t]], \dot{f}[q'[u]])$. □

A condition q' which satisfies (1) $\sim$ (3) in corollary 2.4 is said to be strongly $\dot{f}$-simple.

2.2 Generaters of a condition

Definition 2.1 *Let $q \in \mathbf{LT}$ and $S \subset q \cap [\mathrm{stem}(q)]$. We say that S generates q, if $\{ q[s] \mid s \in S \}$ is a maximal antichain below q in* $\mathbf{LT}$.

Let S and S' generate q. S' is called a refinement of S, if it satisfies that, for any $s \in S'$, there exists $t \in S$ such that $t \subset s$.

For each $S \subset \omega^{<\omega}$, define $\mathrm{Inf}_\alpha(S)$ (for $\alpha \leq \omega_1$) such that

$\mathrm{Inf}_0(S) = \{ s \in S \mid \forall i < |s| \ (s \restriction i \notin S) \}$,

$\mathrm{Inf}_{\alpha+1}(S) = \mathrm{Inf}_\alpha(S) \cup \{ s \in \omega^{<\omega} \mid \exists^\infty i < \omega \ (s^\frown\langle i \rangle \in \mathrm{Inf}_\alpha(S)) \}$,

$\mathrm{Inf}_\alpha(S) = \bigcup_{\xi<\alpha} \mathrm{Inf}_\xi(S)$, *if α is a limit ordinal.*

Set $\mathrm{cl}(S) = \mathrm{Inf}_{\omega_1}(S)$.

Note that

*If $S \subset q \in$ **LT** then* $\mathrm{cl}(S) \subset q$,

$\forall s \in q \cap [\mathrm{stem}(q)] \setminus \mathrm{cl}(S) \ \exists^\infty t \in \mathrm{succ}_q(s) \ (t \notin \mathrm{cl}(S))$,

*If S generates $q \in$ **LT** then* $\mathrm{stem}(q) \in \mathrm{cl}(S)$.

Lemma 2.5 *Let $h \in \omega^\omega$, and $\dot{f}$ an **LT**-name such that $\Vdash \dot{f} \in \mathcal{F}_h$. Then, for any $q \in$ **LT**, there exists $q' \leq_0^* q$ such that*

(1) *q' is strongly $\dot{f}$-simple, and*

(2) *the following* (a) *or* (b) *holds.*

(a) $q' \Vdash \dot{f} = \dot{f}[q']$.

(b) $\{ s \in q' \cap [\mathrm{stem}(q')] \mid \dot{f}[q'[s]] = \dot{f}[q']$ *and* $\dot{f}[q'[s] \neq \dot{f}[q'[t]]$, *for some/all* $t \in \mathrm{succ}_{q'}(s) \}$ *generates* q'.

Proof Without loss of generality, we may assume that q is strongly $\dot{f}$-simple. Put

$k = |\mathrm{stem}(q)|$, and

$S = \{ s \in q' \cap [\mathrm{stem}(q')] \mid \dot{f}[q'[s]] = \dot{f}[q']$ and $\dot{f}[q'[s] \neq \dot{f}[q'[t]]$, for some/all $t \in \mathrm{succ}_{q'}(s) \}$.

Case 1. $\mathrm{stem}(q) \in \mathrm{cl}(S)$.

Put $S' = \mathrm{cl}(S) \bigcup_{s \in S} (q \cap [s])$. By induction on $n < \omega$, define $U_n \subset q \cap \omega^{k+n}$ by

$U_0 = \{ \mathrm{stem}(q) \}$,

$U_{n+1} = \{ s^\frown\langle j \rangle \in q \cap \omega^{k+n} \mid s \in U_n \text{ and } s^\frown\langle j \rangle \in S' \}$.

Then, q' = the condition generated by $\bigcup_{n<\omega} U_n$ satisfies (b).

Case 2. $\mathrm{stem}(q) \notin \mathrm{cl}(S)$.

Note that

$\forall s \in q \cap [\mathrm{stem}(q)] \ (\text{if } s \notin \mathrm{cl}(S) \text{ then } \exists^\infty j < \omega \ (s^\frown\langle j \rangle \in q \setminus \mathrm{cl}(S)))$.

By induction on $n < \omega$, define $U_n \subset q \cap \omega^{k+n}$ by

$U_0 = \{ \mathrm{stem}(q) \}$,

$U_{n+1} = \{ s^\frown\langle j \rangle \in q \setminus \mathrm{cl}(S) \mid s \in U_n \}$.

Then, q' = the condition generated by $\bigcup_{n<\omega} U_n$ satisfies (a). □

Definition 2.2 *Let $S \subset \omega^{<\omega}$. Define the relation $\leq_S$ on **LT** by*

$q \leq_S q'$ if and only if $q \leq_0^* q'$ and $S \subset q \cap q' \cap [\mathrm{stem}(q)]$.

Definition 2.3 *Let $q \in \mathbf{LT}$. A sequence $\langle (q_i, S_i) \mid i < \omega \rangle$ is called a fusion sequence below q, if it satisfies that, for all $i < \omega$,*
(1) $q_i \in \mathbf{LT}$ *and* S_i *generates* q_i.
(2) $q_0 \leq_0^* q$ *and* $q_{i+1} \leq_{S_i} q_i$.
(3) S_{i+1} *is a refinement of* $\mathrm{Succ}_{q_{i+1}}(S_i)$.

Lemma 2.6 *Let $\langle (q_i, S_i) \mid i < \omega \rangle$ be a fusion sequence. Put $\widetilde{q} = \bigcap_{i<\omega} q_i$. Then, it holds that*
$\widetilde{q} \in \mathbf{LT}$ *and* $\widetilde{q} \leq_{S_i} q_i$, *for all* $i < \omega$. □

Lemma 2.7 *Let $q \in \mathbf{LT}$ and $\dot{x}$ a $\mathbf{LT}$-name such that $q \Vdash \dot{x} \in [\omega]^\omega$. Then, there exist $q' \leq_0^* q$, $S \subset q' \cap [\mathrm{stem}(q')]$, and $m_s < \omega$ (for $s \in S$) such that*
(1) S *generates* q' *and* $\forall s \in S\ (\ |\mathrm{stem}(q')| < |s|\ )$,
(2) $q'[s] \Vdash m_s \in \dot{x}$, *for all* $s \in S$,
(3) $m_s \neq m_t$, *for all distinct* $s,\ t \in S$.

Proof Let $\dot{f}$ be the $\mathbf{LT}$-name such that $\Vdash \dot{f}$ is the characteristic function of $\dot{x}$. Without loss of generality, we may assume that q is strongly $\dot{f}$-simple. Set $y = \{ j < \omega \mid \dot{f}[q](j) = 1 \}$.
Case 1. y is infinite.
By induction on $k < \omega$, take $n_k \in y$, $s_k \in \mathrm{succ}_q(\mathrm{stem}(q))$ such that
$n_k < n_{k+1}$ and $\forall l < k\ (\ s_k \neq s_l\ )$ and $\dot{f}[q[s_k]] \restriction (n_k + 1) = \dot{f}[q] \restriction (n_k + 1)$.
For each $k < \omega$, take $r_k \leq_0^* q[s_k]$ such that
$r_k \Vdash \dot{f}[q[s_k]] \restriction (n_k + 1) \subset \dot{f}$.
Let $q' = \bigcup_{k<\omega} r_k$ and $S = \{ s_k \mid k < \omega \}$, and $m_{s_k} = n_k$, for $k < \omega$. Then, these are as required.
Case 2. y is finite
Let $T = \{ s \in q \cap [\mathrm{stem}(q)] \mid \dot{f}[q[s]] = \dot{f}[q]$ and $\dot{f}[q[s]] \neq \dot{f}[q[t]]$, for some/any $t \in \mathrm{succ}_q(s) \}$.
Since $\Vdash \dot{x} \in [\omega]^\omega$, T generates q. For each $t \in T$, let
$a_t = \{ j < \omega \mid \exists s \in \mathrm{succ}_q(t)\ (\ \dot{f}[q[s]](j) = 1\ ) \}$.

Claim 1 *a_t is infinite, for all $t \in T$.*

Proof This is directly followed from the fact that
$\sup\{ \Delta(\dot{f}[q[s]], \dot{f}[q[t]]) \mid s \in \mathrm{succ}_q(t) \} = \omega$. QED of Claim 1
By using this claim, we can take $r_t \leq_0^* q[t]$ (for $t \in T$), and $m_s < \omega$ (for $s \in \bigcup_{t \in T} \mathrm{succ}_{r_t}(\mathrm{stem}(r_t)))$ such that

$r_t[s] \Vdash m_s \in \dot{x}$, for all $s \in \mathrm{succ}_{r_t}(\mathrm{stem}(r_t))$, for all $t \in T$, and,
$m_s \neq m_{s'}$, for all distinct $s,\ s' \in \bigcup_{t\in T} \mathrm{succ}_{r_t}(\mathrm{stem}(r_t))$.

So, $q' = \bigcup_{t\in T} r_t$, $S = \bigcup_{t\in T} \mathrm{succ}_{r_t}(\mathrm{stem}(r_t))$, and m_s (for $s \in S$) are as required. □

3 The skip splitting property

In this section, we give a first application of the previous section. We begin with the definition of the skip splitting property. A tree $H \subset 2^{<\omega}$ is called a skip splitting tree, if it holds that

$\forall t \in \mathrm{succ}_H(s)$ (t is not a splitting node), for any splitting node s of H.

A forcing notion P has the skip splitting property, if it holds that

$\Vdash_P \ \forall f \in 2^\omega \ \exists H \in \mathbf{V}$ (H is a skip splitting tree and $f \in \mathrm{Lim}(H)$).

Theorem 3.1 **LT** *has the skip splitting property.*

To show this theorem, we use the following fact which is easily checked.

Fact 3 *For any* $\{ x_j \mid j < \omega \} \subset [\omega]^\omega$, *there exist* $\{ y_j \mid j < \omega \}$ *such that*
(1) $y_j \in [x_j]^\omega$, *for all* $j < \omega$.
(2) $y_j \cap y_k = \phi$, *for all distinct* $j,\ k < \omega$.
(3) $\forall m \in \bigcup_{j<\omega} y_j \ (m + 1 \notin \bigcup_{j<\omega} y_j)$

Proof of Theorem 3.1 Let $\dot{f}$ be a **LT**-name such that $\Vdash \dot{f} \in 2^\omega$ and $q \in \mathbf{LT}$. We show that there exist $q^* \leq_0^* q$ and a skip splitting tree $H \subset 2^{<\omega}$ which satisfy

(*) $q^* \Vdash \dot{f} \in \mathrm{Lim}(H)$.

Replace q by a certain strong condition, if necessary, we may assume that q is strongly $\dot{f}$-simple. Put $S = \{ s \in q \cap [\mathrm{stem}(q)] \mid \dot{f}[q[s]] \neq \dot{f}[q[t]]$, for some/any $t \in \mathrm{succ}_q(s) \}$. For each $s \in S$, put

$x_s = \{ \Delta(\dot{f}[q[s]], \dot{f}[q[s^\frown\langle j\rangle]]) \mid s^\frown\langle j\rangle \in q \}$.

By the above fact, we can take $y_s \in [x_s]^\omega$ (for $s \in S$) such that

$y_s \cap y_t = \phi$, for all distinct $s,\ t \in S$ and $\forall m \in \bigcup_{s\in S} y_s \ (m + 1 \notin \bigcup_{s\in S} y_s)$.

Put $k = |\mathrm{stem}(q)|$. Define $U_n \subset q \cap \omega^{k+n}$ (for $n < \omega$) by

$U_0 = \mathrm{stem}(q)$,

$s^\frown\langle j\rangle \in U_{n+1}$ if and only if $s \in U_n$ and if $s \in S$ then $\Delta(\dot{f}[q[s]], \dot{f}[q[s^\frown\langle j\rangle]]) \in y_s$.

Let q^* be the tree generated by $\bigcup_{n<\omega} U_n$ and H the tree generated by

$\{\dot{f}[q[s]] \mid s \in q^*\}$. Since y_s is infinite for all $s \in S$, it holds that $q^* \in \mathbf{LT}$ and $q^* \leq_0^* q$. So, it holds that $q^* \Vdash \dot{f} \in \mathrm{Lim}(H)$. Since it holds that

$$|\delta| \in \bigcup_{s \in S} y_s, \text{ for every splitting node } \delta \in H,$$

H is a skip splitting tree. □

The skip splitting property is concerned with the cardinal invariant θ_2 which is associated with predictors. We call a function from $\omega^{<\omega}$ to ω a predictor. A predictor π constantly predicts $f \in \omega^\omega$, if there exists an $n < \omega$ such that

$$\forall j < \omega \, \exists k \in [jn, (j+1)n) \, (f(k) = \pi(f \restriction k)).$$

Let θ_2 denote the smallest cardinality of a set of predictors Π such that every $f \in 2^\omega$ is predicted constantly by some $\pi \in \Pi$. It is easy to check that if a forcing notion P has the skip splitting property, then it holds that, in $\mathbf{V}^P$, every function $f \in 2^\omega$ is predicted constantly by some predictor in the ground model $\mathbf{V}$. So, if the skip splitting property was preserved by countable support iterations, we could get a generic model of $\theta_2 = \omega_1 < \mathbf{b} = \omega_2$ by using Laver forcing. Unfortunately, countable support iterations do not preserve this property, in general. In fact, it holds that, in a generic model which is obtained from the ω-stage countable support iteration by Laver forcing, there exists $g \in 2^\omega$ which is not predicted constantly by any predictor in the ground model [2].

Question Is $\theta_2 < \mathbf{b}$ consistent with ZFC?

4 The intersection property

In this section, we introduce the intersection property, and show that the countable support iterations of Laver forcing satisfies this property. As a corollary, we show the following Theorem.

Theorem 4.1 *Assume that* $\mathbf{V} \models$ CH. *Let P be the ω_2-stage countable support iteration by Laver forcing. Then, it holds that*

$$\mathbf{V}^P \models \textit{the splitting number } \mathbf{s} \textit{ of } [\omega]^\omega = \omega_1.$$

This corollary is a known result. Professor Kada at Kitami Institute of Technology informed me that this followed from the fact $\mathbf{s} \leq \mathrm{non}(\mathcal{N})$ and a result of Shelah $\mathbf{V}^P \models \mathrm{non}(\mathcal{N}) = \omega_1$.

Throughtout this section, we use the standard notations and notion of proper forcing (see e.g., [1]). Let λ denote an arbitrary but fixed sufficiently large regular cardinal. We denote by $H(\lambda)$ the set of all sets with hereditary cardinality $< \lambda$.

We begin with the definition of the intersection property.

Definition 4.1 *A forcing notion P has the intersection property, if the following holds.*

For any coutable elementary substructure N of $H(\lambda)$, and any $\{ a_j \mid j < \omega \} \subset [\omega]^\omega$, if

$P \in N$ *and* $\forall x \in N \cap [\omega]^\omega \, (x \cap a_j \in [\omega]^\omega)$, *for all* $j < \omega$,

then, for any $p \in N \cap P$, there exists $p' \le p$ such that

p' *is (N, P)-generic and* $p' \Vdash \forall x \in N[\dot{\mathcal{G}}_P] \cap [\omega]^\omega \, (x \cap a_j \in [\omega]^\omega)$, *for all* $j < \omega$.

The next lemma can be proved by a standard argment. We give a proof for a convenience to the reader.

Lemma 4.2 *Countable support iterations preserve the intersection property. I.e., for any countable support iteration $\langle P_\alpha \mid \alpha \le \beta \rangle$, $\langle \dot{Q}_\alpha \mid \alpha < \beta \rangle$, if*

$\Vdash_\alpha \dot{Q}_\alpha$ *has the intersection property, for all* $\alpha < \beta$,

then P_β has the intersection property.

Proof By induction on $\beta \in \mathbf{On}$. The case that β is a successor ordinal is easily proved. We only treat with the case that β is a limit ordinal. So, let β be a limit ordinal, N a countable elementary substructure of $H(\lambda)$, and $\{ a_j \mid j < \omega \} \subset [\omega]^\omega$ satisfy

$\langle P_\alpha \mid \alpha \le \beta \rangle \in N$ and $\forall x \in N \cap [\omega]^\omega \, (x \cap a_j \in [\omega]^\omega)$, for all $j < \omega$.

Take an increasing sequence $\langle \beta_n \mid n < \omega \rangle$ of ordinals in N such that $\sup_{n<\omega} \beta_n = \sup(\beta \cap N)$. Take a surjection $\pi \in N$ from ω to $\omega \cup \omega^2$ such that

$\forall j, k < \omega \, \exists^\infty n < \omega \, (\pi(n) = (j, k))$.

Let $\langle \dot{\xi}_j \mid j < \omega \rangle$ and $\langle \dot{x}_k \mid k < \omega \rangle$ be enumerations of $\{ \dot{\xi} \in N \mid \dot{\xi}$ is P_β-name and $\Vdash \dot{\xi} \in \mathbf{On} \}$ and $\{ \dot{x} \in N \mid \dot{x}$ is P_β-name and $\Vdash \dot{x} \in [\omega]^\omega \}$, respectively. To prove this lemma, let $p \in P_\beta \cap N$.

Claim 2 *There exist $p_n \in P_{\beta_n}$ and a P_{β_n}-name $\dot{r}_n$ (for $n < \omega$) such that*

(1) $p_n \le p \restriction \beta_n$ *and p_n is (N, P_{β_n})-generic.*

(2) $\Vdash \dot{r}_n \in N[\dot{\mathcal{G}}_{P_{\beta_n}}] \cap P_\beta / P_{\beta_n}$ *and* $\dot{r}_n \le p \restriction [\beta_n, \beta)$.

(3) $p_{n+1} \restriction \beta_n = p_n$ *and* $p_{n+1} \Vdash \dot{r}_{n+1} \le \dot{r}_n \restriction [\beta_n, \beta)$.

(4) $p_n \Vdash p_{n+1} \restriction [\beta_n, \beta_{n+1}) \le \dot{r}_n \restriction [\beta_n, \beta_{n+1})$.

(5) $p_n \Vdash \forall x \in N[\dot{\mathcal{G}}_{P_{\beta_n}}] \cap [\omega]^\omega \, (x \cap a_j \in [\omega]^\omega)$, *for all* $j < \omega$.

(6.a) *If $\tau(n) = j$, then*

$p_{n+1} \Vdash \dot{r}_{n+1}$ *decides the value of* $\dot{\xi}_j$.

(6.b) *If $\tau(n) = (j, k)$, then*

$p_{n+1} \Vdash \exists u \in [a_j]^n \, (\dot{r}_{n+1} \Vdash u \subset \dot{x}_k)$.

Proof of Claim 2 By induction on $n < \omega$.

Case 1. $n = 0$.

Since $p \restriction \beta_0 \in N \cap P_{\beta_0}$, by induction hypothesis, take $p_0 \leq p \restriction \beta_0$ which is (N, P_{β_0})-generic and satisfies (5). Put $\dot{r}_0 = p \restriction [\beta_0, \beta)$.
Case 2. $n = m + 1$.
Case 2.1. $\tau(m) = j$.

Work in $\mathbf{V}^{P_{\beta_m}}$ below p_m. Since $\dot{r}_m \in P_\beta / P_{\beta_m} \cap N[\dot{\mathcal{G}}_{P_{\beta_m}}]$ and $\dot{\xi}_j \in N[\dot{\mathcal{G}}_{P_{\beta_m}}]$, we can take $\dot{r} \in N[\dot{\mathcal{G}}_{P_{\beta_m}}] \cap P_\beta / P_{\beta_m}$ such that

$\dot{r} \leq \dot{r}_m$ and $\dot{r}$ decides the value of $\dot{\xi}_j$.

Put $\dot{r}_n = \dot{r} \restriction [\beta_n, \beta)$. Take $\dot{u} \in P_{\beta_n} / P_{\beta_m}$ such that

(7) $\dot{u}$ is $(N[\dot{\mathcal{G}}_{P_{\beta_m}}], P_{\beta_n}/P_{\beta_m})$-generic and $\dot{u} \leq \dot{r} \restriction [\beta_m, \beta_n)$.

(8) $\dot{u} \Vdash \forall x \in N[\dot{\mathcal{G}}_{P_{\beta_m}}][\dot{\mathcal{G}}_{P_{\beta_n}/P_{\beta_m}}] \cap [\omega]^\omega$ ($x \cap a_k \in [\omega]^\omega$), for all $k < \omega$.

(9) $\mathrm{support}(\dot{u}) \subset N[\dot{\mathcal{G}}_{P_{\beta_m}}]$.

By (9), we can take $p_n \in P_{\beta_n}$ such that

(10) $p_n \restriction \beta_m = p_m$ and $p_m \Vdash \dot{u} = p_n \restriction [\beta_m, \beta_n)$.

Then, p_n and $\dot{r}_n$ satisfy (1) $\sim$ (6).
Case 2.2. $\tau(m) = (j, k)$.

In $N[\dot{\mathcal{G}}_{P_{\alpha_m}}]$, take an interpretation $\dot{y}$ of $\dot{x}_k$ below $\dot{r}_m$ such that $\dot{y}$ is infinite. By induction hypothesis (5), $\dot{y} \cap a_j$ is infinite. So, take $\dot{u} \in [\dot{y} \cap a_j]^n$. Since $\dot{u} \in N[\dot{\mathcal{G}}_{P_{\alpha_m}}]$, there exists $\dot{r}' \leq \dot{r}_m$ such that $\dot{r}' \in N[\dot{\mathcal{G}}_{P_{\alpha_m}}]$ and $\dot{r}' \Vdash \dot{u} \subset \dot{x}_k$. Let $\dot{r}_n = \dot{r}' \restriction [\beta_m, \beta_n)$. By using a similar argment of the case 2.1, take p_n which satisfy (1) and (4). Then, p_n and $\dot{r}_n$ are as required. QED of Claim 2

Put $p' = \bigcup_{n<\omega} p_n$. By (1) and (6.1), p' is (N, P_β)-generic. We complete the proof by showing that

$p' \Vdash \dot{x}_k \cap a_j \in [\omega]^\omega$, for all $k, j < \omega$.

So, let $k, j < \omega$. It suffices to show that

$p' \Vdash |\dot{x}_k \cap a_j| \geq m$, for all $m < \omega$.

So, let $m < \omega$. Take $n < \omega$ such that $m < n$ and $\pi(n) = (j, k)$. Then, by (6.2), it holds that

$p_{n+1} \Vdash \dot{r}_{n+1} \Vdash |\dot{x}_k \cap a_j| \geq n$.

By this, since $p' \restriction \beta_{n+1} \Vdash p' \restriction [\beta_{n+1}, \beta) \leq \dot{r}_{n+1}$, we have that

$p' \Vdash |\dot{x}_k \cap a_j| \geq m$. □

In order to give a proof of that **LT** has the intersection property, we need the following two lemmas.

Lemma 4.3 *Let $q \in$ **LT** and $\dot{x}$ **LT**-name such that $\Vdash \dot{x} \in \mathbf{V}$. Then, there exist $q^* \leq_0^* q$ and $S \subset q^*$ such that*

(1) S generates q^*.

(2) $q^*[s]$ decides the value of $\dot{x}$, for all $s \in S$.

Proof Let

$S_0 = \{\, s \in q \cap [\mathrm{stem}(q)] \mid$ there is an $r \leq_0^* q[s]$ such that r decides $\dot{x}\,\}$,

$S_1 = \{ s \in S_0 \mid s \restriction j \notin S_0, \text{ for all } j < |s| \}$.

It is not difficult to check $\mathrm{stem}(q) \in \mathrm{cl}(S_1)$. So, we can take $q' \leq_0^* q$ such that

(3) $q' \cap S_1$ generates q'.

(4) $q'[s] = q[s]$, for all $s \in S_1 \cap q'$.

For each $s \in S_1 \cap q'$, take $r_s \leq_0^* q[s]$ such that r_s decides $\dot{x}$. Then, $S = S_1 \cap q'$ and $q^* = \bigcup_{s \in S} r_s$ are as required. □

Lemma 4.4 *Suppose that N is a countable elementary substructure of $H(\lambda)$, and $a \in [\omega]^\omega$ satisfies*

$\forall x \in N \cap [\omega]^\omega \ (x \cap a \in [\omega]^\omega)$.

Let $\dot{x} \in N$ be a **LT***-name such that $\Vdash \dot{x} \in [\omega]^\omega$. Then, for any $m < \omega$ and any $q \in N \cap \mathbf{LT}$, there exist $q' \leq_0^* q$ and $S \subset q' \cap [\mathrm{stem}(q')]$ such that*

(1) *S generates q'.*

(2) *$q'[s] \in N$ and $\exists u \in [a]^m \ (q'[s] \Vdash u \subset \dot{x})$, for any $s \in S$.*

Proof We first deal with the case $m = 1$. Work in N. Let $\dot{f}$ be the name of the characteristic function of $\dot{x}$. By Lemma 2.5, take $q' \leq_0^* q$ such that q is strongly $\dot{f}$-simple, and the following (a) or (b) holds.

(a) $q' \Vdash \dot{f}[q'] = \dot{f}$.

(b) $S_0 = \{ s \in q' \cap [\mathrm{stem}(q')] \mid \dot{f}[q'[s]] = \dot{f}[q'] \text{ and } \dot{f}[q'[s]] \neq \dot{f}[q'[t]], \text{ for some/any } t \in \mathrm{succ}_{q'}(s) \}$ generates q'.

Let $g = \dot{f}[q]$ and $y = g^{-1}\{1\}$.

Case 1. (a) holds.

Returne to $\mathbf{V}$. Since y is infinite and belongs to N, there exists $k \in a \cap y$. Note that $q' \Vdash k \in \dot{x}$. So, q' and $\{ \mathrm{stem}(q') \}$ are as required.

Case 2. (b) holds.

Case 2.1. y is infinite.

Returne to $\mathbf{V}$. Take $k \in a \cap y$. In N, take $q'' \leq_0^* q'$ such that

$q'' \Vdash g \restriction (k+1) \subset \dot{f}$.

Then, it holds that $q'' \Vdash k \in \dot{x}$. So, q'' and $\{ \mathrm{stem}(q'') \}$ are as required.

Case 2.2. y is finite.

Take $n < \omega$ such that $y \subset n$. Put $T = \mathrm{Succ}_{q'}(S_0)$. For each $t \in T$, set

$k_t = \min\{ k < \omega \mid n \leq k \text{ and } \dot{f}[q'[t]](k) = 1 \}$,

and for each $s \in S_0$, set

$x_s = \{ k_t \mid t \in \mathrm{succ}_{q'}(s) \}$.

Return to $\mathbf{V}$. For each $s \in S_0$, since $x_s \in N \cap [\omega]^\omega$, by the assumption, it holds that $\exists^\infty t \in \mathrm{succ}_{q'}(s) \ (k_t \in a)$. For each $s \in S_0$, set

$b_s = \{ t \in \mathrm{succ}_{q'}(s) \mid k_t \in a \}$.

For each $s \in S_0$ and $t \in b_s$, take $r_t \leq_0^* q'[t]$ such that $r_t \in N$ and $r_t \Vdash k_t \in \dot{x}$. Set $q'' = \bigcup_{s \in S_0} \bigcup_{t \in b_s} r_t$ and $S = \mathrm{Succ}_{q''}(S_0)$. Then, q'' and S are as required.

Now, we deal the case $m = n + 1$. By induction hypothesis, take $q' \leq_0^* q$

and $S \subset q'$ which satisfy (1) and (2). Let $s \in S$. Take $u_s \in [a]^n$ such that $q'[s] \Vdash u_s \subset \dot{x}$. By using the result of $m = 1$, take $r_s \leq_0^* q'[s]$ and $T_s \subset r_s$ such that

(1)' T_s generates r_s.

(2)' $r_s[t] \in N$ and $\exists k \in a$ ($r_s[t] \Vdash k \in \dot{x} \setminus u_s$), for any $t \in T_s$.

Then, $q'' = \bigcup_{s \in S} r_t$ and $S' = \bigcup_{s \in S} T_s$ are as required. □

Lemma 4.5 *Laver forcing* **LT** *has the intersection property.*

Proof Let N be a countable elementary substructure of $H(\lambda)$ and $\{ a_j \mid j < \omega \} \subset [\omega]^\omega$ such that

$$\forall x \in N \cap [\omega]^\omega \; (x \cap a_j \in [\omega]^\omega), \text{ for all } j < \omega.$$

Take enumerations $\langle \dot{\xi}_j \mid j < \omega \rangle$ and $\langle \dot{x}_k \mid k < \omega \rangle$ of the sets $\{ \dot{\xi} \in N \mid \dot{\xi}$ is a **LT**-name and $\Vdash \dot{\xi} \in \mathbf{On} \}$ and $\{ \dot{s} \in N \mid \dot{x}$ is a **LT**-name and $\Vdash \dot{x} \in [\omega]^\omega \}$,respectively. Take a surjection $\pi : \omega \to \omega \cup \omega^2$ such that

$$\forall k, j < \omega \; \exists^\infty n < \omega \; (\pi(n) = (j, k)).$$

To show this lemma, let $q \in \mathbf{LT} \cap N$. We first show that, by induction on $n < \omega$, we can construct a fusin sequence $\langle (q_n, S_n) \mid n < \omega \rangle$ below q which satisfies that, for all $s \in S_n$,

(1) $q_n[s] \in N$,

(2) if $\pi(n) = j$, then $\forall t \in S_{n+1}$ ($q_{n+1}[t]$ decides $\dot{\xi}_j$),

(3) if $\pi(n) = (j, k)$, then $\forall t \in S_{n+1} \exists u \in [a_j]^n$ ($q_{n+1}[t] \Vdash u \subset \dot{x}_k$).

Case 1. $n = 0$.

The pair $q_0 = q$ and $S_0 = \{ \mathrm{stem}(q_0) \}$ satisfy the requirement.

Case 2. $n = m + 1$.

Let $S' = \mathrm{Succ}_{q_m}(S_m)$.

Cade 2.1. $\pi(m) = j$.

For each $s \in S'$, by using Lemma 4.3, take $r_s \leq_0^* q_m[s]$ and $T_s \subset r_s$ such that

(4) T_s generates r_s

(5) $r_s[t] \in N$ and $r_s[t]$ decides the value of $\dot{\xi}_j$, for all $t \in T_s$.

Then, $q_n = \bigcup_{s \in S'} \bigcup_{t \in T_s} r_s$ and $S_n = \bigcup_{s \in S'} T_s$ are as required.

Case 2.2. $\pi(m) = (j, k)$.

For each $s \in S'$, by using Lemma 4.4, take $r_s \leq_0^* q_m[s]$ and $T_s \subset r_s$ such that

(6) T_s generates r_s

(7) $r_s[t] \in N$ and $\exists u \in [a_j]^n$ ($r_s[t] \Vdash u \subset \dot{x}_k$), for all $t \in T_s$.

Then, $q_n = \bigcup_{s \in S'} \bigcup_{t \in T_s} r_s$ and $S_n = \bigcup_{s \in S'} T_s$ are as required.

Let $q^* = \bigcap_{n<\omega} q_n \in \mathbf{LT}$. By (1) $\sim$ (3), q^* is as required. $\square$

Corollary 4.6 *Ever countable support iteration by Laver forcing has the intersection property.* $\square$

Corollary 4.7 (CH) *Let P be a countable support iteration by Laver forcing. Then, it holds that*

$\mathbf{V}^P \models$ *the splitting number of* $[\omega]^\omega = \omega_1$. $\square$

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