<table>
<thead>
<tr>
<th>Title</th>
<th>A weak basis theorem for $\Pi_2^1$ sets of positive measure</th>
</tr>
</thead>
<tbody>
<tr>
<td>Author(s)</td>
<td>Fujita, Hiroshi</td>
</tr>
<tr>
<td>Citation</td>
<td>数理解析研究所講究録 (2000), 1143: 55-59</td>
</tr>
<tr>
<td>Issue Date</td>
<td>2000-04</td>
</tr>
<tr>
<td>URL</td>
<td><a href="http://hdl.handle.net/2433/63898">http://hdl.handle.net/2433/63898</a></td>
</tr>
<tr>
<td>Type</td>
<td>Departmental Bulletin Paper</td>
</tr>
<tr>
<td>Textversion</td>
<td>publisher</td>
</tr>
</tbody>
</table>

Kyoto University
A weak basis theorem for $\Pi_2^1$ sets of positive measure

Hiroshi Fujita, Ehime University
(愛媛大学理学部 藤田博司)

Abstract

We give a weak basis result for $\Pi_2^1$ sets of positive measure, which is closely related to our previous paper [2] in which we have assumed the existence of $0^\#$. This note is devoted to the following

Theorem 1 Let $s \in 2^\omega$ be a real such that $\aleph_1^L$ is a recursive-in-$s$ ordinal. Then every $\Pi_2^1$ set of positive measure contains a $\Delta_1^1(s)$ member.

This theorem is closely related to the main theorem of our previous paper [2]: if $0^\#$ exists, then every $\Pi_2^1$ set of positive measure contains a member which is arithmetical in $0^\#$. Indeed, letting $s = 0^\#$ the hypothesis of our present theorem is achieved and this almost (but not literally) proves our older theorem. The hypothesis in the present result is weaker than that of the "$0^\#$ version." Therefore, it seems to be applicable to wider context — See Section 3 for some discussion on $L$-generic models in which there is a $\Pi_2^1$ singleton $s$ satisfying the hypothesis of Theorem 1.

1 Tools

Let us fix, once for all, a recursive bijection between $\omega \times \omega$ and $\omega$. By the notation $\langle i, j \rangle$ we mean both the ordered pair and the integer which is assigned to this ordered pair by the fixed bijection. Each real $r \in 2^\omega$ codes a binary relation $\leq_r$ defined as

$$i \leq_r j \iff r(\langle i, j \rangle) = 1$$

Let $\text{WO}$ be the set of reals $r \in 2^\omega$ such that $\leq_r$ well-orders $\omega$. For $r \in \text{WO}$, let $\|r\|$ be the order-type of the wellordering $\leq_r$. A countable ordinal $\xi$ is said to be recursive-in-$a$ if $\xi = \|r\|$ for some real $r \in \text{WO}$ which is recursive in $a$.

The smallest ordinal which is not recursive-in-$a$ is denoted by $\omega_1^a$. Then $\omega_1^a$ equals the smallest ordinal $\xi > \omega$ such that the structure $(L_\xi(a), \in, a)$ is
admissible. A real $x$ is hyperarithmetic in $a$ if and only if it is $\Delta^1_1(a)$ if and only if it belongs to $L_{\omega^a}$. 

For a countable ordinal $\xi$ let $\text{WO}^{*}(\xi)$ be the set of $r \in \text{WO}$ with $\|r\| < \xi$.

For each countable $\xi$, the set $\text{WO}^{*}(\xi)$ is Borel. Indeed we have:

**Lemma 1.1** Let $s \in 2^\omega$. Let $\xi$ be a recursive-in-$s$ ordinal. Then $\text{WO}^{*}(\xi)$ is a $\Delta^1_1(s)$ set.

**Proof.** Let $r \in \text{WO}$ be a real which is recursive in $s$ and satisfies $\xi = \|r\|$. Then a real $x$ belongs to $\text{WO}^{*}(\xi)$ if and only if it is an order-preserving mapping of $(\omega, \leq_x)$ into an initial segment of $(\omega, \leq_r)$, if and only if $x \in \text{WO}$ and there is no order-preserving mapping of $(\omega, \leq_r)$ into $(\omega, \leq_x)$. This gives a $\Delta^1_1(r)$ characterization of $\text{WO}^{*}(\xi)$.

Let $s \in 2^\omega$ be a real such that $\aleph^L_1$ is a recursive-in-$s$ ordinal. This readily implies $\aleph^L_1$ is countable. Under this assumption, every $\Pi^1_2$ set of reals is Lebesgue measurable. The main theorem is proved by examining how this measurability is realized in a certain effective way. To this end, we need two $\Delta^1_1(s)$ sets: Lemma 1.1 implies that the set $\text{WO}(\aleph^L_1)$ of codes of constructibly countable well-ordering is $\Delta^1_1(s)$. Next we see that there is a $\Delta^1_1(s)$ set $C$ of measure one consisting of random reals over $L$.

For a real $t \in 2^\omega$ and an integer $n \in \omega$, let $(t)_n$ be the real defined by: $(t)_n(i) = t((n,i))$. Each real codes a countable sequence of reals in this way.

**Lemma 1.2** There is a $\Delta^1_1(s)$ real $t$ such that

$$\{ (t)_n : n \in \omega \} = 2^\omega \cap L.$$

**Proof.** For $2^\omega \cap L = 2^\omega \cap L_{\aleph^L_1}$, this set belongs to $L_{\omega^1}[s]$, the smallest admissible set containing $s$. Since $L_{\omega^1}[s]$ models "every set is countable," there exists in it a surjection $f : \omega \rightarrow 2^\omega \cap L_{\aleph^L_1}$. Let $t((n,i)) = f(n)(i)$.

Let $U \subset 2^\omega \times 2^\omega$ be a $\Pi^1_2$ set which is universal for $\Pi^1_2$. Let $t$ be a real as in Lemma 1.2. Let $C \subset 2^\omega$ be the following set

$$C = \{ x \in 2^\omega : (\forall y \in 2^\omega \cap L)[\mu(U_y) = 0 \implies x \notin U_y] \}$$

$$= \{ x \in 2^\omega : (\forall n)[\mu(U_{(t)_n}) = 0 \implies x \notin U_{(t)_n}] \},$$

where $\mu$ denotes the Lebesgue measure. Then $C$ is a $\Delta^1_1(s)$ set such that $\mu(C) = 1$.

**Lemma 1.3** Every $x \in C$ is random over $L$. Consequently the equality $\aleph^L_1[x] = \aleph^L_1$ holds for all $x \in C$.

2 Reducing $\Pi^1_2$ sets to $\Pi^1_1(s)$

Let $P$ be a $\Sigma^1_2$ set of reals, then there is a recursive function $f : 2^\omega \times 2^\omega \rightarrow 2^\omega$ such that

$$x \in P \iff (\exists y)[f(x,y) \in \text{WO}].$$
By the Shoenfield Absoluteness Lemma, it is equivalent to say
\[ x \in P \iff (\exists y \in 2^\omega \cap L[x]) [ f(x, y) \in \text{WO}(\kappa_1^L[x])]. \]

In such a case, we have \( f(x, y) \in L[x] \). So \( \| f(x, y) \| < \kappa_1^L[x] \). It follows that
\[ x \in P \iff (\exists y \in 2^\omega \cap L[x]) [ f(x, y) \in \text{WO}(\kappa_1^L[x])]. \]

By these observations, we have:

**Lemma 2.1** Let \( P \) be a \( \Sigma_2^1 \) set of reals, then there is a recursive function \( f : 2^\omega \times 2^\omega \to 2^\omega \) such that
\[ x \in P \iff (\exists y)[ f(x, y) \in \text{WO}(\kappa_1^L[x])]. \]

Now let \( A \) be a \( \Pi_2^1 \) set of reals. Put \( P = 2^\omega \setminus A \), then by Lemmas 1.3 and 2.1, there is a recursive function \( f : 2^\omega \times 2^\omega \to 2^\omega \) such that
\[ x \in C \implies [ x \in A \iff (\forall y)[ f(x, y) \notin \text{WO}(\kappa_1^L)]]. \]

Therefore we have

**Lemma 2.2** Let \( A \) and \( f \) as above. Then
\[ A \cap C = \{ x \in 2^\omega : x \in C \& (\forall y)[ f(x, y) \notin \text{WO}(\kappa_1^L)] \}. \]

Consequently, \( A \cap C \) is a \( \Pi_1^1(\mathfrak{s}) \) set.

If \( A \) has positive Lebesgue measure, so is \( A \cap C \), for \( C \) contains almost all reals. Being a \( \Pi_1^1(\mathfrak{s}) \) set of positive measure, \( A \cap C \) contains a \( \Delta_1^1(\mathfrak{s}) \) real by the Sacks-Tanaka Basis Theorem ([4], Chap.IV, 2.2). Thus we have proved the main theorem.

### 3 Some remarks

Theorem 1 would be of no interest unless there exists a definable real which makes \( R_1^L \) countable. The simplest way to make \( R_1^L \) countable is to add to \( L \) a generic function on \( \omega \) onto \( R_1^L \) by forcing with finite partial functions. This forcing adds no ordinal-definable reals. Hence in the generic extension the non-constructible reals form a \( \Pi_2^1 \) set of positive measure which does not contain any ordinal-definable real.

Much finer method to force \( R_1^L \) countable have been invented by Jensen and Solovay. In [3] they give a forcing notion \( P \in L \) and a \( \Pi_2^1 \) formula \( \varphi \) such that if \( G \subset P \) is generic then there exists a real \( a \in V[G] \) such that

1. \( L[a] \models (\forall x \subset \omega)[ \varphi(x) \iff x = a]; \)
2. every constructible real is recursive in \( a \).
Clause 2 implies that the real $a$ is non-constructible. Hence, in $L[a]$, $a$ is a non-constructible $\Pi^1_2$ singleton. (See Theorem B of [1] for a yet sharper result along this line.)

Now let $a$ be as above and $s = \mathcal{O}^a$, the hyperjump of $a$. That is to say, $s$ is the set of notations of constructive ordinals relative to $a$. (See Chapter I of [4].) If you are not familiar with theory of hyperarithmetic hierarchy, you can use here the set $\{ e \in \omega : \{ e \}^a \in \text{WO} \}$ instead of $\mathcal{O}^a$. Since every ordinal below $\aleph_1^L$ is recursive-in-$a$, we have $\aleph_1^s \leq \omega_1^a < \omega_1^s$. In $L[a]$, on the other hand, $s$ is a $\Pi^1_2$ singleton for, in $L[a]$,

$$ x = s \iff (\forall y)[y = \{ e_0 \}^x \implies \varphi(y) \& x = \mathcal{O}^y], $$

where $e_0$ is a universal Gödel number which retrieves $y$ from $\mathcal{O}^y$. Thus in the Jensen-Solovay model, there is a $\Pi^1_2$ singleton $s$ such that $\aleph_1^s$ is a recursive-in-$s$ ordinal:

**Theorem 2** There is a model of ZFC in which $0^d$ does not exist while every $\Pi^1_2$ set of reals is Lebesgue measurable and every positive-measure $\Pi^1_2$ set contains $\Delta^1_3$ members.

In this model, however, exists a $\Delta^1_3$ real $r$ such that there exists a non-measurable $\Pi^1_2(r)$ set. Can we somehow multiply the Solovay-Jensen method to obtain an $L$-generic model of: For every real $r$ every $\Pi^1_2(r)$ set is Lebesgue measurable and if it has positive measure then it contains $\Delta^1_3(r)$ members?

Our hypothesis of Theorem 1 "$\aleph_1^s$ is a recursive-in-$s$ ordinal" seems quite essential, for otherwise $\text{WO}(\aleph_1^s)$ is not a $\Sigma^1_1(s)$ set. We do not know whether this hypothesis can be weakened to "every ordinal below $\aleph_1^L$ is recursive in $s$," or equivalently, "every constructible real is $\Delta^1_1(s)$." Let us note here that this condition is strictly weaker than the one in Theorem 1:

**Theorem 3** There is a real $s \in 2^\omega$ in which every constructible real is recursive whereas $\aleph_1^s$ is not a recursive-in-$s$ ordinal.

*Proof.* A model $\mathcal{M} = (M, \in_M)$ of set theory is called an $\omega$-model if all $\mathcal{M}$-integers are standard. Let us say an $\omega$-model $\mathcal{M}$ to be nice if $M = \omega$ and the natural sequence $((n)^M : n \in \omega)$ of the $\mathcal{M}$-integers is recursive in the real world. Every countable $\omega$-model has an isomorphic copy which is nice.

Let $a \subset \omega$ be a real such that $\aleph_1^a = \omega_1^a$. Then let $\Psi$ be the set of reals $r \in 2^\omega$ which codes the $\in$-relation of a non-wellfounded nice $\omega$-model of KP set theory in which an instance of $a$ exists. Then $\Psi$ is a non-empty $\Sigma^1_1(a)$ set. Therefore by the Gandy Basis Theorem (see, [4] Chap.III, 1.5), there is an $s \in \Psi$ such that $\omega_1^{(a,s)} = \omega_1^a = \aleph_1^s$.

Let $M$ be the model coded by $s$. Since $M$ contains an instance of $a$, it follows that $\omega_1^s \leq \omega_1^s$. Hence $\aleph_1^s = \aleph_1^L$. Each non-standard ordinal in $M$ has order type $\omega_1^s \times (1 + \text{OrderType}(Q, \prec)) + \rho$ for some $\rho < \omega_1^s$. Therefore for each ordinal $\xi < \omega_1^s$ the set $L_\xi$ is isomorphic to an initial part of the constructible hierarchy.

\[ \text{ZFC} \]
in $M$. It follows that $M$ contains instances of all sets in $L_{\aleph_1^L}$. From this it follows that every constructible real is recursive in $s$.

□

References


