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A weak basis theorem for $\Pi^1_2$ sets of positive measure

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Abstract

We give a weak basis result for $\Pi^1_2$ sets of positive measure, which is closely related to our previous paper [2] in which we have assumed the existence of $0^\sharp$. This note is devoted to the following

Theorem 1 Let $s \in 2^\omega$ be a real such that $\aleph_1^L$ is a recursive-in-$s$ ordinal. Then every $\Pi^1_2$ set of positive measure contains a $\Delta^1_1(s)$ member.

This theorem is closely related to the main theorem of our previous paper [2]: if $0^\sharp$ exists, then every $\Pi^1_2$ set of positive measure contains a member which is arithmetical in $0^\sharp$. Indeed, letting $s = 0^\sharp$ the hypothesis of our present theorem is achieved and this almost (but not literally) proves our older theorem. The hypothesis in the present result is weaker than that of the "$0^\sharp$ version." Therefore, it seems to be applicable to wider context — See Section 3 for some discussion on $L$-generic models in which there is a $\Pi^1_2$ singleton $s$ satisfying the hypothesis of Theorem 1.

1 Tools

Let us fix, once for all, a recursive bijection between $\omega \times \omega$ and $\omega$. By the notation $\langle i,j \rangle$ we mean both the ordered pair and the integer which is assigned to this ordered pair by the fixed bijection. Each real $r \in 2^\omega$ codes a binary relation $\leq_r$ defined as

$$i \leq_r j \iff r(\langle i, j \rangle) = 1$$

Let $WO$ be the set of reals $r \in 2^\omega$ such that $\leq_r$ well-orders $\omega$. For $r \in WO$, let $||r||$ be the order-type of the wellordering $\leq_r$. A countable ordinal $\xi$ is said to be recursive-in-$a$ if $\xi = ||r||$ for some real $r \in WO$ which is recursive in $a$.

The smallest ordinal which is not recursive-in-$a$ is denoted by $\omega^a_1$. Then $\omega^a_1$ equals the smallest ordinal $\xi > \omega$ such that the structure $(L_\xi(a), \in, a)$ is...
admissible. A real $x$ is hyperarithmetical in $a$ if and only if it is $\Delta_1^1(a)$ if and only if it belongs to $L_{\omega_1^a}(a)$.

For a countable ordinal $\xi$ let $\text{WO}(\xi)$ be the set of $r \in \text{WO}$ with $\|r\| < \xi$. For each countable $\xi$, the set $\text{WO}(\xi)$ is Borel. Indeed we have:

Lemma 1.1 Let $s \in 2^\omega$. Let $\xi$ be a recursive-in-$s$ ordinal. Then $\text{WO}(\xi)$ is a $\Delta_1^1(s)$ set.

Proof. Let $r \in \text{WO}$ be a real which is recursive in $s$ and satisfies $\xi = \|r\|$. Then a real $x$ belongs to $\text{WO}(\xi)$ if and only if there is an order-preserving mapping of $(\omega, \leq_s)$ into an initial segment of $(\omega, \leq_r)$, if and only if $x \in \text{WO}$ and there is no order-preserving mapping of $(\omega, \leq_r)$ into $(\omega, \leq_s)$. This gives a $\Delta_1^1(r)$ characterization of $\text{WO}(\xi)$.

Let $s \in 2^\omega$ be a real such that $\aleph_1^L$ is a recursive-in-$s$ ordinal. This readily implies $\aleph_1^L$ is countable. Under this assumption, every $\Pi_2^1$ set of reals is Lebesgue measurable. The main theorem is proved by examining how this measurability is realized in a certain effective way. To this end, we need two $\Delta_1^1(s)$ sets: Lemma 1.1 implies that the set $\text{WO}(\aleph_1^L)$ of codes of constructively countable well-ordering is $\Delta_1^1(s)$. Next we see that there is a $\Delta_1^1(s)$ set $C$ of measure one consisting of random reals over $L$.

For a real $t \in 2^\omega$ and an integer $n \in \omega$, let $(t)_n$ be the real defined by: $(t)_n(i) = t((n,i))$. Each real codes a countable sequence of reals in this way.

Lemma 1.2 There is a $\Delta_1^1(s)$ real $t$ such that

$$\{(t)_n : n \in \omega\} = 2^\omega \cap L.$$

Proof. For $2^\omega \cap L = 2^\omega \cap L_{\aleph_1^L}$, this set belongs to $L_{\omega_1^L}[s]$, the smallest admissible set containing $s$. Since $L_{\omega_1^L}[s]$ models “every set is countable,” there exists in it a surjection $f : \omega \rightarrow 2^\omega \cap L_{\aleph_1^L}$. Let $t(n,i) = f(n)(i)$.

Let $U \subset 2^\omega \times 2^\omega$ be a $\Pi_2^0$ set which is universal for $\Pi_2^0$. Let $t$ be a real as in Lemma 1.2. Let $C \subset 2^\omega$ be the following set

$$C = \{ x \in 2^\omega : (\forall y \in 2^\omega \cap L)[\mu(U_y) = 0 \implies x \notin U_y] \} = \{ x \in 2^\omega : (\forall n)[\mu(U_{(t)_n}) = 0 \implies x \notin U_{(t)_n}] \}.$$

where $\mu$ denotes the Lebesgue measure. Then $C$ is a $\Delta_1^1(s)$ set such that $\mu(C) = 1$.

Lemma 1.3 Every $x \in C$ is random over $L$. Consequently the equality $\aleph_1^L[x] = \aleph_1^L$ holds for all $x \in C$.

2 Reducing $\Pi_2^1$ sets to $\Pi_1^1(s)$

Let $P$ be a $\Sigma_2^1$ set of reals, then there is a recursive function $f : 2^\omega \times 2^\omega \rightarrow 2^\omega$ such that

$$x \in P \iff (\exists y)[f(x,y) \in \text{WO}].$$
By the Shoenfield Absoluteness Lemma, it is equivalent to say

\[ x \in P \iff (\exists y \in 2^\omega \cap L[x])[f(x, y) \in \text{WO}]. \]

In such a case, we have \( f(x, y) \in L[x] \). So \( \|f(x, y)\| < \aleph_1^{L[x]} \). It follows that

\[ x \in P \iff (\exists y \in 2^\omega \cap L[x])[f(x, y) \in \text{WO}(\aleph_1^{L[x]})]. \]

By these observations, we have:

**Lemma 2.1** Let \( P \) be a \( \Sigma_2^1 \) set of reals, then there is a recursive function \( f : 2^\omega \times 2^\omega \to 2^\omega \) such that

\[ x \in P \iff (\exists y)[f(x, y) \in \text{WO}(\aleph_1^{L[x]})]. \]

Now let \( A \) be a \( \Pi_2^1 \) set of reals. Put \( P = 2^\omega \setminus A \), then by Lemmas 1.3 and 2.1, there is a recursive function \( f : 2^\omega \times 2^\omega \to 2^\omega \) such that

\[ x \in C \Rightarrow [x \in A \iff (\forall y)[f(x, y) \notin \text{WO}(\aleph_1^{L})]]. \]

Therefore we have

**Lemma 2.2** Let \( A \) and \( f \) as above. Then

\[ A \cap C = \{ x \in 2^\omega : x \in C \land (\forall y)[f(x, y) \notin \text{WO}(\aleph_1^{L})] \}. \]

Consequently, \( A \cap C \) is a \( \Pi_1^1 \) set.

If \( A \) has positive Lebesgue measure, so is \( A \cap C \), for \( C \) contains almost all reals. Being a \( \Pi_1^1 \) set of positive measure, \( A \cap C \) contains a \( \Delta_1^1 \) real by the Sacks-Tanaka Basis Theorem ([4], Chap.IV, 2.2). Thus we have proved the main theorem.

### 3 Some remarks

Theorem 1 would be of no interest unless there exists a definable real which makes \( R_1^L \) countable. The simplest way to make \( R_1^L \) countable is to add to \( L \) a generic function on \( \omega \) onto \( R_1^L \) by forcing with finite partial functions. This forcing adds no ordinal-definable reals. Hence in the generic extension the non-constructible reals form a \( \Pi_2^1 \) set of positive measure which does not contain any ordinal-definable real.

Much finer method to force \( R_1^L \) countable have been invented by Jensen and Solovay. In [3] they give a forcing notion \( P \in L \) and a \( \Pi_2^1 \) formula \( \varphi \) such that if \( G \subset P \) is generic then there exists a real \( a \in V[G] \) such that

1. \( L[a] \models (\forall x \subset \omega)[\varphi(x) \iff x = a]; \)
2. every constructible real is recursive in \( a \).
Clause 2 implies that the real $a$ is non-constructible. Hence, in $L[a]$, $a$ is a non-constructible $\Pi^1_2$ singleton. (See Theorem B of [1] for a yet sharper result along this line.)

Now let $a$ be as above and $s = O^a$, the hyperjump of $a$. That is to say, $s$ is the set of notations of constructive ordinals relative to $a$. (See Chapter I of [4].)

If you are not familiar with theory of hyperarithmetic hierarchy, you can use here the set $\{ e \in \omega : \{ e \}^a \in WO \}$ instead of $O^a$.) Since every ordinal below $\aleph_1^L$ is recursive-in-$a$, we have $\aleph_1^L \leq \omega_1^a < \omega_1^s$. In $L[a]$, on the other hand, $s$ is a $\Pi^1_2$ singleton for, in $L[a]$,

$$x = s \iff (\forall y)[y = \{ e_0 \}^x \implies \varphi(y) \& x = O^y],$$

where $e_0$ is a universal Gödel number which retrieves $y$ from $O^y$. Thus in the Jensen-Solovay model, there is a $\Pi^1_2$ singleton $s$ such that $\aleph_1^L$ is a recursive-in-$s$ ordinal:

**Theorem 2** There is a model of ZFC in which $0^d$ does not exist while every $\Pi^1_2$ set of reals is Lebesgue measurable and every positive-measure $\Pi^1_2$ set contains $\Delta^1_3$ members.

In this model, however, exists a $\Delta^1_3$ real $r$ such that there exists a non-measurable $\Pi^1_2(r)$ set. Can we somehow multiply the Solovay-Jensen method to obtain an $L$-generic model of: For every real $r$ every $\Pi^1_2(r)$ set is Lebesgue measurable and if it has positive measure then it contains $\Delta^1_3(r)$ members?

Our hypothesis of Theorem 1 “$\aleph_1^L$ is a recursive-in-$s$ ordinal” seems quite essential, for otherwise $WO(\aleph_1^L)$ is not a $\Sigma^1_1(s)$ set. We do not know whether this hypothesis can be weakened to “every ordinal below $\aleph_1^L$ is recursive in $s$,” or equivalently, “every constructible real is $\Delta^1_1(s)$.” Let us note here that this condition is strictly weaker than the one in Theorem 1:

**Theorem 3** There is a real $s \in 2^\omega$ in which every constructible real is recursive whereas $\aleph_1^L$ is not a recursive-in-$s$ ordinal.

**Proof.** A model $\mathcal{M} = (M, \in_M)$ of set theory is called an $\omega$-model if all $\mathcal{M}$-integers are standard. Let us say an $\omega$-model $\mathcal{M}$ to be nice if $M = \omega$ and the natural sequence $\langle (n)^\mathcal{M} : n \in \omega \rangle$ of the $\mathcal{M}$-integers is recursive in the real world. Every countable $\omega$-model has an isomorphic copy which is nice.

Let $a \subset \omega$ be a real such that $\aleph_1^L = \omega_1^a$. Then let $\Psi$ be the set of reals $r \in 2^\omega$ which codes the $\in$-relation of a non-wellfounded nice $\omega$-model of KP set theory in which an instance of $a$ exists. Then $\Psi$ is a non-empty $\Sigma^1_1(a)$ set. Therefore by the Gandy Basis Theorem (see, [4] Chap.III, 1.5), there is an $s \in \Psi$ such that $\omega_1^{(a,s)} = \omega_1^a = \aleph_1^L$.

Let $M$ be the model coded by $s$. Since $M$ contains an instance of $a$, it follows that $\omega_1^a \leq \omega_1^s$. Hence $\omega_1^a = \aleph_1^L$. Each non-standard ordinal in $M$ has order type $\omega_1^a \times (1 + \text{OrderType}(\mathbb{Q}, <)) + \rho$ for some $\rho < \omega_1^a$. Therefore for each ordinal $\xi < \omega_1^a$ the set $L_\xi$ is isomorphic to an initial part of the constructible hierarchy
in $M$. It follows that $M$ contains instances of all sets in $L_{\aleph_1^L}$. From this it follows that every constructible real is recursive in $s$.

References


