TOPOLOGICAL PROPERTIES OF
PRODUCTS OF ORDINAL NUMBERS

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INTRODUCTION

The greek letters $\alpha, \beta, \gamma, \ldots$ denote ordinal numbers with the usual order topologies. A space means a $T_1$ (every one point set is closed) topological space. A space $X$ is regular if every point $x \in X$ and every closed set $F$ with $x \notin F$ are separated by disjoint open sets. It is easy to verify that:

1. If $X$ and $Y$ are regular, then so is the product space $X \times Y$.
2. If $X$ is regular and $Y \subset X$, then $Y$ is regular.

A space $X$ is normal if every pair of disjoint closed sets are separated by disjoint open sets. All compact spaces are normal and all subspaces of ordinal numbers are also normal. On the other hand, $X = (\omega_1 + 1) \times \omega_1$ is not normal. Indeed, using the Pressing Down Lemma, we can show that the diagonal $\{\langle \alpha, \alpha \rangle \in X : \alpha < \omega_1 \}$ and the set $(\omega_1) \times \omega_1$ cannot be separated by disjoint open sets. Therefore:

1. $X = (\omega_1 + 1)$ and $Y = \omega_1$ are normal but $X \times Y$ is not normal.
2. $X = (\omega_1 + 1)^2$ is compact so normal, but the subspace $Y = (\omega_1 + 1) \times \omega_1$ of $X$ is not normal.

Thus the notion of normality is completely different from that of regularity.

The simplest non-trivial space is $\omega + 1$, that is, the convergent sequence with its unique limit point. The following famous result was proved by Dowker [Do]:

Dowker's Theorem. If $X$ is normal, then $X \times (\omega + 1)$ is normal iff $X$ is countably paracompact.

Here a space $X$ is said to be countably paracompact (countably metacompact) if for every countable open cover $U = \{U_n : n \in \omega \}$, there is a locally finite (point finite, respectively) open refinement $V$ of $U$, where $V$ is locally finite (point finite) if for every $x \in X$, there is a neighborhood $U$ of $x$ such that $\{V \in V : V \cap U \neq \emptyset \}$ is finite (if for every $x \in X$, $\{V \in V : x \in V \}$ is finite, respectively), moreover an open cover $V$ is said to be an open refinement of $U$ if for every $V \in V$, there is $U \in U$ with $V \subset U$.

Dowker asked in [Do] whether there exists a normal space which is not countably paracompact. More than twenty years later, M. E. Rudin constructed in [Ru] such a space in ZFC.

In these connections, we present more definitions. A space $X$ is CollectionWise Normal (abbreviated as CWN) if for every discrete collection $F$ of closed sets of $X$, there exists a disjoint (equivalently, discrete) collection $U = \{U(F) : F \in F \}$.
of open sets with \( F \subset U(F) \), where a collection \( \mathcal{F} \) is discrete if for every \( x \in X \), there is a neighborhood \( U \) of \( x \) with \( |\{F \in \mathcal{F} : U \cap F \neq \emptyset\}| \leq 1 \). A space \( X \) is expandable if for every locally finite collection \( \mathcal{F} \) of closed sets, there is a locally finite collection \( \mathcal{U} = \{U(F) : F \in \mathcal{F}\} \) of open sets with \( F \subset U(F) \). For every space, it is not difficult to verify:

1. \( \text{CWN} \rightarrow \text{normal} \).
2. \( \text{expandable} \rightarrow \text{countably paracompact} \rightarrow \text{countably metacompact} \).
3. \( \text{normal} + \text{countably metacompact} \rightarrow \text{countably paracompact} \).
4. \( \text{normal} + \text{expandable} \Leftrightarrow \text{CWN} + \text{countably paracompact} \).
5. \( (\omega_1 + 1) \times \omega_1 \) is expandable but not normal.

2. Results

In the past 10 years, we have investigated such topological properties described in section 1 of product spaces of ordinal numbers. In this section, \( \alpha \) denotes an arbitrary large ordinal number. First we proved in [KOT]:

**Theorem 1.** For every pair of subspaces \( A \) and \( B \) of \( \alpha \),

1. \( A \times B \) is normal iff it is CWN.
2. \( A \times B \) is countably paracompact iff it is expandable.
3. If \( A \times B \) is normal, then it is countably paracompact. Note that \( (\omega_1 + 1) \times \omega_1 \) is countably paracompact but not normal.
4. For every pair of subspaces \( A \) and \( B \) of \( \omega_1 \), \( A \times B \) is normal iff it is countably paracompact iff \( A \) or \( B \) are non-stationary or \( A \cap B \) is stationary. Thus, if \( A \) and \( B \) are disjoint stationary sets of \( \omega_1 \), then \( A \times B \) is neither normal nor countably paracompact.

We asked in [KOT]:

(a) Is \( A \times B \) countably metacompact for every pair of subspaces \( A \) and \( B \) of \( \alpha \)?
(b) Are normality and CWN equivalent for all subspaces of \( \alpha^2 \)?
(c) Are countable paracompactness and expandability equivalent for all subspaces of \( \alpha^2 \)?

On (a), we got in [KS1] and [KS2]:

**Theorem 2.**

1. All subspaces of \( \alpha^2 \) are countably metacompact.
2. All subspaces of \( \omega_1^n \) are countably metacompact for every \( n \in \omega \).
3. There is a subspace of \( \omega_1^\omega \) which is not countably metacompact.

After then we got an affirmative answer of (b) in [KNSY]:

**Theorem 3.** Normality and CWN are equivalent for all subspaces of \( \alpha^2 \).

However, the question (c) still remains open.

In connection with (4) of Theorem 1, we asked in [KNSY]:

(d) Are normality and countable paracompactness equivalent for all subspaces of \( \omega_2^n \)?

On (d), we proved in [KSS]:
Theorem 4. For every subspace $X$ of $\omega_1^2$,

1. $X$ is normal iff $X$ is expandable iff $X$ is countably paracompact and strongly collectionwise Hausdorff, where a space is strongly collectionwise Hausdorff (collectionwise Hausdorff) if for every subset $F$ of $X$ with the collection \{\{x\} : x \in F\} discrete, there is a discrete (disjoint, respectively) collection $\mathcal{U} = \{U(x) : x \in F\}$ of open sets with $x \in U(x)$.

2. If $V=L$ or the Product Measure Extension Axiom are assumed, then $X$ is normal iff $X$ is countably paracompact.

3. $X$ is collectionwise Hausdorff.

This theorem also says that the question (c) is closely related to (d).

3. On the question (d)

Now we conjecture that there is a model in which (d) is not true, that is, there is a countably paracompact but not normal subspace of $\omega_1^2$. American young mathematicians Eisworth, Just, Pavlov, Smith, Szeptycki are working on this problem. In discussion with them, we have had a candidate of such a subspace. The remaining is an unpublished work with them.

Let $\text{Lim} = \{\alpha < \omega_1 : \alpha \text{ is limit }\}$ and $\text{Succ} = \omega_1 \setminus \text{Lim}$. For each $\alpha \in \text{Lim}$, fix a strictly increasing $\omega$-sequence $L_\alpha$ cofinal in $\alpha$, moreover for simplicity of our discussion we assume $L_\alpha \subset \text{Succ}$. Then we call $\mathcal{L} = \{L_\alpha : \alpha \in \text{Lim}\}$ a ladder system. Set $L(\mathcal{L}) = \bigcup_{\alpha \in \text{Lim}} L_\alpha$, then $L(\mathcal{L}) \subset \text{Succ}$. The ladder space $X(\mathcal{L})$ determined by $\mathcal{L}$ is defined as follows:

$$X(\mathcal{L}) = \left[ \bigcup_{\alpha \in L(\mathcal{L})} \{\alpha\} \times \{\beta \in \text{Lim} : \alpha < \beta\} \right] \cup \left[ \bigcup_{\alpha \in \text{Lim}} (\{\alpha\} \cup L_\alpha) \times \{\alpha + 1\} \right].$$

This is our candidate. The following are proved in our discussion:

1. In ZFC, $X(\mathcal{L})$ is not normal for every ladder system $\mathcal{L}$.
2. If $\text{MA}(\omega_1)$ is assumed, then for every ladder system $\mathcal{L}$, $X(\mathcal{L})$ is not countably paracompact. In fact, $\text{MA}(\omega_1)$ destroys the property $(+)$ below.
3. In ZFC, $X(\mathcal{L})$ is not countably paracompact for some ladder system $\mathcal{L}$.

So our conjecture is:

(d') In some model, there is a ladder system $\mathcal{L}$ such that $X(\mathcal{L})$ is countably paracompact.

Finally we present a combinatorial equivalent property due to Pavlov and Szeptycki, independently.

4. Let $\mathcal{L}$ be a ladder system. Then $X(\mathcal{L})$ is countably paracompact iff $\mathcal{L}$ satisfies the following two properties (WU) and $(+)$:

(WU) $\forall f : \text{Lim} \to \omega \exists g : L(\mathcal{L}) \to [\omega]^{\omega} \forall \alpha \in \text{Lim}(\{\beta \in L_\alpha : f(\alpha) \neq g(\beta)\} < \omega)$.

$(+)$ $\forall f : L(\mathcal{L}) \to \omega(\{\alpha \in \text{Lim} : |f''L_\alpha| = \omega\} \text{ is not stationary})$.

So the conjecture (d') can be written as:

(d'') In some model, there is a ladder system $\mathcal{L}$ satisfying both (WU) and $(+)$. 

REFERENCES


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