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LOGARITHMIC ORDER AND DUAL LOGARITHMIC ORDER

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Abstract. We shall define the following four orders for strictly positive operators $A$ and $B$ on a Hilbert space $H$ such that $1 \not\in \sigma(A), \sigma(B)$.

- **Strictly logarithmic order** (denoted by $A \succ_{sl} B$) is defined by \[ \frac{A - I}{\log A} > \frac{B - I}{B \log B} \]
- **Logarithmic order** (denoted by $A \succ_{l} B$) is defined by \[ \frac{A - I}{\log A} \geq \frac{B - I}{\log B} \]
- **Strictly dual logarithmic order** (denoted by $A \succ_{sdl} B$) is defined by \[ \frac{A \log A}{A - I} > \frac{B \log B}{B - I} \]
- **Dual Logarithmic order** (denoted by $A \succ_{dl} B$) is defined by \[ \frac{A \log A}{A - I} \geq \frac{B \log B}{B - I} \]

Firstly we shall show direct and simplified proofs of operator monotonicity of logarithmic function $f(t) = \frac{t - 1}{\log t}$ and dual logarithmic function $f^{*}(t) = \frac{t \log t}{t - 1}$.

In what follows, let $A$ and $B$ be strictly positive operators on a Hilbert space $H$ such that $1 \not\in \sigma(A), \sigma(B)$. Secondary we shall show the following:

(\ast) $\log A > \log B \implies$ there exists $\beta \in (0, 1]$ such that $A^{\alpha} \succ_{sl} B^{\alpha}$ holds for all $\alpha \in (0, \beta)$.

(\dag) $\log A > \log B \implies$ there exists $\beta \in (0, 1]$ such that $A^{\alpha} \succ_{sdl} B^{\alpha}$ holds for all $\alpha \in (0, \beta)$.

By using these two results (\ast) and (\dag), we summarize the following interesting contrast among $A > B > 0, A \geq B > 0, \log A > \log B$ and $\log A \geq \log B$.

(i) $A > B > 0 \implies$ there exists $\beta \in (0, 1]$ such that $A^{\alpha} \succ_{sl} B^{\alpha}$ holds for all $\alpha \in (0, \beta)$.

(ii) $A \geq B > 0 \implies A^{\alpha} \succ_{l} B^{\alpha}$ for all $\alpha \in (0, 1]$.

(iii) $\log A \geq \log B \implies$ for any $\delta \in (0, 1]$, there exists $\beta = \beta_{\delta} \in (0, 1]$ such that $(e^{\delta} A)^{\alpha} \succ_{sl} B^{\alpha}$ holds for all $\alpha \in (0, \beta)$.

(iv) $\log A \geq \log B \implies$ for any $p \geq 0$ there exists $K_{p} > 1$ such that $K_{p} \to 1$ as $p \to +0$ and $(K_{p} A)^{p \alpha} \succ_{l} B^{p \alpha}$ for all $\alpha \in (0, 1]$.

(i-ii) $A > B > 0 \implies$ there exists $\beta \in (0, 1]$ such that $A^{\alpha} \succ_{sdl} B^{\alpha}$ holds for all $\alpha \in (0, \beta)$.

(i-ii) $A \geq B > 0 \implies A^{\alpha} \succ_{dl} B^{\alpha}$ for all $\alpha \in (0, 1]$.

(i-iii) $\log A \geq \log B \implies$ for any $\delta \in (0, 1]$, there exists $\beta = \beta_{\delta} \in (0, 1]$ such that $(e^{\delta} A)^{\alpha} \succ_{sdl} B^{\alpha}$ holds for all $\alpha \in (0, \beta)$. 

(ii-ii) $A > B > 0 \implies$ there exists $\beta \in (0, 1]$ such that $A^{\alpha} \succ_{dl} B^{\alpha}$ holds for all $\alpha \in (0, \beta)$.
(dl-iv) \( \log A \geq \log B \implies \) for any \( p \geq 0 \) there exists \( K_p > 1 \) such that \( K_p \to 1 \) as \( p \to +0 \) and \( (K_p)^{p\alpha} \succ_{dl} B^{p\alpha} \) for all \( \alpha \in (0,1] \).

Finally we cite a counterexample related to (l-iii) and (dl-iii).

1. Introduction

A capital letter means a bounded linear operator on a complex Hilbert space \( H \). An operator \( T \) is said to be positive (denoted by \( T \geq 0 \)) if \( (Tx,x) \geq 0 \) for all \( x \in H \) and also an operator \( T \) is said to be strictly positive (denoted by \( T > 0 \)) if \( T \) is positive and invertible. The strictly chaotic order is defined by \( \log A > \log B \) for strictly positive operators \( A \) and \( B \).

It is well known that the usual order \( A \geq B > 0 \) ensures the chaotic order \( \log A \geq \log B \) since \( \log t \) is operator monotone function.

Also it is known by [Theorem, 6] and [Example 5.1.12 and Corollary 5.1.11, 5] that
\[
A \geq B > 0 \text{ ensures } \frac{A-I}{\log A} \geq \frac{B-I}{\log B} \tag{d1}
\]
and
\[
A \geq B > 0 \text{ ensures } \frac{A \log A}{A-I} \geq \frac{B \log B}{B-I} \tag{d4}
\]
since \( f(t) = \frac{t-1}{\log t} \) \((t > 0, t \neq 1)\) and \( f^*(t) = \frac{t \log t}{t-1} \) \((t > 0, t \neq 1)\) are both operator monotone functions (see Theorem A underbelow). The function \( f(t) = \frac{t-1}{\log t} \) \((t > 0, t \neq 1)\) is said to be "logarithmic function" which is widely used in the theory of heat transfer of the heat engineering and fluid mechanics. Also the function \( f^*(t) = \frac{t \log t}{t-1} \) \((t > 0, t \neq 1)\) is said to be "dual logarithmic function". Related to these two operator inequalities, we shall define the following four orders for strictly positive operators \( A \) and \( B \) such that \( 1 \notin \sigma(A), \sigma(B) \).

**Definition 1.** Let \( A \) and \( B \) be strictly positive operators on a Hilbert space \( H \) such that \( 1 \notin \sigma(A), \sigma(B) \).

(d1) *Strictly logarithmic order* (denoted by \( A \succ_{sl} B \)) is defined by
\[
\frac{A-I}{\log A} > \frac{B-I}{\log B}.
\]

(d2) *Logarithmic order* (denoted by \( A \succ_{l} B \)) is defined by
\[
\frac{A-I}{\log A} \geq \frac{B-I}{\log B}.
\]

(d3) *Strictly dual logarithmic order* (denoted by \( A \succ_{sdl} B \)) is defined by
\[
\frac{A \log A}{A-I} > \frac{B \log B}{B-I}.
\]

(d4) *Dual Logarithmic order* (denoted by \( A \succ_{dl} B \)) is defined by
\[
\frac{A \log A}{A-I} \geq \frac{B \log B}{B-I}.
\]

2. Simplified proofs of operator monotonicity of logarithmic function and dual logarithmic function
We shall show a direct and simplified proof of the following result [Theorem, 6] and [Example 5.1.12 and Corollary 5.1.11, 5] without use of Löwner general result.

**Theorem A.** The function $f$ and $f^*$ given by

$$f(t) = \begin{cases} \frac{t - 1}{\log t} & (t > 0, t \neq 1) \\ 1 & (t = 1) \\ 0 & (t = 0) \end{cases}$$

and

$$f^*(t) = \begin{cases} \frac{t \log t}{t - 1} & (t > 0, t \neq 1) \\ 1 & (t = 1) \\ 0 & (t = 0) \end{cases}$$

are operator monotone functions satisfying the symmetry condition:

$$f(t) = tf\left(\frac{1}{t}\right) \text{ and } f^*(t) = tf^*\left(\frac{1}{t}\right).$$

**Proof.** Let $A$ and $B$ be strictly positive operators such that $1 \not\in \sigma(A), \sigma(B)$. We have only to show the following (i) and (ii) since the latter half is obvious.

(i) If $A \geq B$, then

$$\frac{A - I}{\log A} \geq \frac{B - I}{\log B}.$$

(ii) If $A \geq B$, then

$$A \log A \geq \frac{B \log B}{A - I}.$$

First of all, we cite the following obvious result:

(1) $T - I = (T^{\frac{1}{n}} - I)(T^{1 - \frac{1}{n}} + T^{1 - \frac{2}{n}} + \ldots + T^{\frac{1}{n}} + I)$ for $T \geq 0$ and for any natural number $n$.

(2) $\lim_{n \to \infty} n(T^{\frac{1}{n}} - I) = \log T$ holds for any $T \geq 0$.

(3) If $A \geq B \geq 0$, then $A^\alpha \geq B^\alpha$ holds for any $\alpha \in [0, 1]$. (Löwner-Heinz inequality)

(i). $\frac{A - I}{n(A^{\frac{1}{n}} - I)} = \frac{1}{n}(A^{1 - \frac{1}{n}} + A^{1 - \frac{2}{n}} + \ldots + A^{\frac{1}{n}} + I)$ by (1) for any natural number $n$

$$\geq \frac{1}{n}(B^{1 - \frac{1}{n}} + B^{1 - \frac{2}{n}} + \ldots + B^{\frac{1}{n}} + I)$$

by (3) for any natural number $n$

$$= \frac{B - I}{n(B^{\frac{1}{n}} - I)}$$

for any natural number $n$ by (1) tending $n$ to $\infty$, so we obtain (i) by (2).

(ii). $\frac{n(A^{\frac{1}{n}} - I)A}{A - I} = \frac{n}{(A^{\frac{1}{n}} + A^{\frac{2}{n}} + \ldots + A^{-1})}$ by (1) for any natural number $n$

$$\geq \frac{n}{(B^{\frac{1}{n}} + B^{\frac{2}{n}} + \ldots + B^{-1})}$$

by (3) for any natural number $n$

$$= \frac{n(B^{\frac{1}{n}} - I)B}{B - I}$$

by (1) tending $n$ to $\infty$, so we obtain (ii) by (2).

**Remark 1.** It is well known that (i) is equivalent to (ii) in Theorem A. Alternative proof of (i) in the proof of Theorem A is cited in [5]. Related to Theorem A, we remark that the following
result in [Corollary 2.6, 4], [Theorem 2, 7] and [Corollary 5.11, 5]: let $g(t)$ be a continuous positive function such that $(0, \infty) \to (0, \infty)$. Then $g(t)$ is operator monotone function if and only if $g^*(t) = \frac{t}{g(t)}$ is operator monotone function. Actually, $f(t)$ and $f^*(t)$ in Theorem A satisfy this condition $f^*(t) = \frac{t}{f(t)}$.

3. Strictly logarithmic order $A \succ_{sl} B$ and logarithmic order $A \succ_{l} B$

Let $A$ and $B$ be strictly positive operators such that $1 \not\in \sigma(A), \sigma(B)$. Firstly we shall give Theorem 1 asserting the following

\[(*) \quad \log A > \log B \implies \text{there exists } \beta \in (0, 1] \text{ such that } A^{\alpha} \succ_{sl} B^{\alpha} \text{ holds for all } \alpha \in (0, \beta).\]

Secondary, we shall give Corollary 2 showing that there exists an interesting contrast between $A \geq B > 0$ and $A > B > 0$ related to $A \succ_{sl} B$ and $A \succ_{l} B$. Thirdly, we shall give some applications of two characterizations (Theorem A and Theorem B under below) of chaotic order to $A \succ_{sl} B$ and $A \succ_{l} B$ in Corollary 3.

**Lemma 1.** Let $A$ and $B$ be invertible self adjoint operators on a Hilbert space $H$. If $A > B$, then there exists $\beta \in (0, 1]$ such that the following inequality holds for all $\alpha \in (0, \beta)$;

\[
\frac{e^{\alpha A} - I}{\alpha A} > \frac{e^{\alpha B} - I}{\alpha B}, \text{ i.e., } e^{\alpha A} \succ_{sl} e^{\alpha B}.
\]

**Proof.** There exists $\varepsilon$ such that $A - B \geq \varepsilon > 0$. Choose $\alpha$ and $\beta$ such that

\[(4) \quad 0 < \alpha < \text{Min} \left\{ \frac{\varepsilon}{2}, \left( \frac{e^{||A||}}{||A||} + \frac{e^{||B||}}{||B||} \right)^{-1} \right\} = \beta.
\]

By an easy calculation, we obtain

\[
\frac{e^{\alpha A} - I}{A} - \frac{e^{\alpha B} - I}{B} = \sum_{n=1}^{\infty} \frac{\alpha^n}{n!} A^{n-1} - \sum_{n=1}^{\infty} \frac{\alpha^n}{n!} B^{n-1}
\]

\[
= \sum_{n=2}^{\infty} \frac{\alpha^n}{n!} (A^{n-1} - B^{n-1})
\]

\[
= \frac{\alpha^2}{2!} (A - B) + \sum_{n=3}^{\infty} \frac{\alpha^n}{n!} (A^{n-1} - B^{n-1})
\]

\[
\geq \frac{\alpha^2}{2!} \varepsilon - \alpha^3 \sum_{n=3}^{\infty} \frac{1}{n!} (||A||^{n-1} + ||B||^{n-1})
\]

\[
\geq \frac{\alpha^2}{2!} \varepsilon - \alpha^3 \left( \frac{e^{||A||}}{||A||} + \frac{e^{||B||}}{||B||} \right) > 0 \quad \text{by (4)},
\]
so that \( \frac{e^{\alpha A} - I}{\alpha A} - \frac{e^{\alpha B} - I}{\alpha B} \) holds, i.e., there exists \( \beta \in (0, 1] \) such that \( e^{\alpha A} \succ_{sl} e^{\alpha B} \) holds for all \( \alpha \in (0, \beta) \) and the proof is complete.

**Theorem 1.** Let \( A \) and \( B \) be strictly positive operators such that \( 1 \not\in \sigma(A), \sigma(B) \).

If \( \log A > \log B \), then there exists \( \beta \in (0, 1] \) such that \( A^\alpha \succ_{sl} B^\alpha \) holds for all \( \alpha \in (0, \beta) \).

**Proof.** We have only to replace \( A \) by \( \log A \) and also \( B \) by \( \log B \) respectively in Lemma 1.

**Corollary 2.** Let \( A \) and \( B \) be strictly positive operators such that \( 1 \not\in \sigma(A), \sigma(B) \). Then

(i) If \( A > B > 0 \), then there exists \( \beta \in (0, 1] \) such that \( A^\alpha \succ_{sl} B^\alpha \) holds for all \( \alpha \in (0, \beta) \).

(ii) If \( A \geq B > 0 \), then \( A^\alpha \succ_{l} B^\alpha \) holds for all \( \alpha \in (0, 1] \).

In Corollary 2, It is interesting to point out the contrast between \( A > B > 0 \) and \( A \geq B > 0 \).

**Proof of Corollary 2.** (i). We cite the following obvious and fundamental result (5)

\[
\text{If } A > B > 0, \text{ then } \log A > \log B.
\]

In fact if \( A > B > 0 \), then \( A \geq B + \epsilon > B \) for some \( \epsilon > 0 \), so that \( \log A \geq \log(B + \epsilon) > \log B \), that is, (5) holds. (i) follows by (5) and Theorem 1.

(ii). If \( A \geq B > 0 \), then \( A^\alpha \geq B^\alpha \) for all \( \alpha \in (0, 1] \) by Löwner-Heinz inequality and (ii) follows by the result that the function \( f(t) = \frac{t - 1}{\log t} \) \( (t > 0, t \neq 1) \) is an operator monotone function by Theorem A, i.e., \( f(A^\alpha) \geq f(B^\alpha) \) for all \( \alpha \in (0, 1] \), so we have (ii).

**Corollary 3.** Let \( A \) and \( B \) be strictly positive operators such that \( 1 \not\in \sigma(A), \sigma(B) \) and \( \log A \geq \log B \). Then

(i) For any \( \delta \in (0, 1] \) there exists \( \beta = \beta_\delta \in (0, 1] \) such that \( (e^\delta A)^\alpha \succ_{sl} B^\alpha \) holds for all \( \alpha \in (0, \beta_\delta) \).

(ii) For any \( p \geq 0 \) there exists \( K_p > 1 \) such that \( K_p \to 1 \text{ as } p \to +0 \) and \( (K_p A)^{p\alpha} \succ_{l} B^{p\alpha} \) for all \( \alpha \in (0, 1] \).

We cite the following two results in order to give a proof of Corollary 3.

**Theorem A** [1][3]. Let \( A \) and \( B \) be invertible positive operators on a Hilbert space \( H \). \( \log A \geq \log B \) holds if and only if for any \( \delta \in (0, 1] \) there exists \( \alpha = \alpha_\delta > 0 \) such that \( (e^\delta A)^\alpha > B^\alpha \).

**Theorem B** [8]. Let \( A \) and \( B \) be invertible positive operators on a Hilbert space \( H \). \( \log A \geq \log B \) if and only if for any \( p \geq 0 \) there exists a \( K_p > 1 \) such that \( K_p \to 1 \text{ as } p \to +0 \) and \( (K_p A)^p \geq B^p \).
Proof of Corollary 3.

(i). As \( \log A \geq \log B \) holds, then for any \( \delta \in (0, 1] \), there exists \( \alpha' = \alpha'_\delta > 0 \) such that \( (e^{\delta}A)^{\alpha'} > B^{\alpha'} \) by Theorem A. Then \( e^{\delta}A > \log B \) by (5), so that there exists \( \beta = \beta_\delta \in (0, 1] \) such that \( (e^{\delta}A)^{\alpha} >_{dl} B^{\alpha} \) holds for all \( \alpha \in (0, \beta) \) by Theorem 1.

(ii). As \( \log A \geq \log B \) holds, then for any \( p \geq 0 \) there exists a there exists \( K_p > 1 \) such that \( K_p \rightarrow 1 \) as \( p \rightarrow +0 \) and \( (K_pA)^p \geq B^p \) by Theorem B, so we have \( (K_pA)^{p\alpha} \geq B^{p\alpha} \) for all \( \alpha \in (0, 1] \) by (ii) of Corollary 2.

4. Strictly dual logarithmic order \( A >_{sdl} B \) and dual logarithmic order \( A >_{dl} B \)

Let \( A \) and \( B \) be strictly positive operators such that \( 1 \notin \sigma(A), \sigma(B) \). Firstly we shall give Theorem 4 asserting the following:

\[(1) \quad \log A > \log B \implies \text{there exists } \beta \in (0, 1] \text{ such that } A^\alpha >_{sdl} B^\alpha \text{ holds for all } \alpha \in (0, \beta).\]

Secondary, we shall give Corollary 5 showing that there exists an interesting contrast between \( A \geq B > 0 \) and \( A > B > 0 \) related to \( A >_{sdl} B \) and \( A >_{dl} B \). Thirdly, we shall give some applications of Theorem A and Theorem B to \( A >_{sdl} B \) and \( A >_{dl} B \) in Corollary 6.

Lemma 2. Let \( A \) and \( B \) be invertible self-adjoint operators on a Hilbert space \( H \). If \( A > B \), then there exists \( \beta \in (0, 1] \) such that the following inequality holds for all \( \alpha \in (0, \beta) \):

\[
\frac{\alpha Ae^{\alpha A}}{e^{\alpha A} - I} > \frac{\alpha Be^{\alpha B}}{e^{\alpha B} - I}, \quad \text{i.e., } e^{\alpha A} >_{sdl} e^{\alpha B}.
\]

Proof. As \( -B > -A \) holds by applying Lemma 1, there exists \( \beta \in (0, 1] \) such that

\[
\frac{e^{-\alpha B} - I}{-\alpha B} > \frac{e^{-\alpha A} - I}{-\alpha A}.
\]

holds for all \( \alpha \in (0, \beta) \). That is, \( \frac{e^{\alpha B} - I}{\alpha Be^{\alpha B}} > \frac{e^{\alpha A} - I}{\alpha Ae^{\alpha A}} \) holds if \( \frac{\alpha Ae^{\alpha A}}{e^{\alpha A} - I} > \frac{\alpha Be^{\alpha B}}{e^{\alpha B} - I} \) holds, i.e.,

there exists \( \beta \in (0, 1] \) such that \( e^{\alpha A} >_{sdl} e^{\alpha B} \) holds for all \( \alpha \in (0, \beta) \) and the proof is complete.

Theorem 4. Let \( A \) and \( B \) be strictly positive operators such that \( 1 \notin \sigma(A), \sigma(B) \).

If \( \log A > \log B \), then there exists \( \beta \in (0, 1] \) such that \( A^\alpha >_{sdl} B^\alpha \) holds for all \( \alpha \in (0, \beta) \).

Proof. We have only to replace \( A \) by \( \log A \) and also \( B \) by \( \log B \) respectively in Lemma 2.

Corollary 5. Let \( A \) and \( B \) be strictly positive operators such that \( 1 \notin \sigma(A), \sigma(B) \). Then

(i) If \( A > B > 0 \), then there exists \( \beta \in (0, 1] \) such that \( A^\alpha >_{sdl} B^\alpha \) holds for all \( \alpha \in (0, \beta) \).

(ii) If \( A \geq B > 0 \), then \( A^\alpha >_{dl} B^\alpha \) for all \( \alpha \in (0, 1] \).
In Corollary 5, it is interesting to point out the contrast between $A > B > 0$ and $A \geq B > 0$.

**Proof of Corollary 5.** By the same way as a proof of Corollary 2, we shall give the following proofs of (i) and (ii).

(i). If $A > B > 0$, then $\log A > \log B$ holds by (5), so that (i) follows by Theorem 4.

(ii). If $A \geq B > 0$, then $A^\alpha \geq B^\alpha$ for all $\alpha \in (0, 1]$ by Löwner-Heinz inequality. The function $f^*(t) = \frac{\log t}{t - 1}$ ($t > 0, t \neq 1$) is also an operator monotone function by Theorem A, so that $f^*(A^\alpha) \geq f^*(B^\alpha)$ for all $\alpha \in (0, 1]$, so we have (ii).

**Corollary 6.** Let $A$ and $B$ be strictly positive operators such that $1 \not\in \sigma(A), \sigma(B)$ and $\log A \geq \log B$. Then

(i). For any $\delta \in (0, 1]$ there exists $\beta = \beta_\delta \in (0, 1]$ such that $(e^A)^\alpha \succ_{sdl} B^\alpha$ holds for all $\alpha \in (0, \beta)$.

(ii). For any $p \geq 0$ there exists $K_p > 1$ such that $K_p \rightarrow 1$ as $p \rightarrow +0$ and $(K_p A)^p \geq_{dl} B^p$ for all $\alpha \in (0, 1]$.

**Proof of Corollary 6.** We shall obtain Corollary 6 by the same way as one in Corollary 3.

(i). As $\log A \geq \log B$ holds, then for any $\delta \in (0, 1]$, there exists $\alpha' = \alpha'_\delta > 0$ such that $(e^A)^\alpha > B^\alpha'$ by Theorem A. Then $\log e^A > \log B$ by (5), so that there exists $\beta = \beta_\delta \in (0, 1]$ such that $(e^A)^\alpha \succ_{sdl} B^\alpha$ holds for all $\alpha \in (0, \beta)$ by Theorem 4.

(ii). As $\log A \geq \log B$ holds, then for any $p \geq 0$ there exists $a$ a there exists $K_p > 1$ such that $K_p \rightarrow 1$ as $p \rightarrow +0$ and $(K_p A)^p \geq B^p$ by Theorem B, so that $(K_p A)^p \geq_{dl} B^p$ for all $\alpha \in (0, 1]$ by (ii) of Corollary 5.

**5. An example related to strictly logarithmic order $A \succ_{sl} B$ and strictly dual logarithmic order $A \succ_{sdl} B$**

Related to (i) of Corollary 3, we consider the following problem:

(Q1) “Does $\log A \geq \log B$ ensure that there exists an $\alpha > 0$ such that $A^\alpha \succ_{l} B^\alpha$?”

Also related to (i) of Corollary 6, we consider the following problem too;

(Q2) “Does $\log A \geq \log B$ ensure that there exists an $\alpha > 0$ such that $A^\alpha \succ_{dl} B^\alpha$?”

In fact, we cite a counterexample to (Q1) and (Q2) as follows.
Example 1. Take $A$ and $B$ as follows:

\[
\log A = \begin{pmatrix} 2 & 2 \\ 2 & -1 \end{pmatrix} \quad \text{and} \quad \log B = \begin{pmatrix} 1 & 0 \\ 0 & -5 \end{pmatrix}.
\]

Then $\log A \geq \log B$ holds, but

(i) $A^\alpha \succ_{l} B^\alpha$ does not hold for any $\alpha > 0$.

(ii) $A^\alpha \succ_{d/} B^\alpha$ does not hold for any $\alpha > 0$.

(iii) $A^\alpha \geq B^\alpha$ does not hold for any $\alpha > 0$.

In fact, $\log A$ is diagonalized by $U = \frac{1}{\sqrt{5}} \begin{pmatrix} -1 & 2 \\ 2 & 1 \end{pmatrix}$ as follows;

\[
U(\log A)U = \begin{pmatrix} -2 & 0 \\ 0 & 3 \end{pmatrix}, \quad \text{and} \quad UAU = \begin{pmatrix} e^{-2} & 0 \\ 0 & e^3 \end{pmatrix},
\]

so that we have

\[
A^\alpha = U \begin{pmatrix} e^{-2\alpha} & 0 \\ 0 & e^{3\alpha} \end{pmatrix} U \quad \text{and} \quad B^\alpha = \begin{pmatrix} e^{\alpha} & 0 \\ 0 & e^{-5\alpha} \end{pmatrix}.
\]

Put $x = e^\alpha > 1$ since $\alpha > 0$. At first we show (i). By a slight elaborate calculation, we have

\[
\det \left( \frac{A^\alpha - I}{\log A} - \frac{B^\alpha - I}{\log B} \right) = -(x - 1)^6 \frac{(10x^5 + 33x^4 + 48x^3 + 38x^2 + 18x + 3)}{150x^7} < 0 \text{ since } x > 1.
\]

Whence $A^\alpha \succ_{l} B^\alpha$ does not hold for any $\alpha > 0$, so the proof of (i) is complete.

Next we show (ii). By more elaborate calculation than (i), we obtain

\[
\det \left( \frac{A^\alpha \log A}{A^\alpha - I} - \frac{B^\alpha \log B}{B^\alpha - I} \right)
\]
\[
\begin{align*}
&\frac{7x^3 + 9x^2 + x - 2}{5(x^3 + 2x^2 + 2x + 1)} \quad \frac{2(3x^3 + 6x^2 + 4x + 2)}{5(x^3 + 2x^2 + 2x + 1)} \\
&= \left( \frac{7x^3 + 9x^2 + x - 2}{5(x^3 + 2x^2 + 2x + 1)} \right) \left( \frac{3x^4 + 3x^3 + 8x^2 + 8x + 8}{5(x^4 + x^3 - x - 1)} + \frac{5(x-1)}{-x^6 + x^5 + x - 1} \right) \\
&\quad - \left( \frac{2(3x^3 + 6x^2 + 4x + 2)}{5(x^3 + 2x^2 + 2x + 1)} \right)^2
\end{align*}
\]
\[
= \frac{-(x-1)^2(3x^5 + 18x^4 + 38x^3 + 48x^2 + 33x + 10)}{5(x+1)(x^2 + x + 1)(x^4 + x^3 + x^2 + x + 1)} < 0 \text{ since } x > 1
\]

Whence \( A^\alpha \succ_{sl} B^\alpha \) does not hold for any \( \alpha > 0 \), so the proof of (ii) is complete.

Incidentally, we remark that this example also shows that \( \log A \geq \log B \) does not ensure \( A^\alpha \geq B^\alpha \) for any \( \alpha > 0 \). Actually we have

\[
\det(A^\alpha - B^\alpha)
\]
\[
= \left| \begin{array}{cc}
\frac{1}{5x^2} - x + \frac{4x^3}{5} & \frac{2(x^5 - 1)}{5x^2} \\
\frac{2(x^5 - 1)}{x^8 + 4x^3 - 5} & \frac{5x^5}{5x^5}
\end{array} \right|
\]
\[
= \frac{-1}{5x^7} + x^{-4} - \frac{4}{5x^2} - \frac{4}{5x} + x - \frac{x^4}{5}
\]
\[
= \frac{-(x-1)^4(x+1)(x^2 + x + 1)(x^4 + 2x^3 + 4x^2 + 2x + 1)}{5x^7} < 0 \text{ since } x > 1,
\]

that is, \( A^\alpha \geq B^\alpha \) does not hold for any \( \alpha > 0 \), so (iii) is shown. In [2], there is another nice example that \( \log A \geq \log B \) does not ensure \( A^\alpha \geq B^\alpha \) for any \( \alpha > 0 \). In fact, we construct Example 1 inspired by an excellent method in [2].

6. Concluding remarks

Let \( A \) and \( B \) be strictly positive operators such that \( 1 \notin \sigma(A), \sigma(B) \). We can obtain the following interesting contrast among \( A \succ B \succ 0, \ A \geq B \succ 0, \ \log A \succ \log B \) and \( \log A \geq \log B \) by summarizing our results in this paper.

(*) \( \log A \succ \log B \implies \text{there exists } \beta \in (0,1) \text{ such that } A^\alpha \succ_{sl} B^\alpha \text{ holds for all } \alpha \in (0, \beta) \).

(\dagger) \( \log A \succ \log B \implies \text{there exists } \beta \in (0,1) \text{ such that } A^\alpha \succ_{sl} B^\alpha \text{ holds for all } \alpha \in (0, \beta) \).

(\ddagger) \( A \succ B \succ 0 \implies \text{there exists } \beta \in (0,1) \text{ such that } A^\alpha \succ_{stl} B^\alpha \text{ holds for all } \alpha \in (0, \beta) \).

\( A \succ B \succ 0 \implies \text{there exists } \beta \in (0,1) \text{ such that } A^\alpha \succ_{sl} B^\alpha \text{ holds for all } \alpha \in (0, \beta) \).
(li) $A \geq B > 0 \implies A^\alpha \succ_{l} B^\alpha$ for all $\alpha \in (0, 1]$.

(lii) $\log A \geq \log B \implies$ for any $\delta \in (0, 1]$, there exists $\beta = \beta_\delta \in (0, 1]$ such that $(e^\delta A)^\alpha \succ_{sl} B^\alpha$ holds for all $\alpha \in (0, \beta)$.

(liii) $A \geq B > 0 \implies$ for any $p \geq 0$ there exists $K_p > 1$ such that $K_p \to 1$ as $p \to +0$ and $(K_p A)^{p\alpha} \succ_{dl} B^{p\alpha}$ for all $\alpha \in (0, 1]$. 

(liv) $\log A \geq \log B \implies$ for any $p \geq 0$ there exists $K_p > 1$ such that $K_p \to 1$ as $p \to +0$ and $(K_p A)^{p\alpha} \succ_{sl} B^{p\alpha}$ for all $\alpha \in (0, 1]$. 

(dli) $A > B > 0 \implies$ there exists $\beta \in (0, 1]$ such that $A^\alpha \succ_{sdl} B^\alpha$ holds for all $\alpha \in (0, \beta)$.

(dlii) $A \geq B > 0 \implies A^\alpha \succ_{dl} B^\alpha$ for all $\alpha \in (0, 1]$. 

(dliii) $\log A \geq \log B \implies$ for any $\delta \in (0, 1]$, there exists $\beta = \beta_\delta \in (0, 1]$ such that $(e^\delta A)^\alpha \succ_{sdl} B^\alpha$ holds for all $\alpha \in (0, \beta)$.

(dliv) $\log A \geq \log B \implies$ for any $p \geq 0$ there exists $K_p > 1$ such that $K_p \to 1$ as $p \to +0$ and $(K_p A)^{p\alpha} \succ_{dl} B^{p\alpha}$ for all $\alpha \in (0, 1]$. 

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References


7. Appendix

Simple proof of the concavity on operator entropy $f(A) = -A\log A$

A capital letter means a bounded linear and strictly positive operator on a Hilbert space. Here we shall give a simple proof of the following well known and excellent result obtained by [1] and [2] independently.

**Theorem A.** $f(A) = -A\log A$ is concave function for any $A > 0$.

**Proof.** Firstly we recall the following obvious result

(*) $\lim_{n\to\infty} (T^{\frac{1}{n}} - I)n = -\log T$ for any $T > 0$.

As $g(t) = t^q$ is operator concave for $q \in [0, 1]$, then for $A > 0$, $B > 0$ and $\alpha, \beta \in [0, 1]$ such that $\alpha + \beta = 1$

$$(\alpha A + \beta B)^{1-\frac{1}{n}} \geq \alpha A^{1-\frac{1}{n}} + \beta B^{1-\frac{1}{n}}$$

for any natural number $n$ so we obtain

$$(\alpha A + \beta B)\left((\alpha A + \beta B)^{-\frac{1}{n}} - I\right)n \geq \alpha A(A^{-\frac{1}{n}} - I)n + \beta B(B^{-\frac{1}{n}} - I)n$$

tending $n \to \infty$, we have

$$-(\alpha A + \beta B)\log(\alpha A + \beta B) \geq (-\alpha A\log A - \beta B\log B)$$

by (*)

that is,

$$f(\alpha A + \beta B) \geq \alpha f(A) + \beta f(B)$$

so the proof is complete.

**References**


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