<table>
<thead>
<tr>
<th>項目</th>
<th>内容</th>
</tr>
</thead>
<tbody>
<tr>
<td>Title</td>
<td>LOGARITHMIC ORDER AND DUAL LOGARITHMIC ORDER (Operator Inequalities and Related Areas)</td>
</tr>
<tr>
<td>Author(s)</td>
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</tr>
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LOGARITHMIC ORDER AND DUAL LOGARITHMIC ORDER

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Abstract. We shall define the following four orders for strictly positive operators $A$ and $B$ on a Hilbert space $H$ such that $1 \not\in \sigma(A), \sigma(B)$.

- **Strictly logarithmic order** (denoted by $A \succ_{sl} B$) is defined by \( \frac{A-I}{\log A} > \frac{B-I}{B \log B} \).

- **Logarithmic order** (denoted by $A \succ_{l} B$) is defined by \( \frac{A-I}{\log A} \geq \frac{B-I}{B \log B} \).

- **Strictly dual logarithmic order** (denoted by $A \succ_{sdl} B$) is defined by \( \frac{A \log A}{A-I} > \frac{B \log B}{B-I} \).

- **Dual Logarithmic order** (denoted by $A \succ_{dl} B$) is defined by \( \frac{A \log A}{A-I} \geq \frac{B \log B}{B-I} \).

Firstly we shall show direct and simplified proofs of operator monotonicity of logarithmic function \( f(t) = \frac{t-1}{\log t} \) and dual logarithmic function \( f^*(t) = \frac{t \log t}{t-1} \).

In what follows, let $A$ and $B$ be strictly positive operators on a Hilbert space $H$ such that $1 \not\in \sigma(A), \sigma(B)$. Secondary we shall show the following:

- **(\ast)** \( \log A > \log B \Rightarrow \) there exists $\beta \in (0,1]$ such that $A^\alpha \succ_{sl} B^\alpha$ holds for all $\alpha \in (0, \beta)$.

- **(\dagger)** \( \log A > \log B \Rightarrow \) there exists $\beta \in (0,1]$ such that $A^\alpha \succ_{sdl} B^\alpha$ holds for all $\alpha \in (0, \beta)$.

By using these two results (\ast) and (\dagger), we summarize the following interesting contrast among $A > B > 0$, $A \geq B > 0$, $\log A > \log B$ and $\log A \geq \log B$.

- **(l-i)** $A > B > 0 \Rightarrow$ there exists $\beta \in (0,1]$ such that $A^\alpha \succ_{sl} B^\alpha$ holds for all $\alpha \in (0, \beta)$.

- **(l-ii)** $A \geq B > 0 \Rightarrow A^\alpha \succ_{l} B^\alpha$ for all $\alpha \in (0,1]$.

- **(l-iii)** $\log A \geq \log B \Rightarrow$ for any $\delta \in (0,1]$, there exists $\beta = \beta_\delta \in (0,1]$ such that $(e^{\delta}A)^\alpha \succ_{sl} B^\alpha$ holds for all $\alpha \in (0, \beta)$.

- **(l-iv)** $\log A \geq \log B \Rightarrow$ for any $\delta \in (0,1]$, there exists $\beta = \beta_\delta \in (0,1]$ such that $(e^{\delta}A)^\alpha \succ_{sdl} B^\alpha$ holds for all $\alpha \in (0, \beta)$.

- **(dl-i)** $A > B > 0 \Rightarrow$ there exists $\beta \in (0,1]$ such that $A^\alpha \succ_{sdl} B^\alpha$ holds for all $\alpha \in (0, \beta)$.

- **(dl-ii)** $A \geq B > 0 \Rightarrow A^\alpha \succ_{dl} B^\alpha$ for all $\alpha \in (0,1]$.

- **(dl-iii)** $\log A \geq \log B \Rightarrow$ for any $\delta \in (0,1]$, there exists $\beta = \beta_\delta \in (0,1]$ such that $(e^{\delta}A)^\alpha \succ_{dl} B^\alpha$ holds for all $\alpha \in (0, \beta)$. 


(dl-iv) $\log A \geq \log B \implies$ for any $p \geq 0$ there exists $K_p > 1$ such that $K_p \to 1$ as $p \to +0$ and $(K_p)^{pos} \succ_{dl} B^{pos}$ for all $\alpha \in (0, 1]$.

Finally we cite a counterexample related to (l-iii) and (dl-iii).

1. Introduction

A capital letter means a bounded linear operator on a complex Hilbert space $H$. An operator $T$ is said to be positive (denoted by $T \geq 0$) if $(Tx, x) \geq 0$ for all $x \in H$ and also an operator $T$ is said to be strictly positive (denoted by $T > 0$) if $T$ is positive and invertible. The strictly chaotic order is defined by $\log A > \log B$ for strictly positive operators $A$ and $B$.

It is well known that the usual order $A \geq B > 0$ ensures the chaotic order $\log A \geq \log B$ since $\log t$ is operator monotone function. Also it is known by [Theorem, 6] and [Example 5.1.12 and Corollary 5.1.11, 5] that

$$A \geq B > 0 \text{ ensures } \frac{A - I}{\log A} \geq \frac{B - I}{\log B}$$

and

$$A \geq B > 0 \text{ ensures } \frac{A \log A}{A - I} \geq \frac{B \log B}{B - I}$$

since $f(t) = \frac{t - 1}{\log t}$ ($t > 0, t \neq 1$) and $f^*(t) = \frac{t \log t}{t - 1}$ ($t > 0, t \neq 1$) are both operator monotone functions (see Theorem A underbelow). The function $f(t) = \frac{t - 1}{\log t}$ ($t > 0, t \neq 1$) is said to be "logarithmic function" which is widely used in the theory of heat transfer of the heat engineering and fluid mechanics. Also the function $f^*(t) = \frac{t \log t}{t - 1}$ ($t > 0, t \neq 1$) is said to be "dual logarithmic function". Related to these two operator inequalities, we shall define the following four orders for strictly positive operators $A$ and $B$ such that $1 \not\in \sigma(A), \sigma(B)$.

**Definition 1.** Let $A$ and $B$ be strictly positive operators on a Hilbert space $H$ such that $1 \not\in \sigma(A), \sigma(B)$.

(d1) Strictly logarithmic order (denoted by $A \succ_{sl} B$) is defined by $\frac{A - I}{\log A} > \frac{B - I}{\log B}$.

(d2) Logarithmic order (denoted by $A \succ_{l} B$) is defined by $\frac{A - I}{\log A} \geq \frac{B - I}{\log B}$.

(d3) Strictly dual logarithmic order (denoted by $A \succ_{sdl} B$) is defined by $\frac{A \log A}{A - I} > \frac{B \log B}{B - I}$.

(d4) Dual Logarithmic order (denoted by $A \succ_{dl} B$) is defined by $\frac{A \log A}{A - I} \geq \frac{B \log B}{B - I}$.

2. Simplified proofs of operator monotonicity of logarithmic function and dual logarithmic function
We shall show a direct and simplified proof of the following result [Theorem 6] and [Example 5.1.12 and Corollary 5.1.11, 5] without use of Löwner general result.

**Theorem A.** The function \( f \) and \( f^* \) given by

\[
f(t) = \begin{cases} \frac{t-1}{\log t} & (t > 0, t \neq 1) \\ 1 & (t = 1) \\ 0 & (t = 0) \end{cases}
\]

and

\[
f^*(t) = \begin{cases} \frac{t \log t}{t-1} & (t > 0, t \neq 1) \\ 1 & (t = 1) \\ 0 & (t = 0) \end{cases}
\]

are operator monotone functions satisfying the symmetry condition:

\[ f(t) = tf\left(\frac{1}{t}\right) \text{ and } f^*(t) = tf^*\left(\frac{1}{t}\right). \]

**Proof.** Let \( A \) and \( B \) be strictly positive operators such that \( 1 \notin \sigma(A), \sigma(B) \). We have only to show the following (i) and (ii) since the latter half is obvious.

(i) If \( A \geq B \), then

\[
\frac{A-I}{\log A} \geq \frac{B-I}{\log B}.
\]

(ii) If \( A \geq B \), then

\[
\frac{A \log A}{A-I} \geq \frac{B \log B}{B-I}.
\]

First of all, we cite the following obvious result;

1. \( T - I = (T^{\frac{1}{n}} - I)(T^{1-\frac{1}{n}} + T^{1-\frac{2}{n}} + ... + T^{\frac{1}{n}} + I) \) for \( T \geq 0 \) and for any natural number \( n \).
2. \( \lim_{n \to \infty} n(T^{\frac{1}{n}} - I) = \log T \) holds for any \( T \geq 0 \).
3. If \( A \geq B \geq 0 \), then \( A^\alpha \geq B^\alpha \) holds for any \( \alpha \in [0, 1] \). (Löwner-Heinz inequality)

(i). \( \frac{A-I}{n(A^{\frac{1}{n}} - I)} = \frac{1}{n}(A^{-\frac{1}{n}} + A^{-\frac{2}{n}} + ... + A + I) \) by (1) for any natural number \( n \)

\[
\geq \frac{1}{n}(B^{-\frac{1}{n}} + B^{-\frac{2}{n}} + ... + B + I) \text{ by (3) for any natural number } n
\]

\[ = \frac{B-I}{n(B^{\frac{1}{n}} - I)} \text{ for any natural number } n \text{ by (1)} \]

tending \( n \) to \( \infty \), so we obtain (i) by (2).

(ii). \( \frac{n(A^{\frac{1}{n}} - I)A}{A-I} = \frac{n}{(A^{-\frac{1}{n}} + A^{-\frac{2}{n}} + ... + A^{-1})} \) by (1) for any natural number \( n \)

\[
\geq \frac{n}{(B^{-\frac{1}{n}} + B^{-\frac{2}{n}} + ... + B^{-1})} \text{ by (3) for any natural number } n
\]

\[ = \frac{n(B^{\frac{1}{n}} - I)B}{B-I} \text{ by (1)} \]

tending \( n \) to \( \infty \), so we obtain (ii) by (2).

**Remark 1.** It is well known that (i) is equivalent to (ii) in Theorem A. Alternative proof of (i) in the proof of Theorem A is cited in [5]. Related to Theorem A, we remark that the following
result in [Corollary 2.6, 4], [Theorem 2, 7] and [Corollary 5.11, 5]: let \( g(t) \) be a continuous positive function such that \((0, \infty) \to (0, \infty)\). Then \( g(t) \) is operator monotone function if and only if \( g^*(t) = \frac{t}{g(t)} \) is operator monotone function. Actually, \( f(t) \) and \( f^*(t) \) in Theorem A satisfy this condition \( f^*(t) = \frac{t}{f(t)} \).

3. Strictly logarithmic order \( A \succ_{sl} B \) and logarithmic order \( A \succ_{l} B \)

Let \( A \) and \( B \) be strictly positive operators such that \( 1 \notin \sigma(A), \sigma(B) \). Firstly we shall give Theorem 1 asserting the following

\((*) \quad \log A > \log B \implies \text{there exists } \beta \in (0, 1] \text{ such that } A^\alpha \succ_{sl} B^\alpha \text{ holds for all } \alpha \in (0, \beta)\).

Secondary, we shall give Corollary 2 showing that there exists an interesting contrast between \( A \geq B > 0 \) and \( A > B > 0 \) related to \( A \succ_{sl} B \) and \( A \succ_{l} B \). Thirdly, we shall give some applications of two characterizations (Theorem A and Theorem B under below) of chaotic order to \( A \succ_{sl} B \) and \( A \succ_{l} B \) in Corollary 3.

**Lemma 1.** Let \( A \) and \( B \) be invertible self adjoint operators on a Hilbert space \( H \). If \( A > B \), then there exists \( \beta \in (0, 1] \) such that the following inequality holds for all \( \alpha \in (0, \beta) \):

\[
\frac{e^{\alpha A} - I}{\alpha A} > \frac{e^{\alpha B} - I}{\alpha B}, \quad \text{i.e., } e^{\alpha A} \succ_{sl} e^{\alpha B}.
\]

**Proof.** There exists \( \varepsilon \) such that \( A - B \geq \varepsilon > 0 \). Choose \( \alpha \) and \( \beta \) such that

\[(4) \quad 0 < \alpha < \text{Min}\{\varepsilon \left( \frac{e^{||A||}}{||A||} + \frac{e^{||B||}}{||B||} \right)^{-1}, 1 \} = \beta.
\]

By an easy calculation, we obtain

\[
\frac{e^{\alpha A} - I}{A} - \frac{e^{\alpha B} - I}{B} = \sum_{n=1}^{\infty} \frac{\alpha^n}{n!} A^{n-1} - \sum_{n=1}^{\infty} \frac{\alpha^n}{n!} B^{n-1}
\]

\[
= \sum_{n=2}^{\infty} \frac{\alpha^n}{n!} (A^{n-1} - B^{n-1})
\]

\[
= \frac{\alpha^2}{2!} (A - B) + \sum_{n=3}^{\infty} \frac{\alpha^n}{n!} (A^{n-1} - B^{n-1})
\]

\[
\geq \frac{\alpha^2}{2!} \varepsilon - \alpha^3 \left[ \sum_{n=3}^{\infty} \frac{1}{n!} (||A||^{n-1} + ||B||^{n-1}) \right]
\]

\[
\geq \alpha^3 \left[ \frac{\varepsilon}{2!} - \alpha \left( \frac{e^{||A||}}{||A||} + \frac{e^{||B||}}{||B||} \right) \right] > 0 \quad \text{by (4)},
\]
so that \( \frac{e^{\alpha A} - I}{\alpha A} - \frac{e^{\alpha B} - I}{\alpha B} \) holds, i.e., there exists \( \beta \in (0, 1] \) such that \( e^{\alpha A} \succ_{sl} e^{\alpha B} \) holds for all \( \alpha \in (0, \beta) \) and the proof is complete.

**Theorem 1.** Let \( A \) and \( B \) be strictly positive operators such that \( 1 \not\in \sigma(A), \sigma(B) \). If \( \log A > \log B \), then there exists \( \beta \in (0, 1] \) such that \( A^\alpha \succ_{sl} B^\alpha \) holds for all \( \alpha \in (0, \beta) \).

**Proof.** We have only to replace \( A \) by \( \log A \) and also \( B \) by \( \log B \) respectively in Lemma 1.

**Corollary 2.** Let \( A \) and \( B \) be strictly positive operators such that \( 1 \not\in \sigma(A), \sigma(B) \). Then

(i) If \( A > B > 0 \), then there exists \( \beta \in (0, 1] \) such that \( A^\alpha \succ_{sl} B^\alpha \) holds for all \( \alpha \in (0, \beta) \).

(ii) If \( A \geq B > 0 \), then \( A^\alpha \succ_{l} B^\alpha \) holds for all \( \alpha \in (0, 1] \).

In Corollary 2, it is interesting to point out the contrast between \( A > B > 0 \) and \( A \geq B > 0 \).

**Proof of Corollary 2.** (i). We cite the following obvious and fundamental result (5)

\[
\text{If } A > B > 0, \text{ then } \log A > \log B.
\]

In fact if \( A > B > 0 \), then \( A \geq B + \epsilon > B \) for some \( \epsilon > 0 \), so that \( \log A \geq \log(B + \epsilon) > \log B \), that is, (5) holds. (i) follows by (5) and Theorem 1.

(ii). If \( A \geq B > 0 \), then \( A^\alpha \geq B^\alpha \) for all \( \alpha \in (0, 1] \) by Löwner-Heinz inequality and (ii) follows by the result that the function \( f(t) = \frac{t - 1}{\log t} \) \( (t > 0, t \neq 1) \) is an operator monotone function by Theorem A, i.e., \( f(A^\alpha) \geq f(B^\alpha) \) for all \( \alpha \in (0, 1] \), so we have (ii).

**Corollary 3.** Let \( A \) and \( B \) be strictly positive operators such that \( 1 \not\in \sigma(A), \sigma(B) \) and \( \log A \geq \log B \). Then

(i) For any \( \delta \in (0, 1] \) there exists \( \beta = \beta_\delta \in (0, 1] \) such that \( (e^\delta A)^\alpha \succ_{sl} B^\alpha \) holds for all \( \alpha \in (0, \beta) \).

(ii) For any \( p \geq 0 \) there exists \( K_p > 1 \) such that \( K_p \to 1 \) as \( p \to +0 \) and \( (K_p A)^p \succ_{l} B^p \) for all \( \alpha \in (0, 1] \).

We cite the following two results in order to give a proof of Corollary 3.

**Theorem A** [1][3]. Let \( A \) and \( B \) be invertible positive operators on a Hilbert space \( H \). \( \log A \geq \log B \) holds if and only if for any \( \delta \in (0, 1] \) there exists \( \alpha = \alpha_\delta > 0 \) such that \( (e^\delta A)^\alpha > B^\alpha \).

**Theorem B** [8]. Let \( A \) and \( B \) be invertible positive operators on a Hilbert space \( H \). \( \log A \geq \log B \) if and only if for any \( p \geq 0 \) there exists a \( K_p > 1 \) such that \( K_p \to 1 \) as \( p \to +0 \) and \( (K_p A)^p \geq B^p \).
Proof of Corollary 3.

(i) As $\log A \geq \log B$ holds, then for any $\delta \in (0, 1]$, there exists $\alpha' = \alpha'_\delta > 0$ such that $(e^\delta A)^{\alpha'} > B^{\alpha'}$ by Theorem A. Then $e^\delta A > \log B$ by (5), so that there exists $\beta = \beta_\delta \in (0, 1]$ such that $(e^\delta A)^\alpha \succ_{sd} B^\alpha$ holds for all $\alpha \in (0, \beta)$ by Theorem 1.

(ii) As $\log A \geq \log B$ holds, then for any $p \geq 0$ there exists a there exists $K_p > 1$ such that $K_p \rightarrow 1$ as $p \rightarrow +0$ and $(K_pA)^p \geq B^p$ by Theorem B, so we have $(K_pA)^{p\alpha} \geq B^{p\alpha}$ for all $\alpha \in (0, 1]$ by (ii) of Corollary 2

4. Strictly dual logarithmic order $A \succ_{sd} B$ and dual logarithmic order $A \succ_{dl} B$

Let $A$ and $B$ be strictly positive operators such that $1 \notin \sigma(A), \sigma(B)$. Firstly we shall give Theorem 4 asserting the following

(1) $\log A > \log B \implies$ there exists $\beta \in (0, 1]$ such that $A^\alpha \succ_{sd} B^\alpha$ holds for all $\alpha \in (0, \beta)$.

Secondary, we shall give Corollary 5 showing that there exists an interesting contrast between $A \geq B > 0$ and $A > B > 0$ related to $A \succ_{sd} B$ and $A \succ_{dl} B$. Thirdly, we shall give some applications of Theorem A and Theorem B to $A \succ_{sd} B$ and $A \succ_{dl} B$ in Corollary 6.

**Lemma 2.** Let $A$ and $B$ be invertible self-adjoint operators on a Hilbert space $H$. If $A > B$, then there exists $\beta \in (0, 1]$ such that the following inequality holds for all $\alpha \in (0, \beta)$:
\[
\frac{\alpha A e^{\alpha A}}{e^{\alpha A} - 1} > \frac{\alpha Be^{\alpha B}}{e^{\alpha B} - 1}, \text{ i.e., } e^{\alpha A} \succ_{sd} e^{\alpha B}.
\]

**Proof.** As $-B > -A$ holds, by applying Lemma 1, there exists $\beta \in (0, 1]$ such that
\[
\frac{e^{-\alpha B} - I}{-\alpha B} > \frac{e^{-\alpha A} - I}{-\alpha A},
\]
holds for all $\alpha \in (0, \beta)$. That is, $\frac{e^{-\alpha B} - I}{\alpha Be^{\alpha B}} > \frac{e^{-\alpha A} - I}{\alpha Ae^{\alpha A}}$ holds iff $\frac{\alpha Ae^{\alpha A}}{e^{\alpha A} - I} > \frac{\alpha Be^{\alpha B}}{e^{\alpha B} - I}$ holds, i.e., there exists $\beta \in (0, 1]$ such that $e^{\alpha A} \succ_{sd} e^{\alpha B}$ holds for all $\alpha \in (0, \beta)$ and the proof is complete.

**Theorem 4.** Let $A$ and $B$ be strictly positive operators such that $1 \notin \sigma(A), \sigma(B)$.

If $\log A > \log B$, then there exists $\beta \in (0, 1]$ such that $A^\alpha \succ_{sd} B^\alpha$ holds for all $\alpha \in (0, \beta)$.

**Proof.** We have only to replace $A$ by $\log A$ and also $B$ by $\log B$ respectively in Lemma 2.

**Corollary 5.** Let $A$ and $B$ be strictly positive operators such that $1 \notin \sigma(A), \sigma(B)$. Then

(i) If $A > B > 0$, then there exists $\beta \in (0, 1]$ such that $A^\alpha \succ_{sd} B^\alpha$ holds for all $\alpha \in (0, \beta)$.

(ii) If $A \geq B > 0$, then $A^\alpha \succ_{dl} B^\alpha$ for all $\alpha \in (0, 1]$. 
In Corollary 5, it is interesting to point out the contrast between $A > B > 0$ and $A \geq B > 0$.

**Proof of Corollary 5.** By the same way as a proof of Corollary 2, we shall give the following proofs of (i) and (ii).

(i) If $A > B > 0$, then $\log A > \log B$ holds by (5), so that (i) follows by Theorem 4.

(ii) If $A \geq B > 0$, then $A^\alpha \geq B^\alpha$ for all $\alpha \in (0,1]$ by Löwner-Heinz inequality. The function $f^*(t) = \frac{\log t}{t-1} (t > 0, t \neq 1)$ is also an operator monotone function by Theorem A, so that $f^*(A^\alpha) \geq f^*(B^\alpha)$ for all $\alpha \in (0,1]$, so we have (ii).

**Corollary 6.** Let $A$ and $B$ be strictly positive operators such that $1 \notin \sigma(A), \sigma(B)$ and $\log A \geq \log B$. Then

(i). For any $\delta \in (0,1]$ there exists $\beta = \beta_\delta \in (0,1]$ such that $(e^\delta A)^\alpha\succ_{sdl} B^\alpha$ holds for all $\alpha \in (0,\beta)$.

(ii). For any $p \geq 0$ there exists $K_p > 1$ such that $K_p \rightarrow 1$ as $p \rightarrow +0$ and $(K_p A)^{p\alpha} \succ_{dl} B^{p\alpha}$ for all $\alpha \in (0,1]$.

**Proof of Corollary 6.** We shall obtain Corollary 6 by the same way as one in Corollary 3.

(i). As $\log A \geq \log B$ holds, then for any $\delta \in (0,1]$, there exists $\alpha' = \alpha'_\delta > 0$ such that $(e^\delta A)^\alpha' > B^{\alpha'}$ by Theorem A. Then $\log e^\delta A > \log B$ by (5), so that there exists $\beta = \beta_\delta \in (0,1]$ such that $(e^\delta A)^\alpha \succ_{sdl} B^\alpha$ holds for all $\alpha \in (0,\beta)$ by Theorem 4.

(ii). As $\log A \geq \log B$ holds, then for any $p \geq 0$ there exists a there exists $K_p > 1$ such that $K_p \rightarrow 1$ as $p \rightarrow +0$ and $(K_p A)^p \geq B^p$ by Theorem B, so that $(K_p A)^{p\alpha} \succ_{dl} B^{p\alpha}$ for all $\alpha \in (0,1]$ by (ii) of Corollary 5.

5. An example related to strictly logarithmic order $A \succ_{sl} B$ and strictly dual logarithmic order $A \succ_{sdt} B$

Related to (i) of Corollary 3, we consider the following problem:

(Q1) "Does $\log A \geq \log B$ ensure that there exists an $\alpha > 0$ such that $A^\alpha \succ_{l} B^\alpha$?"

Also related to (i) of Corollary 6, we consider the following problem too;

(Q2) "Does $\log A \geq \log B$ ensure that there exists an $\alpha > 0$ such that $A^\alpha \succ_{dl} B^\alpha$?"

In fact, we cite a counterexample to (Q1) and (Q2) as follows.
Example 1. Take $A$ and $B$ as follows:

$$
\log A = \begin{pmatrix} 2 & 2 \\ 2 & -1 \end{pmatrix} \quad \text{and} \quad \log B = \begin{pmatrix} 1 & 0 \\ 0 & -5 \end{pmatrix}.
$$

Then $\log A \geq \log B$ holds, but

(i) $A^\alpha \succ_l B^\alpha$ does not hold for any $\alpha > 0$.

(ii) $A^\alpha \succ_d B^\alpha$ does not hold for any $\alpha > 0$.

(iii) $A^\alpha \geq B^\alpha$ does not hold for any $\alpha > 0$.

In fact, $\log A$ is diagonalized by $U = \frac{1}{\sqrt{5}} \begin{pmatrix} -1 & 2 \\ 2 & 1 \end{pmatrix}$ as follows;

$$U(\log A)U = \begin{pmatrix} -2 & 0 \\ 0 & 3 \end{pmatrix}, \quad \text{and} \quad UAU = \begin{pmatrix} e^{-2} & 0 \\ 0 & e^3 \end{pmatrix},$$

so that we have

$$A^\alpha = U \begin{pmatrix} e^{-2\alpha} & 0 \\ 0 & e^{3\alpha} \end{pmatrix} U \text{ and } B^\alpha = \begin{pmatrix} e^{\alpha} & 0 \\ 0 & e^{-5\alpha} \end{pmatrix}.$$

Put $x = e^\alpha > 1$ since $\alpha > 0$. At first we show (i). By a slight elaborate calculation, we have

$$\det \left( \frac{A^\alpha - I}{\log A} - \frac{B^\alpha - I}{\log B} \right)$$

$$= \left| \begin{array}{ccc} 5 & 0 & 0 \\ -1 & 1 & 0 \\ 1 & -2 & 1 \end{array} \right|$$

$$= -\frac{1}{50x^7} \left( 10x^5 + 33x^4 + 48x^3 + 38x^2 + 18x + 3 \right) < 0 \text{ since } x > 1.$$

Whence $A^\alpha \succ_l B^\alpha$ does not hold for any $\alpha > 0$, so the proof of (i) is complete.

Next we show (ii). By more elaborate calculation than (i), we obtain

$$\det \left( \frac{A^\alpha \log A}{A^\alpha - I} - \frac{B^\alpha \log B}{B^\alpha - I} \right)$$

$$= -\frac{(x-1)^6(10x^5 + 33x^4 + 48x^3 + 38x^2 + 18x + 3)}{150x^7} < 0 \text{ since } x > 1.$$
\[
\begin{align*}
&= \frac{7x^3 + 9x^2 + x - 2}{5(x^3 + 2x^2 + 2x + 1)} - \frac{2(3x^3 + 6x^2 + 4x + 2)}{5(x^3 + 2x^2 + 2x + 1)} \\
&= \left( \frac{7x^3 + 9x^2 + x - 2}{5(x^3 + 2x^2 + 2x + 1)} \right) - \left( \frac{2(3x^3 + 6x^2 + 4x + 2)}{5(x^3 + 2x^2 + 2x + 1)} \right)^2 \\
&= \frac{(x-1)^2(3x^5 + 18x^4 + 38x^3 + 48x^2 + 33x + 10)}{5(x^3 + x^2 + x + 1)(x^4 + x^3 + x^2 + x + 1)} < 0 \text{ since } x > 1
\end{align*}
\]

Whence \( A^\alpha \succ_{sl} B^\alpha \) does not hold for any \( \alpha > 0 \), so the proof of (ii) is complete.

Incidentally, we remark that this example also shows that \( \log A \geq \log B \) does not ensure \( A^\alpha \geq B^\alpha \) for any \( \alpha > 0 \). Actually we have

\[
\text{det}(A^\alpha - B^\alpha)
\]

\[
= \left| \begin{array}{cc}
1 & \frac{2(x^5 - 1)}{5x^2} \\
\frac{2(x^5 - 1)}{5x^2} & \frac{2(x^5 - 1)}{5x^2}
\end{array} \right|
\]

\[
= -\frac{1}{5x^7} + x^{-4} - \frac{4}{5x^2} + x - x^4
\]

\[
= \frac{-(x - 1)^4(x + 1)(x^2 + x + 1)(x^4 + 2x^3 + 4x^2 + 2x + 1)}{5x^7} < 0 \text{ since } x > 1,
\]

that is, \( A^\alpha \geq B^\alpha \) does not hold for any \( \alpha > 0 \), so (iii) is shown. In [2], there is another nice example that \( \log A \geq \log B \) does not ensure \( A^\alpha \geq B^\alpha \) for any \( \alpha > 0 \). In fact, we construct Example 1 inspired by an excellent method in [2].

6. Concluding remarks

Let \( A \) and \( B \) be strictly positive operators such that \( 1 \notin \sigma(A), \sigma(B) \). We can obtain the following interesting contrast among \( A > B > 0 \), \( A \geq B > 0 \), \( \log A > \log B \) and \( \log A \geq \log B \) by summarizing our results in this paper.

\((*)\) \( \log A > \log B \implies \) there exists \( \beta \in (0, 1] \) such that \( A^\alpha \succ_{sl} B^\alpha \) holds for all \( \alpha \in (0, \beta) \).

\((i)\) \( \log A > \log B \implies \) there exists \( \beta \in (0, 1] \) such that \( A^\alpha \succ_{sdi} B^\alpha \) holds for all \( \alpha \in (0, \beta) \).

\((i)\) \( A > B > 0 \implies \) there exists \( \beta \in (0, 1] \) such that \( A^\alpha \succ_{sl} B^\alpha \) holds for all \( \alpha \in (0, \beta) \).
(l-ii) $A \geq B > 0 \Rightarrow A^\alpha \succ_l B^\alpha$ for all $\alpha \in (0,1].$

(l-iii) $\log A \geq \log B \Rightarrow \text{for any} \, \delta \in (0,1], \text{there exists} \, \beta = \beta_\delta \in (0,1] \text{such that} \, (e^\delta A)^\alpha \succ_{sl} B^\alpha \text{holds for all} \, \alpha \in (0,\beta).$

(l-iv) $\log A \geq \log B \Rightarrow \text{for any} \, p \geq 0 \text{there exists} \, K_p > 1 \text{such that} \, K_p \rightarrow 1 \text{as} \, p \rightarrow +0 \text{and} \, (K_p A)^p \alpha \succ_{dl} B^{p\alpha} \text{for all} \, \alpha \in (0,1].$

(dl-i) $A > B > 0 \Rightarrow \text{there exists} \, \beta \in (0,1] \text{such that} \, A^\alpha \succ_{sdl} B^\alpha \text{holds for all} \, \alpha \in (0,\beta).$

(dl-ii) $A \geq B > 0 \Rightarrow \text{for all} \, \alpha \in (0,1].$

(dl-iii) $\log A \geq \log B \Rightarrow \text{for any} \, \delta \in (0,1], \text{there exists} \, \beta = \beta_\delta \in (0,1] \text{such that} \, (e^\delta A)^\alpha \succ_{sdl} B^\alpha \text{holds for all} \, \alpha \in (0,\beta).$

(dl-iv) $\log A \geq \log B \Rightarrow \text{for any} \, p \geq 0 \text{there exists} \, K_p > 1 \text{such that} \, K_p \rightarrow 1 \text{as} \, p \rightarrow +0 \text{and} \, (K_p A)^p \alpha \succ_{dl} B^{p\alpha} \text{for all} \, \alpha \in (0,1].$

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References


7. Appendix

**Simple proof of the concavity on operator entropy** $f(A) = -A \log A$

A capital letter means a bounded linear and strictly positive operator on a Hilbert space. Here we shall give a simple proof of the following well known and excellent result obtained by [1] and [2] independently.

**Theorem A.** $f(A) = -A \log A$ is concave function for any $A > 0$.

**Proof.** Firstly we recall the following obvious result

\[(*) \quad \lim_{n \to \infty} (T^{-\frac{1}{n}} - I)n = -\log T \quad \text{for any } T > 0.\]

As $g(t) = t^q$ is operator concave for $q \in [0, 1]$, then for $A > 0$, $B > 0$ and $\alpha, \beta \in [0, 1]$ such that $\alpha + \beta = 1$

\[(\alpha A + \beta B)^{1-\frac{1}{n}} \geq \alpha A^{1-\frac{1}{n}} + \beta B^{1-\frac{1}{n}} \quad \text{for any natural number } n\]

so we obtain

\[-(\alpha A + \beta B) \log (\alpha A + \beta B) \geq (-\alpha A \log A - \beta B \log B) \quad \text{by } (*)\]

that is,

\[f(\alpha A + \beta B) \geq \alpha f(A) + \beta f(B)\]

so the proof is complete.

**References**


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