EXTENSIONS OF PARANORMAL OPERATORS AND THEIR PROPERTIES

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ABSTRACT. This report is based on the following papers:


An operator $T$ is said to be paranormal if $\|T^2x\| \geq \|Tx\|^2$ holds for every unit vector $x$. Several extensions of paranormal operators have been considered until now, for example, absolute-$k$-paranormal and $p$-paranormal operators introduced in [6] and [3], respectively. As a further generalization of paranormal operators, we shall introduce absolute-$(p, r)$-paranormal operators for $p > 0$ and $r > 0$ such that $\|\|T^p|T^*|^r x\|^r \geq \|T^*|^r x\|^{p+r}$ for every unit vector $x$. We shall show several properties on absolute-$(p, r)$-paranormal operators as generalizations of the results on absolute-$k$-paranormal and $p$-paranormal operators. We shall also show a characterization of log-hyponormal operators via absolute-$(p, r)$-paranormality, that is, an invertible operator $T$ satisfies $\log T^* T \geq \log TT^*$ if and only if $T$ is absolute-$(p, r)$-paranormal for all $p > 0$ and $r > 0$.

1. Introduction

In this report, an operator means a bounded linear operator on a complex Hilbert space $H$. An operator $T$ is said to be positive (denoted by $T \geq 0$) if $(Tx, x) \geq 0$ for all $x \in H$, and also $T$ is said to be strictly positive (denoted by $T > 0$) if $T$ is positive and invertible.

An operator $T$ is said to be $p$-hyponormal for $p > 0$ if $(T^* T)^p \geq (TT^*)^p$, and $T$ is said to be log-hyponormal if $T$ is invertible and $\log T^* T \geq \log TT^*$. $p$-Hyponormal and log-hyponormal operators were defined as extensions of hyponormal ones, i.e., $T^* T \geq TT^*$. It is easily seen that every $p$-hyponormal operator is $q$-hyponormal for $p \geq q > 0$ by the celebrated Löwner-Heinz theorem "$A \geq B \geq 0$ ensures $A^\alpha \geq B^\alpha$ for any $\alpha \in [0, 1]$," and every invertible $p$-hyponormal operator for $p > 0$ is log-hyponormal since $\log t$ is an operator monotone function.

An operator $T$ is said to be paranormal if

\begin{equation}
\|T^2x\| \geq \|Tx\|^2 \quad \text{for every unit vector } x.
\end{equation}

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Paranormal operators have been studied by many researchers, for example, [1][5] and [7]. Particularly, Ando [1] showed that every log-hyponormal operator is paranormal. Afterward, in [6], we gave another simple proof of this result by introducing class A as a new class of operators given by an operator inequality. An operator $T$ belongs to class A if

$$|T^2| \geq |T|^2,$$

where $|T| = (T^*T)^{\frac{1}{2}},$ and we showed that every log-hyponormal operator belongs to class A and every class A operator is paranormal.

In [6], we introduced class $A(k)$ and absolute-$k$-paranormal operators for $k > 0$ as generalizations of class A and paranormal operators, respectively. An operator $T$ belongs to class $A(k)$ if

$$(T^*|T|^{2k}T)^{\frac{1}{k+1}} \geq |T|^2,$$

and $T$ is said to be absolute-$k$-paranormal if

$$|||T^{k}Tx|| \geq ||Tx||^{k+1} \text{ for every unit vector } x.$$  

(1.2)

It is clear that class $A(1)$ equals class A and absolute-$1$-paranormality equals paranormality since $||S|| = ||Sy||$ for any $S \in B(H)$ and $y \in H$. An operator $T$ is said to be normaloid if $||T|| = r(T)$. We showed inclusion relations among these classes in [6]. Class A and class $A(k)$ operators have been studied in [8][9] and [11].

On the other hand, Fujii, Izumino and Nakamoto [3] introduced $p$-paranormal operators for $p > 0$ as another generalization of paranormal operators. An operator $T$ is said to be $p$-paranormal if

$$|||T^pU|T|^p x|| \geq ||T|^{p}x||^2 \text{ for every unit vector } x,$$

where the polar decomposition of $T$ is $T = U|T|$. It is clear that $1$-paranormality equals paranormality. $p$-Paranormality is based on the following fact [2]: $T = U|T|$ is $p$-hyponormal if and only if $S = U|T|^p$ is hyponormal for $p > 0$. Actually, it was shown in [3] that $T = U|T|$ is $p$-paranormal if and only if $S = U|T|^p$ is paranormal for $p > 0$.

Fujii, Jung, S.H.Lee, M.Y.Lee and Nakamoto [4] introduced class $A(p, r)$ as a further generalization of class $A(k)$. An operator $T$ belongs to class $A(p, r)$ for $p > 0$ and $r > 0$ if

$$||(T^*r|T|^{2p}T^*r^*)^\frac{r}{p+r} \geq |T^*|^{2r},$$

(1.4)

and class $AI(p, r)$ is the class of all invertible operators which belong to class $A(p, r)$. It was pointed out in [11] that class $A(k, 1)$ equals class $A(k)$ for each $k > 0$.

In this report, as a parallel concept to class $A(p, r)$, we shall introduce absolute-$(p, r)$-paranormality which is a further generalization of both absolute-$k$-paranormality and $p$-paranormality. Then we shall generalize the results on absolute-$k$-paranormal and $p$-paranormal operators for absolute-$(p, r)$-paranormal operators. We shall also show a characterization of log-hyponormal operators via absolute-$(p, r)$-paranormality and $p$-paranormality.
2. Definition and properties of absolute-$(p, r)$-paranormal operators

We introduce the following class of operators.

**Definition ([Y2]).** For positive real numbers $p > 0$ and $r > 0$, an operator $T$ is absolute-$(p, r)$-paranormal if

$$(2.1) \quad |||T^p|||T^r||x||^r \geq |||T^r||x||^{p+r} \quad \text{for every unit vector } x,$$

or equivalently,

$$(2.2) \quad |||T^p|||T^r||x||^r||x||^p \geq |||T^r||x||^{p+r} \quad \text{for all } x \in H.$$  

We remark that the definition of absolute-$(p, r)$-paranormal operators (2.1) and (2.2) are expressed in terms of only $T$ and $T^*$, without $U$ which appears in the polar decomposition of $T = U|T|$. 

To consider the relations to absolute-$k$-paranormality and $p$-paranormality, we show another expression of absolute-$(p, r)$-paranormality as follows.

**Proposition 1 ([Y2]).** For each $p > 0$ and $r > 0$, $T$ is absolute-$(p, r)$-paranormal if and only if

$$(2.3) \quad |||T^p|||T^r||x||^r \geq |||T^r||x||^{p+r} \quad \text{for every unit vector } x,$$

where the polar decomposition of $T$ is $T = U|T|$. 

The following result is easily obtained as a corollary of Proposition 1.

**Corollary 2 ([Y2]).**

(i) For each $k > 0$, $T$ is absolute-$k$-paranormal iff $T$ is absolute-$(k, 1)$-paranormal.

(ii) For each $p > 0$, $T$ is $p$-paranormal iff $T$ is absolute-$(p, p)$-paranormal.

(iii) $T$ is paranormal iff $T$ is absolute-$(1, 1)$-paranormal.

It turns out by Corollary 2 that absolute-$(p, r)$-paranormality is a further generalization of paranormality than both absolute-$k$-paranormality and $p$-paranormality.

**Proof of Proposition 1.** It is well known that $|T^*|^r = U|T|^r U^*$ for $r > 0$, so that (2.2) is equivalent to the following (2.4):

$$(2.4) \quad |||T^p|||T^r||U^*x||^r||x||^p \geq ||U|T|^rU^*x||^{p+r} \quad \text{for all } x \in H.$$ 

It is also well known that $N(S^r) = N(S)$ for any $S \geq 0$ and $r > 0$. By using this fact, we have $R(|||T^r||) \subseteq R(|||T||^r||) = N(|||T^r||)^\perp = N(U)^\perp$, so that $||U|T|^rU^*x|| = |||T^r||x||^p$ for all $x \in H$. Hence (2.4) is equivalent to the following (2.5):

$$(2.5) \quad |||T^p|||T^r||U^*x||^r||x||^p \geq |||T^r||U^*x||^{p+r} \quad \text{for all } x \in H.$$ 

Put $x = Uy$ in (2.5), then we have the following (2.6) since $|T|^rU^*U = |T|^r$:

$$(2.6) \quad |||T^p|||T^r||y||^r||Uy||^p \geq |||T^r||y||^{p+r} \quad \text{for all } y \in H.$$
(2.6) yields the following (2.7) since \( \|y\| \geq \|Uy\| \) for all \( y \in H \):

\[
(2.7) \quad \|T^pU|T^r|y\| \|y\| \geq \|T^rU^*x\|^{p+r} \quad \text{for all } y \in H.
\]

Hence (2.5) implies (2.7). Here we show that (2.7) implies (2.5) conversely. Put \( y = U^*x \) in (2.7), then we have

\[
(2.8) \quad \|T^pU|T^r|U^*x\| \|U^*x\| \geq \|T^rU^*x\|^{p+r} \quad \text{for all } x \in H.
\]

(2.8) yields (2.5) since \( \|x\| \geq \|U^*x\| \) for all \( x \in H \). Hence (2.7) implies (2.5), so that (2.5) is equivalent to (2.7). Consequently, the proof of Proposition 1 is complete since (2.7) is equivalent to (2.3).

\( \square \)

**Proof of Corollary 2.** We remark that \( \|S|y\| = \|Sy\| \) holds for any \( S \in B(H) \) and \( y \in H \).

(i) Put \( p = k > 0 \) and \( r = 1 \) in (2.3), then we have (1.2).

(ii) Put \( r = p > 0 \) in (2.3), then we have (1.3).

(iii) Put \( r = p = 1 \) in (2.3), then we have (1.1).

It was shown in [5] and [7] that if \( T \) is invertible and paranormal, then \( T^{-1} \) is also paranormal. Here we show the following generalization of this well-known result.

**Proposition 3 ([Y2]).** The following assertions hold for each \( p > 0 \) and \( r > 0 \):

(i) If \( T \) is invertible and absolute-(\( p, r \))-paranormal, then \( T^{-1} \) is absolute-(\( r, p \))-paranormal.

(ii) If \( T \) is invertible and \( p \)-paranormal, then \( T^{-1} \) is also \( p \)-paranormal.

Proposition 3 can be considered as a parallel result to the following Proposition A for class \( \text{AI}(p, r) \) operators.

**Proposition A ([Y2]).** The following assertions hold for each \( p > 0 \) and \( r > 0 \):

(i) If \( T \) belongs to class \( \text{AI}(p, r) \), then \( T^{-1} \) belongs to class \( \text{AI}(r, p) \).

(ii) If \( T \) belongs to class \( \text{AI}(p, p) \), then \( T^{-1} \) also belongs to class \( \text{AI}(p, p) \).

(iii) If \( T \) is invertible and belongs to class \( A \), then \( T^{-1} \) also belongs to class \( A \).

We prepare the following lemma to give a proof of Proposition 3.

**Lemma 4 ([Y2]).** Let \( T \) be an invertible operator. For each \( p > 0 \) and \( r > 0 \), \( T \) is absolute-(\( p, r \))-paranormal if and only if

\[
(2.9) \quad \|T^p|x\| \|T^{-1}|x\| \geq 1 \quad \text{for every unit vector } x.
\]

**Proof.** (2.2) is equivalent to the following (2.10) by putting \( y = |T^r|x \) since \( R(|T^r|) = H \):

\[
(2.10) \quad \|T^p|y\| \|T^r|y\| \geq \|y\|^{p+r} \quad \text{for all } y \in H.
\]

(2.10) is equivalent to the following (2.11):

\[
(2.11) \quad \|T^p|y\| \|T^r|y\| \geq 1 \quad \text{for every unit vector } y.
\]

(2.11) is equivalent to (2.9) since \( |T^r|^{-1} = |T^{-1}| \), so that the proof is complete. \( \square \)
Proof of Proposition 3.

(i) Obvious by Lemma 4.

(ii) Put \( r = p > 0 \) in (i), then we have (ii) by (ii) of Corollary 2.

\[ \square \]

3. Inclusion relations among the related classes

We cite the following result which plays an important role to give proofs of the results in this section.

**Theorem H-M** (Hölder-McCarthy inequality [10]). Let \( A \) be a positive operator. Then the following inequalities hold for all \( x \in H \):

(i) \((A^r x, x) \leq (Ax, x)^r \|x\|^{2(1-r)} \) for \( 0 < r \leq 1 \).

(i') \( \|A^r x\| \leq \|Ax\|^r \|x\|^{1-r} \) for \( 0 < r \leq 1 \).

(ii) \((A^r x, x) \geq (Ax, x)^r \|x\|^{2(1-r)} \) for \( r \geq 1 \).

(ii') \( \|A^r x\| \geq \|Ax\|^r \|x\|^{1-r} \) for \( r \geq 1 \).

Firstly, we show the monotonicity of the classes of absolute-\((p, r)\)-paranormal operators for \( p > 0 \) and \( r > 0 \) as generalizations of [4, Theorem 4.1] and [6, Theorem 4].

**Theorem 5** ([Y2]). Let \( T \) be absolute-\((p_0, r_0)\)-paranormal for \( p_0 > 0 \) and \( r_0 > 0 \). Then \( T \) is absolute-\((p, r)\)-paranormal for any \( p \geq p_0 \) and \( r \geq r_0 \). Moreover, for each \( r \geq r_0 \) and unit vector \( x \),

\[(3.1) \quad f_r(p) = \|T^p |T^*|^r x\|^{\frac{r}{p+r}} \]

is increasing for \( p \geq p_0 \).

Theorem 5 can be considered as a parallel result to the following Theorem B which states the monotonicity of class \( AI(p, r) \) for \( p > 0 \) and \( r > 0 \).

**Theorem B** ([4]). If \( T \) belongs to class \( AI(p_0, r_0) \) for \( p_0 > 0 \) and \( r_0 > 0 \), then \( T \) belongs to class \( AI(p, r) \) for any \( p \geq p_0 \) and \( r \geq r_0 \).

**Proof of Theorem 5.** Assume that \( T \) is absolute-\((p_0, r_0)\)-paranormal for \( p_0 > 0 \) and \( r_0 > 0 \), i.e.,

\[(3.2) \quad \|T^{p_0} |T^*|^{r_0} y\|^{r_0} \geq \|T^*|^{r_0} y\|^{p_0} \|y\|^{r_0} \quad \text{for all } y \in H. \]

Then for each \( r \geq r_0 \) and unit vector \( x \),

\[
\begin{align*}
\|T^{p_0} |T^*|^{r_0} x\|^{r_0} & = \|T^{p_0} |T^*|^{r_0} |T^*|^{-r_0} x\|^{r_0} \\
& \geq \|T^*|^{r_0} |T^*|^{-r_0} x\|^{p_0+r_0} \|T^*|^{r_0} x\|^{r_0} \quad \text{by (3.2)} \\
& = \|T^*|^{r_0} x\|^{p_0+r_0} \|T^*|^{-r_0} x\|^{-p_0} \\
& \geq \|T^*|^{r_0} x\|^{p_0+r_0} \|T^*|^{r_0} x\|^{-r_0} \quad \text{by (i') of Theorem H-M for } \frac{r-r_0}{r} \in [0, 1) \\
& = \|T^*|^{r_0} x\|^{p_0},
\end{align*}
\]

...
so that we have

\[(3.3) \quad \|\| T^0 | T^* x \|^{p0} \geq \|\| T^* x \|^{p}.\]

Hence for each \( p \geq p_0, r \geq r_0 \) and unit vector \( x \),

\[
\|\| T^p | T^* x \| \geq \|\| T^p | T^* x \|^{p0} \geq \|\| T^* x \|^{p0} \geq \|\| T^* x \|^{p0}.\]

by (ii) of Theorem H-M for \( p \geq 1 \)

\[
\geq \|\| T^p | T^* x \| \geq \|\| T^p | T^* x \|^{r0} \geq \|\| T^* x \|^{r0} \geq \|\| T^* x \|^{r0}.\]

by (3.3)

\[
\geq \|\| T^* x \|^{r0} \geq \|\| T^* x \|^{p0} \geq \|\| T^* x \|^{p0} \geq \|\| T^* x \|^{p0}.\]

so that we have

\[(3.4) \quad \|\| T^p | T^* x \| \geq \|\| T^p | T^* x \|^{p0} \geq \|\| T^* x \|^{p0} \geq \|\| T^* x \|^{p0}.\]

(3.4) assures that \( T \) is absolute-(\( p, r \))-paranormal for any \( p \geq p_0 \) and \( r \geq r_0 \), and for each \( r \geq r_0 \) and unit vector \( x \), \( f_r(p) = \|\| T^p | T^* x \| \geq \|\| T^p | T^* x \|^{r} \geq \|\| T^* x \|^{r} \geq \|\| T^* x \|^{r} \)

by (3.3),

so that we have

\[(2.1) \quad \|\| T^p | T^* x \| \geq \|\| T^p | T^* x \|^{p0} \geq \|\| T^* x \|^{p0} \geq \|\| T^* x \|^{p0}.\]

for every unit vector \( x \), i.e., \( T \) is absolute-(\( p, r \))-paranormal.

Secondly, we show inclusion relations among the class of absolute-(\( p, r \))-paranormal operators and the related classes.

**Theorem 6 ([Y2]).** The following assertions hold for each \( p > 0 \) and \( r > 0 \):

(i) Every class \( A(p, r) \) operator is absolute-(\( p, r \))-paranormal.

(ii) Every absolute-(\( p, r \))-paranormal operator is normaloid.

(i) of Theorem 6 is a generalization of [4, Theorem 3.5] and [6, Theorem 4], and (ii) is a generalization of [6, Theorem 5] and the following result.

**Theorem C ([4]).** Every \( p \)-paranormal operator is normaloid for \( p > 0 \).

**Proof of Theorem 6.**

**Proof of (i).** Assume that \( T \) belongs to class \( A(p, r) \) for \( p > 0 \) and \( r > 0 \), i.e.,

\[(1.4) \quad \|\| T^* x \| \geq \|\| T^* x \|^{2p} \geq \|\| T^* x \|^{p}.\]

Then for every unit vector \( x \),

\[
\|\| T^* x \|^{2p} = \|\| T^* x \|^{p} \geq \|\| T^* x \|^{p} \geq \|\| T^* x \|^{p}.\]

so that we have

\[(2.1) \quad \|\| T^* x \|^{2p} \geq \|\| T^* x \|^{p}.\]

for every unit vector \( x \), i.e., \( T \) is absolute-(\( p, r \))-paranormal.
Proof of (ii). Assume that $T$ is absolute-$(p, r)$-paranormal. Put $q = \max\{p, r\} > 0$, then $T$ is absolute-$(q, q)$-paranormal by Theorem 5, i.e., $T$ is $q$-paranormal by (ii) of Corollary 2. Hence $T$ is normaloid by Theorem C.

4. A characterization of log-hyponormal operators via absolute-$(p, r)$-paranormality and $p$-paranormality

In [4], the following result was shown which is a characterization of log-hyponormal operators in terms of class $\text{AI}(p, r)$.

**Theorem D ([4]).** The following assertions are mutually equivalent:

(i) $T$ is log-hyponormal.
(ii) $T$ belongs to class $\text{AI}(p, p)$ for all $p > 0$.
(iii) $T$ belongs to class $\text{AI}(p, r)$ for all $p > 0$ and $r > 0$.

Theorem D states that the class of log-hyponormal operators can be considered as the limit of class $\text{AI}(p, r)$ as $p \to +0$ and $r \to +0$ since class $\text{AI}(p, r)$ is monotone increasing for $p > 0$ and $r > 0$ by Theorem B.

Here we shall give a characterization of log-hyponormal operators in terms of absolute-$(p, r)$-paranormality and $p$-paranormality.

**Theorem 7 ([Y1][Y2]).** The following assertions are mutually equivalent:

(i) $T$ is log-hyponormal.
(ii) $T$ is invertible and $p$-paranormal for all $p > 0$.
(iii) $T$ is invertible and absolute-$(p, r)$-paranormal for all $p > 0$ and $r > 0$.

Theorem 7 states that the class of log-hyponormal operators can be considered as the limit of the class of invertible and absolute-$(p, r)$-paranormal operators as $p \to +0$ and $r \to +0$ since the class of absolute-$(p, r)$-paranormal operators is monotone increasing for $p > 0$ and $r > 0$ by Theorem 5. It is interesting to remark that class $\text{AI}(p, r)$ and the class of invertible and absolute-$(p, r)$-paranormal operators can be considered to be parallel to each other, but their limits as $p \to +0$ and $r \to +0$ coincide. In fact, Theorem 7 gives a more precise sufficient condition for that an operator $T$ is log-hyponormal than Theorem D by (i) of Theorem 6.

In order to give a proof of Theorem 7, we prepare the following result.

**Proposition 8 ([Y2]).** The following assertions hold for each $p > 0$ and $r > 0$:

(i) $T$ is absolute-$(p, r)$-paranormal if and only if

\begin{equation}
|T^*|^p |T|^{2p} |T^*|^r - (p + r) \lambda^p |T^*|^{2r} + p \lambda^{p+r} I \geq 0 \quad \text{for all } \lambda > 0.
\end{equation}

(ii) $T$ is $p$-paranormal if and only if

\begin{equation}
|T^*|^p |T|^{2p} |T^*|^p - 2 \lambda |T^*|^{2p} + \lambda^2 I \geq 0 \quad \text{for all } \lambda > 0.
\end{equation}
Ando [1] gave a characterization of paranormal operators via an operator inequality as follows: $T$ is paranormal if and only if
\[ T^2T^2 - 2\lambda T^*T + \lambda^2I \geq 0 \]
for all $\lambda > 0$. A generalization of this result for absolute-$k$-paranormal operators was shown in [6, Theorem 6], and Proposition 8 is a further generalization for absolute-$(p, r)$-paranormal operators.

We use the following well-known fact in the proof of Proposition 8.

**Lemma E.** For positive real numbers $a > 0$ and $b > 0$,
\[ \lambda a + \mu b \geq ab^\mu \lambda \]
holds for $\lambda > 0$ and $\mu > 0$ such that $\lambda + \mu = 1$.

**Proof of Proposition 8.**

**Proof of (i).** (2.2) is equivalent to the following (4.3):
\[ \left( |T^*|^p |T|^{2p} |T^*|^r |x,x\rangle \right)^{\frac{r}{p+r}} \geq \left( |T|^2r |x,x\rangle \right) \quad \text{for all } x \in H. \]

By Lemma E,
\[ \left( |T^*|^p |T|^{2p} |T^*|^r |x,x\rangle \right)^{\frac{r}{p+r}} = \left\{ \lambda^{-p} \left( |T^*|^p |T|^{2p} |T^*|^r |x,x\rangle \right) \right\}^{\frac{p}{p+r}} \left\{ \lambda^r \right\}^{\frac{p}{p+r}} \leq \frac{r}{p+r} \cdot \lambda^{-p} \left( |T^*|^p |T|^{2p} |T^*|^r |x,x\rangle \right) + \frac{p}{p+r} \cdot \lambda^r \]
holds for all $x \in H$ and $\lambda > 0$, so that (4.3) implies the following (4.4):
\[ \frac{r}{p+r} \cdot \lambda^{-p} \left( |T^*|^p |T|^{2p} |T^*|^r |x,x\rangle \right) + \frac{p}{p+r} \cdot \lambda^r \geq \left( |T|^2r |x,x\rangle \right) \quad \text{for all } x \in H \quad \text{and } \lambda > 0. \]

Conversely, (4.3) follows from (4.4) by putting $\lambda = \left\{ \left( |T^*|^p |T|^{2p} |T^*|^r |x,x\rangle \right) \right\}^{\frac{1}{p+r}} > 0$ in case
\[ \left( |T^*|^p |T|^{2p} |T^*|^r |x,x\rangle \right) \neq 0, \]
and letting $\lambda \to +0$ in case \( |T^*|^p |T|^{2p} |T^*|^r |x,x\rangle = 0 \). Hence (4.3) is equivalent to (4.4). Consequently, the proof of Proposition 8 is complete since (4.4) is equivalent to (4.1).

**Proof of (ii).** Put $r = p > 0$ and replace $\lambda^p$ with $\lambda$ in (i), then we have (ii) by (ii) of Corollary 2.

**Proof of Theorem 7.** It is pointed out in [4] that every log-hyponormal operator belongs to class AI $(p, r)$ for all $p > 0$ and $r > 0$, so that (i) $\implies$ (iii) holds by (i) of Theorem 6. And (iii) $\implies$ (ii) can be proved by putting $r = p > 0$ by (ii) of Corollary 2. Hence we have only to prove (ii) $\implies$ (i).

Assume that $T$ is $p$-paranormal for all $p > 0$. By (ii) of Proposition 8, (4.2) holds particularly for $\lambda = 1$, that is,
\[ |T^*|^p |T|^{2p} |T^*|^r - 2|T^*|^{2p} + I \geq 0 \quad \text{for all } p > 0. \]
Since $T$ is invertible, (4.5) can be rewritten as the following (4.6):

\[
\frac{|T|^{2p} - I}{p} \geq \frac{|T^{*}|^{-2p} - I}{-p}
\]

for all $p > 0$. By letting $p \to +0$ in (4.6), we have

\[
\log |T|^{2} \geq \log |T^{*}|^{2},
\]

i.e., $T$ is log-hyponormal.

The following diagram represents the inclusion relations among the classes discussed in this report.

References


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