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On powers of class A(k) operators including p-hyponormal and log-hyponormal operators

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ABSTRACT

In [15], we introduced class A(k) as a class of operators including p-hyponormal and log-hyponormal operators. In this report, we shall show that "*if* T *is an invertible* class A(k) *operator for* $k\in(0,1]$, *then* T^{n} *is a* class A$(\frac{k}{n})$ *operator for all positive integer* n."

Moreover, we shall show a similar result on powers of class AI(s,t) operators which were introduced in [7] as extensions of class A(k) operators, that is, "*if* T *is a* class AI(s,t) *operator for* $s,t\in(0,1]$, *then* T^{n} *is a* class AI$(\frac{s}{n},\frac{t}{n})$ *operator for all positive integer* n."

1. INTRODUCTION

In what follows, a capital letter means a bounded linear operator on a complex Hilbert space H. An operator T is said to be positive (denoted by $T\geq 0$) if $(Tx,x)\geq 0$ for all $x\in H$.

An operator T is said to be *p-hyponormal* if $(T^{*}T)^{p}\geq(TT^{*})^{p}$ for a positive number p, and T is said to be log-*hyponormal* if T is invertible and $\log T^{*}T\geq\log TT^{*}$. p-Hyponormal and log-hyponormal operators were defined as extensions of hyponormal operators, i.e., $T^{*}T\geq TT^{*}$. It is well known that "*every p-hyponormal operator is a q-hyponormal operator for* $p\geq q>0$" by the celebrated Löwner-Heinz theorem "$A\geq B\geq 0$ *ensures* $A^{\alpha}\geq B^{\alpha}$ *for any* $\alpha\in[0,1]$," and "*every invertible p-hyponormal operator is a* log-*hyponormal operator*" since $\log t$ is an operator monotone function. It is also well known that there exists a hyponormal operator T such that T^{2} is not a hyponormal operator [16, Problem 209]. Related to this fact, the following result on powers of p-hyponormal operators for $p>0$ was shown by Aluthge-Wang [3] and Ito [19].

Theorem A.1 ([3, 19]). *If T is a p-hyponormal operator for $p > 0$, then T^n is a* $\min\{1, \frac{p}{n}\}$*-hyponormal operator for all positive integer n.*

We remark that Aluthge and Wang [3] showd Theorem A.1 in case $p \in (0,1]$. Then Ito [19] showed Theorem A.1 in case $p > 0$. Moreover, we obtained the following result for log-hyponormal operators.

Theorem A.2 ([23]). *If T is a* log*-hyponormal operator, then T^n is also a* log-*hyponormal operator for all positive integer n.*

We remark that the best possibilities of Theorem A.1 and Theorem A.2 were shown in [14, 19]. Theorem A.1 and Theorem A.2 were shown as nice applications of the following Theorem F.

Theorem F (Furuta inequality [9]).

If $A \geq B \geq 0$, then for each $r \geq 0$,

(i) $$(B^{\frac{r}{2}}A^pB^{\frac{r}{2}})^{\frac{1}{q}} \geq (B^{\frac{r}{2}}B^pB^{\frac{r}{2}})^{\frac{1}{q}}$$

and

(ii) $$(A^{\frac{r}{2}}A^pA^{\frac{r}{2}})^{\frac{1}{q}} \geq (A^{\frac{r}{2}}B^pA^{\frac{r}{2}})^{\frac{1}{q}}$$

hold for $p \geq 0$ and $q \geq 1$ with $(1+r)q \geq p+r$.

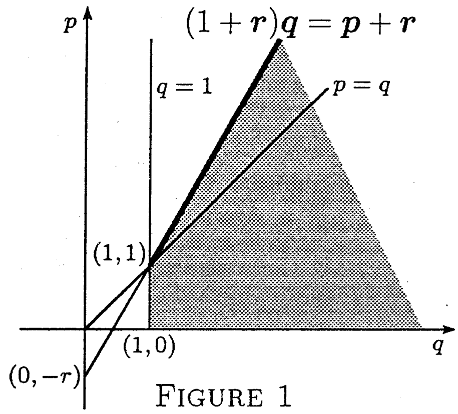

FIGURE 1

We remark that Theorem F yields the Löwner-Heinz theorem when we put $r = 0$. Alternative proofs of Theorem F are given in [6] and [20] and also an elementary one page proof in [10]. Tanahashi [22] showed that the domain drawn for p, q and r in the Figure 1 is the best possible one for Theorem F.

On the other hand, related to p-hyponormal, log-hyponormal and paranormal (i.e., $\|T^2x\| \geq \|Tx\|^2$ for every unit vector $x \in H$) operators, we introduced classes of operators in [15] as follows:

Definition 1 ([15]).

(i) *An operator T belongs to* class A *if $|T^2| \geq |T|^2$ where $|T| = (T^*T)^{\frac{1}{2}}$.*

(ii) *For each $k > 0$, an operator T belongs to* class A(k) *if*

(1.1) $$(T^*|T|^{2k}T)^{\frac{1}{k+1}} \geq |T|^2.$$

(iii) *For each $k > 0$, an operator T is said to be absolute-k-paranormal if*

$$\||T|^kTx\| \geq \|Tx\|^{k+1}$$

holds for every unit vector $x \in H$.

We remark that class A(1) equals class A, and absolute-1-paranormal equals paranormal. Related to these classes, we obtained the following result on inclusion relations among them in [15].

Theorem B ([15]).

(i) *Every* log-*hyponormal operator is a* class A(k) *operator for all* $k > 0$.
(ii) *Every invertible* class A(k) *operator is a* class A(l) *operator for* $l \geq k > 0$.
(iii) *Every absolute-k-paranormal operator is a absolute-l-paranormal operator for* $l \geq k > 0$.
(iv) *For each* $k > 0$, *every* class A(k) *operator is a absolute-k-paranormal operator.*

Inclusion relations among the classes of operators mentioned above can be expressed as follows:

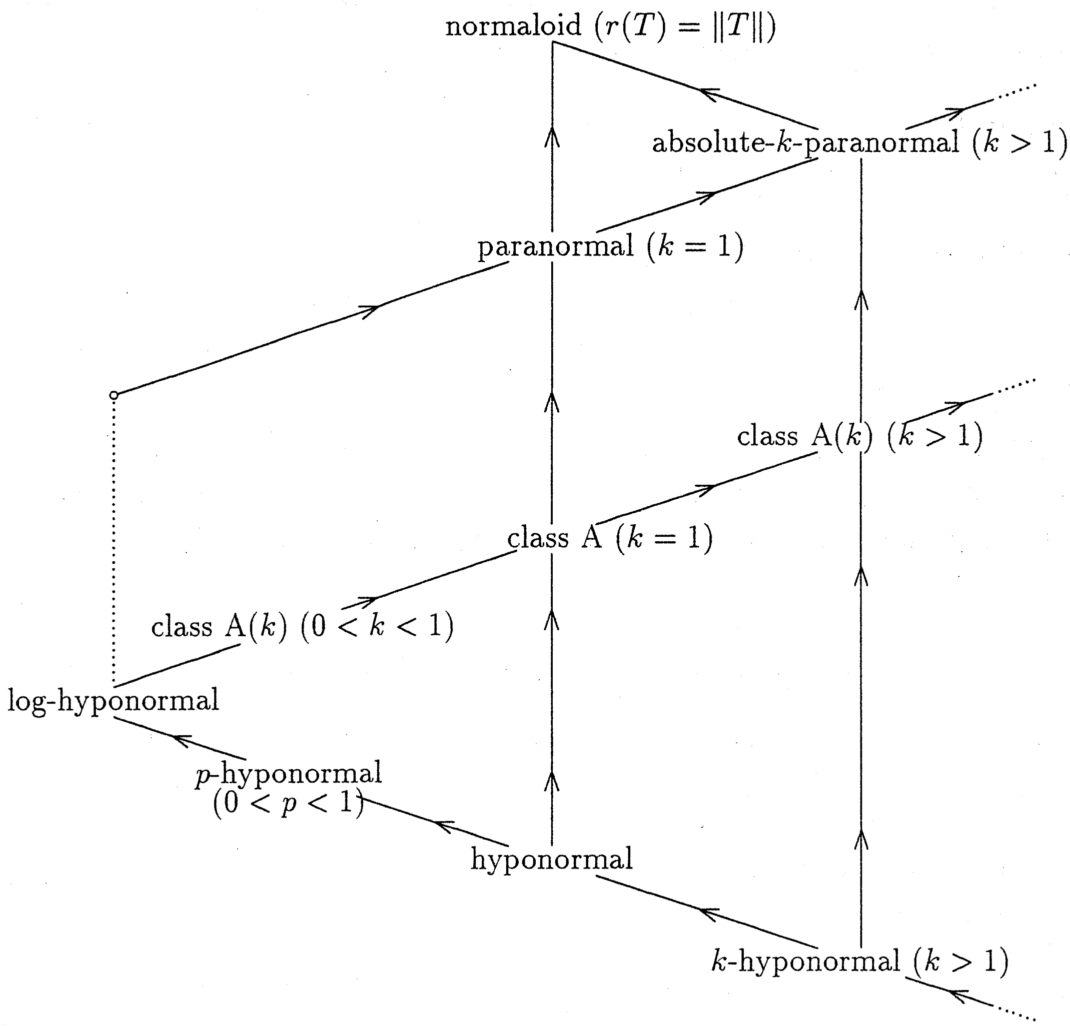

FIGURE 2

Related to Theorem A.1 and Theorem A.2 on powers of p-hyponormal and log-hyponormal operators, Ito [18] showed the following results on powers of class A operators as follows:

Theorem C.1 ([18]). *If T is an invertible* class A *operator, then T^n is also a* class A *operator for all positive integer n.*

Theorem C.2 ([18]). *Let T be an invertible* class A *operator. Then*

(i) $|T^n|^{\frac{2}{n}} \geq \cdots \geq |T^2| \geq |T|^2$

and

(ii) $|T^*|^2 \geq |T^{2^*}| \geq \cdots \geq |T^{n*}|^{\frac{2}{n}}$

hold for all positive integer n.

It is interesting to point out that these theorems are parallel results to the following theorems on paranormal operators.

Theorem C.3 ([8]). *If T is a paranormal operator, then T^n is also a paranormal operator for all positive integer n.*

Theorem C.4 ([8, 18]). *Let T be a paranormal operator. Then*

$$\|T^n x\|^{\frac{2}{n}} \geq \cdots \geq \|T^2 x\| \geq \|Tx\|^2$$

hold for every unit vector $x \in H$ and all positive integer n

In this report, firstly, we shall show a result on powers of invertible class A(k) operators for $k \in (0,1]$ in Theorem 1, which is more precise result than Theorem C.1. Secondly, we shall show similar results to Theorem 1 for related classes to class A(k).

2. Powers of class A(k) operators

Theorem 1. *If T is an invertible* class A(k) *operator for $k \in (0,1]$, then T^n is a* class A$(\frac{k}{n})$ *operator for all positive integer n.*

Corollary 2. *If T is an invertible* class A *operator, then T^n is a* class A$(\frac{1}{n})$ *operator for all positive integer n.*

By using (ii) of Theorem B, Corollary 2 yields Theorem C.1 since class A$(\frac{1}{n})$ is included in class A, so that Corollary 2 is a more precise result than Theorem C.1.

To prove Theorem 1, we prepare the following Proposition 3 and Lemma F.

Proposition 3. *T is a* class A(k) *operator for $k > 0$ if and only if*

$$(|T^*||T|^{2k}|T^*|)^{\frac{1}{k+1}} \geq |T^*|^2. \tag{2.1}$$

Lemma F ([11, 15]). *Let A and B be invertible operators. Then*

$$(BAA^*B^*)^{\lambda} = BA(A^*B^*BA)^{\lambda-1}A^*B^*$$

holds for any real number λ.

Proof of Proposition 3. Let $T = U|T|$ be the polar decomposition of T. Then $T^* = U^*|T^*|$ is also the polar decomposition of T^*. Suppose that T is a class A(k) operator. Then

$$\begin{aligned}(|T^*||T|^{2k}|T^*|)^{\frac{1}{k+1}} &= UU^*(|T^*||T|^{2k}|T^*|)^{\frac{1}{k+1}}UU^* \\ &= U(U^*|T^*||T|^{2k}|T^*|U)^{\frac{1}{k+1}}U^* \\ &= U(T^*|T|^{2k}T)^{\frac{1}{k+1}}U^* \\ &\geq U|T|^2U^* \qquad \text{since } T \text{ is a class A}(k) \text{ operator} \\ &= |T^*|^2.\end{aligned}$$

Hence (2.1) holds.

Conversely, suppose that (2.1) holds. Then

$$\begin{aligned}(T^*|T|^{2k}T)^{\frac{1}{k+1}} &= (U^*|T^*||T|^{2k}|T^*|U)^{\frac{1}{k+1}} \\ &= U^*(|T^*||T|^{2k}|T^*|)^{\frac{1}{k+1}}U \\ &\geq U^*|T^*|^2U \qquad \text{by (2.1)} \\ &= |T|^2.\end{aligned}$$

Hence T is a class A(k) operator.

Whence the proof of Proposition 3 is complete. □

Proof of Theorem 1. Suppose that T is an invertible class A(k) operator for $k \in (0, 1]$. Then we have

$$(|T^*||T|^{2k}|T^*|)^{\frac{1}{k+1}} \geq |T^*|^2 \tag{2.1}$$

by Proposition 3. By (ii) of Theorem B, T is a class A operator, and also we have the following inequalities by Theorem C.2:

$$|T^n|^{\frac{2}{n}} \geq |T|^2, \tag{2.2}$$

$$|T^*|^2 \geq |T^{n*}|^{\frac{2}{n}}. \tag{2.3}$$

Then we have

$$(|T^*||T^n|^{\frac{2k}{n}}|T^*|)^{\frac{1}{k+1}} \geq (|T^*||T|^{2k}|T^*|)^{\frac{1}{k+1}} \geq |T^*|^2 \tag{2.4}$$

by (2.1), (2.2) and Löwner-Heinz theorem. (2.4) implies the following (2.5) by Lemma F:

$$(2.5) \qquad |T^*||T^n|^{\frac{k}{n}}(|T^n|^{\frac{k}{n}}|T^*|^2|T^n|^{\frac{k}{n}})^{\frac{1}{k+1}-1}|T^n|^{\frac{k}{n}}|T^*| \geq |T^*|^2.$$

(2.5) is equivalent to

$$(2.6) \qquad |T^n|^{\frac{2k}{n}} \geq (|T^n|^{\frac{k}{n}}|T^*|^2|T^n|^{\frac{k}{n}})^{\frac{k}{k+1}}.$$

By (2.3), (2.6) and Löwner-Heinz theorem, we have

$$(2.7) \qquad |T^n|^{\frac{2k}{n}} \geq (|T^n|^{\frac{k}{n}}|T^*|^2|T^n|^{\frac{k}{n}})^{\frac{k}{k+1}} \geq (|T^n|^{\frac{k}{n}}|T^{n*}|^{\frac{2}{n}}|T^n|^{\frac{k}{n}})^{\frac{k}{k+1}}.$$

(2.7) implies the following by Lemma F:

$$|T^n|^{\frac{2k}{n}} \geq |T^n|^{\frac{k}{n}}|T^{n*}|^{\frac{1}{n}}(|T^{n*}|^{\frac{1}{n}}|T^n|^{\frac{2k}{n}}|T^{n*}|^{\frac{1}{n}})^{\frac{k}{k+1}-1}|T^{n*}|^{\frac{1}{n}}|T^n|^{\frac{k}{n}}.$$

Then we have

$$(|T^{n*}|^{\frac{1}{n}}|T^n|^{\frac{2k}{n}}|T^{n*}|^{\frac{1}{n}})^{\frac{1}{k+1}} \geq |T^{n*}|^{\frac{2}{n}}.$$

Put $A = (|T^{n*}|^{\frac{1}{n}}|T^n|^{\frac{2k}{n}}|T^{n*}|^{\frac{1}{n}})^{\frac{1}{k+1}}$ and $B = |T^{n*}|^{\frac{2}{n}}$, then $A \geq B > 0$. By using (i) of Theorem F, we have

$$(2.8) \qquad (B^{\frac{r}{2}}A^pB^{\frac{r}{2}})^{\frac{1+r}{p+r}} \geq B^{1+r} \qquad \text{for } p \geq 1 \text{ and } r \geq 0.$$

Put $p = k+1 \geq 1$ and $r = n-1 \geq 0$ in (2.8). Then we have

$$(2.9) \qquad (B^{\frac{n-1}{2}}A^{k+1}B^{\frac{n-1}{2}})^{\frac{n}{k+n}} \geq B^n.$$

(2.9) is equivalent to

$$\left\{|T^{n*}|^{\frac{n-1}{n}}(|T^{n*}|^{\frac{1}{n}}|T^n|^{\frac{2k}{n}}|T^{n*}|^{\frac{1}{n}})^{\frac{k+1}{k+1}}|T^{n*}|^{\frac{n-1}{n}}\right\}^{\frac{n}{k+n}} \geq |T^{n*}|^2.$$

Then we have

$$(2.10) \qquad (|T^{n*}||T^n|^{\frac{2k}{n}}|T^{n*}|)^{\frac{1}{\frac{k}{n}+1}} \geq |T^{n*}|^2.$$

Hence T^n is a class A$(\frac{k}{n})$ operator by Proposition 3. □

Proof of Corollary 2. Put $k = 1$ in Theorem 1. □

3. Powers of class AI(s,t) operators

Very recently, the following classes of operators were defined in [7] as extensions of class A(k).

Definition 2 ([7]).

(i) *For each* $s > 0$ *and* $t > 0$*, an operator* T *belongs to* class $A(s,t)$ *if*

$$(|T^*|^t|T|^{2s}|T^*|^t)^{\frac{t}{s+t}} \geq |T^*|^{2t}. \tag{3.1}$$

(ii) *For each* $s > 0$ *and* $t > 0$*, an operator* T *belongs to* class $AI(s,t)$ *if* T *is an invertible* class $A(s,t)$ *operator.*

We remark that class $A(k)$ coincides with class $A(k,1)$ by Proposition 3. Related to class $A(s,t)$ operators, the following theorem was obtained in [7] as a nice application of Theorem F.

Theorem D ([7]).

(i) *For each* $s > 0$ *and* $t > 0$*, every* class $A(s,t)$ *operator is a* class $A(s,r)$ *operator for* $r \geq t > 0$.

(ii) *For each* $s > 0$ *and* $t > 0$*, every* class $AI(s,t)$ *operator is a* class $AI(p,r)$ *operator for* $p \geq s > 0$ *and* $r \geq t > 0$.

On the other hand, Aluthge and Wang defined w-hyponormal operators in [4] which was related to hyponormal operators as follows: An operator T is said to be w-hyponormal if $|\tilde{T}| \geq |T| \geq |(\tilde{T})^*|$ where $\tilde{T} = |T|^{\frac{1}{2}}U|T|^{\frac{1}{2}}$ and $T = U|T|$ is the polar decomposition of T. $\tilde{T} = |T|^{\frac{1}{2}}U|T|^{\frac{1}{2}}$ is called Aluthge transformation of T. Aluthge transformation was studied in [1, 2, 12, 13, 17, 21, 24]. Related to w-hyponormal operators, the following result was shown in [5].

Theorem E.1 ([5]). *An operator* T *is a* w*-hyponormal operator if and only if*

$$(|T^*|^{\frac{1}{2}}|T||T^*|^{\frac{1}{2}})^{\frac{1}{2}} \geq |T^*| \quad \textit{and} \quad |T| \geq (|T|^{\frac{1}{2}}|T^*||T|^{\frac{1}{2}})^{\frac{1}{2}}.$$

By Theorem E.1, an invertible w-hyponormal operator coincides with a class $AI(\frac{1}{2},\frac{1}{2})$ operator since the first inequality in Theorem E.1 is equivalent to the second inequality in Theorem E.1 in case T is invertible by Lemma F. Moreover, the following Theorem E.2 for w-hyponormal operators was shown by Aluthge and Wang [5].

Theorem E.2 ([5]). *If* T *is an invertible* w*-hyponormal operator, then* T^2 *is also a* w*-hyponormal operator.*

In this section, we shall show the following results for class $AI(s,t)$ operators and w-hyponormal operators as parallel results to Theorem 1 for class $A(k)$ operators.

Theorem 4. *If* T *is a* class $AI(s,t)$ *operator for* $s,t \in (0,1]$*, then* T^n *is a* class $AI(\frac{s}{n},\frac{t}{n})$ *operator for all positive integer* n.

Corollary 5. *If T is an invertible w-hyponormal operator, then T^n is a* class $\mathrm{AI}(\frac{1}{2n}, \frac{1}{2n})$ *operator for all positive integer n.*

By (ii) of Theorem D and Theorem E.1, Corollary 5 yields Theorem E.2 since class $\mathrm{AI}(\frac{1}{2n}, \frac{1}{2n})$ is included in w-hyponormal for all positive integer n.

Proof of Theorem 4. Suppose that T is a class $\mathrm{AI}(s,t)$ operator for $s, t \in (0,1]$, i.e.,

$$(|T^*|^t|T|^{2s}|T^*|^t)^{\frac{t}{s+t}} \geq |T^*|^{2t}. \tag{3.1}$$

By (ii) of Theorem D and Proposition 3, T is a class A operator, and also we have the following inequalities by Theorem C.2:

$$|T^n|^{\frac{2}{n}} \geq |T|^2, \tag{2.2}$$

$$|T^*|^2 \geq |T^{n*}|^{\frac{2}{n}}. \tag{2.3}$$

(2.2) and (2.3) imply the following (3.2) and (3.3), respectively, by Löwner-Heinz theorem for $s \in (0,1]$ and $t \in (0,1]$:

$$|T^n|^{\frac{2s}{n}} \leq |T|^{2s}, \tag{3.2}$$

$$|T^*|^{2t} \geq |T^{n*}|^{\frac{2t}{n}}. \tag{3.3}$$

Then we have

$$(|T^*|^t|T^n|^{\frac{2s}{n}}|T^*|^t)^{\frac{t}{s+t}} \geq (|T^*|^t|T|^{2s}|T^*|^t)^{\frac{t}{s+t}} \geq |T^*|^{2t} \tag{3.4}$$

by (3.1), (3.2) and Löwner-Heinz theorem. (3.4) implies the following (3.5) by Lemma F:

$$|T^*|^t|T^n|^{\frac{s}{n}}(|T^n|^{\frac{s}{n}}|T^*|^{2t}|T^n|^{\frac{s}{n}})^{\frac{t}{s+t}-1}|T^n|^{\frac{s}{n}}|T^*|^t \geq |T^*|^{2t}. \tag{3.5}$$

(3.5) is equivalent to

$$|T^n|^{\frac{2s}{n}} \geq (|T^n|^{\frac{s}{n}}|T^*|^{2t}|T^n|^{\frac{s}{n}})^{\frac{s}{s+t}}. \tag{3.6}$$

By (3.3), (3.6) and Löwner-Heinz theorem, we have

$$|T^n|^{\frac{2s}{n}} \geq (|T^n|^{\frac{s}{n}}|T^*|^{2t}|T^n|^{\frac{s}{n}})^{\frac{s}{s+t}} \geq (|T^n|^{\frac{s}{n}}|T^{n*}|^{\frac{2t}{n}}|T^n|^{\frac{s}{n}})^{\frac{s}{s+t}}. \tag{3.7}$$

(3.7) implies the following by Lemma F:

$$|T^n|^{\frac{2s}{n}} \geq |T^n|^{\frac{s}{n}}|T^{n*}|^{\frac{t}{n}}(|T^{n*}|^{\frac{t}{n}}|T^n|^{\frac{2s}{n}}|T^{n*}|^{\frac{t}{n}})^{\frac{s}{s+t}-1}|T^{n*}|^{\frac{t}{n}}|T^n|^{\frac{s}{n}}.$$

Then we have

$$(|T^{n*}|^{\frac{t}{n}}|T^n|^{\frac{2s}{n}}|T^{n*}|^{\frac{t}{n}})^{\frac{\frac{t}{n}}{\frac{s}{n}+\frac{t}{n}}} \geq |T^{n*}|^{\frac{2t}{n}}.$$

Hence T^n is a class $\mathrm{AI}(\frac{s}{n}, \frac{t}{n})$ operator. □

Proof of Corollary 5. Put $s = \frac{1}{2}$ and $t = \frac{1}{2}$ in Theorem 4. Then we obtain Corollary 5 since the class of all invertible w-hyponormal operators equals class $\mathrm{AI}(\frac{1}{2}, \frac{1}{2})$. □

4. Concluding remarks

Firstly, it is interesting to point out the contrast between the following two facts: Theorem A.1 asserts that if T is a p-hyponormal operator for $p \in (0, 1]$, then T^n belongs to the class of $(\frac{p}{n})$-hyponormal operators which is a larger class of operators than the class of p-hyponormal operators to which T belongs. Contrary to Theorem A.1, Theorem 1 asserts that if T is an invertible class $\mathrm{A}(k)$ operator for $k \in (0, 1]$, then T^n belongs to class $\mathrm{A}(\frac{k}{n})$ which is a smaller class of operators than class $\mathrm{A}(k)$ to which T belongs.

Secondly, it is shown in (i) of Theorem B that every log-hyponormal operator is a class $\mathrm{A}(k)$ operator for all $k > 0$. Here, we shall discuss a more precise relation between class $\mathrm{A}(k)$ and the class of log-hyponormal operators than (i) of Theorem B. Assume that T is an invertible class $\mathrm{A}(k)$ operator. Then we have

$$(|T^*||T|^{2k}|T^*|)^{\frac{1}{k+1}} \geq |T^*|^2 \tag{2.1}$$

by Proposition 3. Then by using Lemma F, (2.1) is equivalent to the following (4.1):

$$|T^*||T|^k(|T|^k|T^*|^2|T|^k)^{\frac{-k}{k+1}}|T|^k|T^*| \geq |T^*|^2. \tag{4.1}$$

Hence an invertible class $\mathrm{A}(k)$ operator satisfies the following inequality by (4.1):

$$|T|^{2k} \geq (|T|^k|T^*|^2|T|^k)^{\frac{k}{k+1}}.$$

Then

$$\log|T|^{2k} \geq \log(|T|^k|T^*|^2|T|^k)^{\frac{k}{k+1}} \tag{4.2}$$

holds since $\log t$ is an operator monotone function. (4.2) is equivalent to

$$\log|T|^{2(k+1)} \geq \log(|T|^k|T^*|^2|T|^k). \tag{4.3}$$

Let $k \to +0$ in (4.3). Then we have $\log T^*T \geq \log TT^*$. Briefly speaking, the class of log-hyponormal operators can be regarded as invertible class $\mathrm{A}(0)$. And it is well known that log-hyponormal also can be regarded as 0-hyponormal. It is interesting to point out this contrast.

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