<table>
<thead>
<tr>
<th>Title</th>
<th>Generalizations of the results on powers of $p$-hyponormal operators (Operator Inequalities and Related Area)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Author(s)</td>
<td>Ito, Masatoshi</td>
</tr>
<tr>
<td>Citation</td>
<td>数理解析研究所講究録 (2000), 1144: 114-126</td>
</tr>
<tr>
<td>Issue Date</td>
<td>2000-04</td>
</tr>
<tr>
<td>URL</td>
<td><a href="http://hdl.handle.net/2433/63910">http://hdl.handle.net/2433/63910</a></td>
</tr>
<tr>
<td>Type</td>
<td>Departmental Bulletin Paper</td>
</tr>
<tr>
<td>Textversion</td>
<td>publisher</td>
</tr>
</tbody>
</table>

Kyoto University
Generalizations of the results on powers of $p$-hyponormal operators

東京理科大学 伊藤公智 (Masatoshi Ito)

This report is based on the following two papers:


Abstract

We shall show that "if $T$ is a $p$-hyponormal operator for $p > 0$, then $T^n$ is $\min\{1, \frac{p}{n}\}$-hyponormal for any positive integer $n"$ and related results as generalizations of the results by Aluthge-Wang [2] and Furuta-Yanagida [11].

1 Introduction

A capital letter means a bounded linear operator on a complex Hilbert space $H$. An operator $T$ is said to be positive (denoted by $T \geq 0$) if $(Tx, x) \geq 0$ for all $x \in H$.

An operator $T$ is said to be $p$-hyponormal for $p > 0$ if $(T^*T)^p \geq (TT^*)^p$. $p$-Hyponormal operators were defined as an extension of hyponormal ones, i.e., $T^*T \geq TT^*$. It is easily obtained that every $p$-hyponormal operator is $q$-hyponormal for $p \geq q > 0$ by Löwner-Heinz theorem "$A \geq B \geq 0$ ensures $A^\alpha \geq B^\alpha$ for any $\alpha \in [0, 1]$,” and it is well known that there exists a hyponormal operator $T$ such that $T^2$ is not hyponormal [13], but paranormal [7], i.e., $\|T^2x\| \geq \|Tx\|^2$ for every unit vector $x \in H$. We remark that every $p$-hyponormal operator for $p > 0$ is paranormal [3] (see also [1][5][10]).

Recently, Aluthge and Wang [2] showed the following results on powers of $p$-hyponormal operators.

**Theorem A.1** ([2]). Let $T$ be a $p$-hyponormal operator for $p \in (0, 1]$. The inequalities

$$(T^n T^n)^\frac{p}{n} \geq (T^*T)^p \geq (TT^*)^p \geq (T^n T^n)^\frac{p}{n}$$

hold for all positive integer $n$.

**Corollary A.2** ([2]). If $T$ is a $p$-hyponormal operator for $p \in (0, 1]$, then $T^n$ is $\frac{p}{n}$-hyponormal for any positive integer $n$. 
By Corollary A.2, if $T$ is a hyponormal operator, then $T^2$ belongs to the class of $\frac{1}{2}$-hyponormal operators which is smaller than that of paranormal operators.

As a more precise result than Theorem A.1, Furuta and Yanagida [11] obtained the following result.

**Theorem A.3 ([11, Theorem 1]).** Let $T$ be a $p$-hyponormal operator for $p \in (0, 1]$. Then
\[
(T^{n^*}T^n)^{\frac{p+1}{n}} \geq (T^*T)^{p+1} \quad \text{and} \quad (TT^*)^{p+1} \geq (T^nT^n^*)^{\frac{p+1}{n}}
\]
hold for all positive integer $n$.

Theorem A.3 asserts that the first and third inequalities of Theorem A.1 hold for the larger exponents $\frac{p+1}{n}$ than $\frac{p}{n}$ in Theorem A.1. In fact, Theorem A.3 ensures Theorem A.1 by Löwner-Heinz theorem for $\frac{p}{p+1} \in (0, 1)$ and $p$-hyponormality of $T$.

On the other hand, Fujii and Nakatsu [6] showed the following result.

**Theorem A.4 ([6]).** For each positive integer $n$, if $T$ is an $n$-hyponormal operator, then $T^n$ is hyponormal.

We remark that Theorem A.1, Corollary A.2 and Theorem A.3 are results on $p$-hyponormal operators for $p \in (0, 1]$, and Theorem A.4 is a result on $n$-hyponormal operators for positive integer $n$. In this report, more generally, we shall discuss powers of $p$-hyponormal operators for all positive real number $p > 0$.

## 2 Main results

**Theorem 1.** Let $T$ be a $p$-hyponormal operator for $p > 0$. Then the following assertions hold:

1. $T^{n^*}T^n \geq (T^*T)^n$ and $(TT^*)^n \geq T^nT^n^*$ hold for positive integer $n$ such that $n < p + 1$.

2. $(T^{n^*}T^n)^{\frac{p+1}{n}} \geq (T^*T)^{p+1}$ and $(TT^*)^{p+1} \geq (T^nT^n^*)^{\frac{p+1}{n}}$ hold for positive integer $n$ such that $n \geq p + 1$.

**Corollary 2.** Let $T$ be a $p$-hyponormal operator for $p > 0$. Then the following assertions hold:

1. $T^{n^*}T^n \geq T^nT^n^*$ holds for positive integer $n$ such that $n < p$. 

(2) \((T^{n}T^{*})^\frac{p}{n} \geq (T^{n}T^{n^{*}})^\frac{p}{n}\) holds for positive integer \(n\) such that \(n \geq p\). In other words, if \(T\) is a \(p\)-hyponormal operator for \(p > 0\), then \(T^n\) is \(\min\{1, \frac{p}{n}\}\)-hyponormal for any positive integer \(n\).

In case \(p \in (0, 1]\), Theorem 1 (resp. Corollary 2) means Theorem A.3 (resp. Corollary A.2). Corollary 2 also yields Theorem A.4 in case \(p = n\). Theorem 1 and Corollary 2 can be rewritten into the following Theorem 1' and Corollary 2', respectively. We shall prove Theorem 1' and Corollary 2'.

**Theorem 1'.** For some positive integer \(m\), let \(T\) be a \(p\)-hyponormal operator for \(m-1 < p \leq m\). Then the following assertions hold:

(1) \(T^nT^m \geq (T^*T)^n\) and \((TT^*)^n \geq T^nT^m\) hold for \(n = 1, 2, \cdots, m\).

(2) \((T^nT^m)^\frac{p+1}{n} \geq (T^*T)^{p+1}\) and \((TT^*)^{p+1} \geq (T^nT^m)^\frac{p+1}{n}\) hold for \(n = m+1, m+2, \cdots\).

**Corollary 2'.** For some positive integer \(m\), let \(T\) be a \(p\)-hyponormal operator for \(m-1 < p \leq m\). Then the following assertions hold:

(1) \(T^nT^m \geq T^nT^m\) holds for \(n = 1, 2, \cdots, m-1\).

(2) \((T^nT^m)^\frac{p}{n} \geq (T^nT^m)^\frac{p}{n}\) holds for \(n = m, m+1, \cdots\).

We need the following theorem in order to give a proof of Theorem 1'.

**Theorem B.1 (Furuta inequality [8]).**

If \(A \geq B \geq 0\), then for each \(r \geq 0\),

(i) \((B^\frac{r}{2}AB^\frac{r}{2})^\frac{1}{2} \geq (B^\frac{r}{2}B^pB^\frac{r}{2})^\frac{1}{2}\)

and

(ii) \((A^\frac{r}{2}APA^\frac{r}{2})^\frac{1}{2} \geq (A^\frac{r}{2}B^pA^\frac{r}{2})^\frac{1}{2}\)

hold for \(p \geq 0\) and \(q \geq 1\) with \((1 + r)q \geq p + r\).

We remark that Theorem B.1 yields Löwner-Heinz theorem when we put \(r = 0\) in (i) or (ii) stated above. Alternative proofs of Theorem B.1 are given in [4] and [14] and also an elementary one page proof in [9]. It is shown in [15] that the domain drawn for \(p, q\) and \(r\) in the Figure is the best possible one for Theorem B.1.

**Proof of Theorem 1'.** We shall prove Theorem 1' by induction.
Proof of (1). We shall prove
\[ T^{n^*}T^n \geq (T^*T)^n \] (2.1)
and
\[ (TT^*)^{n} \geq T^nT^* \] (2.2)
for \( n = 1, 2, \ldots, m \). (2.1) and (2.2) always hold for \( n = 1 \). Assume that (2.1) and (2.2) hold for some \( n \leq m - 1 \). Then we have
\[ T^nT^n \geq (T^*T)^n \geq (TT^*)^n \geq T^nT^n^* \] (2.3)
since the second inequality holds by \( p \)-hyponormality of \( T \) and Löwner-Heinz theorem for \( \frac{n}{p} \in (0, 1) \). By (2.3), we have
\[ T^{n^*}T^n \geq (TT^*)^n \] (2.4)
and
\[ (T^*T)^n \geq T^nT^* \] (2.5)
(2.4) ensures
\[ T^{n+1^*}T^{n+1} = T^{*}(TT^*)^nT \geq T^{*}(TT^*)^nT = (T^*T)^{n+1} \]
and (2.5) ensures
\[ (TT^*)^{n+1} = T(T^*T)^nT^* \geq T(T^*T)^nT^* = T^{n+1}T^{n+1^*} \]
Hence (2.1) and (2.2) hold for \( n + 1 \), so that the proof of (1) is complete.

Proof of (2). We shall prove
\[ (T^{n^*}T^n)^{\frac{p+1}{n}} \geq (T^*T)^{p+1} \] (2.6)
and
\[ (TT^*)^{p+1} \geq (T^nT^{*})^{\frac{p+1}{n}} \] (2.7)
for \( n = m + 1, m + 2, \ldots \). Let \( T = U|T| \) be the polar decomposition of \( T \) where \( |T| = (T^*T)^{\frac{1}{2}} \) and put \( A_n = |T|^\frac{2p}{n} \) and \( B_n = |T^{*}|^\frac{2p}{n} \) for each positive integer \( n \). We remark that \( T^* = U^*|T^*| \) is also the polar decomposition of \( T^* \).
(a) Case \( n = m + 1 \). (2.1) and (2.2) for \( n = m \) ensure
\[ (T^{m^*}T^m)^{\frac{p}{n}} \geq (T^*T)^p \geq (TT^*)^p \geq (T^nT^{*m})^{\frac{p}{n}} \] (2.8)
since the first and third inequalities hold by (2.1), (2.2) and Löwner-Heinz theorem for $\frac{p}{m} \in (0,1]$, and the second inequality holds by $p$-hyponormality of $T$. (2.8) ensures the following (2.9) and (2.10).

$$A_m = (T^{m^*}T^m)^\frac{p}{m} \geq (TT^*)^p = B_1. \quad (2.9)$$

$$A_1 = (T^*T)^p \geq (T^mT^{*m})^\frac{p}{m} = B_m. \quad (2.10)$$

By using (i) of Theorem B.1 for $\frac{m}{p} \geq 1$ and $\frac{1}{p} \geq 0$, we have

$$\begin{align*}
(T^{m+1}T^m)^{\frac{p+1}{m+1}} &= \left(U^*|T^*|T^{m*}T^m|T^*U\right)^{\frac{p+1}{m+1}} \\
&= U^*\left(|T^*|T^{m*}T^m|T^*\right)^{\frac{p+1}{m+1}}U \\
&= U^*(B_1^{\frac{1}{p}}A_m^{\frac{m}{p}}B_1^{\frac{1}{p}})^{\frac{m+1}{p+1}}U \\
&\geq U^*B_1^{1+\frac{1}{p}}U \\
&= U^*|T^*|2(p+1)U \\
&= |T^*|^{2(p+1)} \\
&= (T^*T)^{p+1},
\end{align*}$$

so that (2.6) holds for $n = m + 1$.

By using (ii) of Theorem B.1 for $\frac{m}{p} \geq 1$ and $\frac{1}{p} \geq 0$, we have

$$\begin{align*}
(T^{m+1}T^{m+1})^{\frac{p+1}{m+1}} &= \left(U|T|T^{m*}T^m|T|\right)^{\frac{p+1}{m+1}} \\
&= U\left(|T|T^{m*}T^m|T|\right)^{\frac{p+1}{m+1}}U* \\
&= U\left(A_1^{\frac{1}{2p}}B_m^{\frac{m}{p}}A_1^{\frac{1}{2p}}\right)^{\frac{m+1}{p+1}}U* \\
&\leq UA_1^{1+\frac{1}{p}}U* \\
&= U|T|^{2(p+1)}U* \\
&= |T^*|^{2(p+1)} \\
&= (T^*T)^{p+1},
\end{align*}$$

so that (2.7) holds for $n = m + 1$.

(b) Assume that (2.6) and (2.7) hold for some $n \geq m + 1$. Then (2.6) and (2.7) for $n$ ensure

$$\begin{align*}
(T^mT^n)^{\frac{p}{n}} \geq (T^*T)^p \geq (TT^*)^p \geq (T^nT^m)^{\frac{p}{n}} \quad (2.11)
\end{align*}$$

since the first and third inequalities hold by (2.6) and (2.7) for $n$ and Löwner-Heinz theorem for $\frac{p}{p+1} \in (0,1)$, and the second inequality holds by $p$-hyponormality of $T$. (2.11) ensures the following (2.12) and (2.13).

$$A_n = (T^nT^m)^{\frac{p}{n}} \geq (TT^*)^p = B_1. \quad (2.12)$$
$A_1 = (T^*T)^p \geq (T^nT^n^*)^\frac{p}{n} = B_n$. \hspace{2cm} (2.13)

By using (i) of Theorem B.1 for $\frac{n}{p} \geq 1$ and $\frac{1}{p} \geq 0$, we have

$$(T^{n+1}T^{n+1})^\frac{p+1}{n+1} = (U^*|T|T^mT^n|T^*U)^\frac{p+1}{n+1}$$

$$= U^*|T^mT^mT^n|T^*U|^\frac{p+1}{n+1}$$

$$= U^*(B_1^{\frac{1}{p^2}}A_1^{\frac{n}{p}}B_1^{\frac{1}{p^2}})^\frac{1+p}{n+1}U$$

$$\geq U^*B_1^{1+\frac{1}{p}}U$$

$$= U^*|T^*|^2(p+1)U$$

$$= |T|^{2(p+1)}$$

$$= (T^*T)^{p+1},$$

so that (2.6) holds for $n+1$.

By using (ii) of Theorem B.1 for $\frac{n}{p} \geq 1$ and $\frac{1}{p} \geq 0$, we have

$$(T^{n+1}T^{n+1}^*)^\frac{p+1}{n+1} = (U|T^mT^n|T^*U)^\frac{p+1}{n+1}$$

$$= U(|T^mT^mT^n|T^*)^\frac{p+1}{n+1}U^*$$

$$= U(A_1^{\frac{1}{p^2}}B_m^{\frac{n}{p}}A_1^{\frac{1}{p^2}})^\frac{1+p}{n+1}U^*$$

$$\leq U A_1^{1+\frac{1}{p}}U^*$$

$$= U|T|^{2(p+1)}U^*$$

$$= |T^*|^2(p+1)U^*$$

$$= (TT^*)^{p+1},$$

so that (2.7) holds for $n+1$.

By (a) and (b), (2.6) and (2.7) hold for $n = m+1, m+2, \cdots$, that is, the proof of (2) is complete.

Consequently the proof of Theorem 1' is complete. \qed

Proof of Corollary 2'.

Proof of (1). By (1) of Theorem 1', for $n = m+1, m+2, \cdots, m-1$,

$$T^mT^n \geq (T^*T)^n \geq (TT^*)^n \geq T^nT^n^*$$

hold since the second inequality holds by $p$-hyponormality of $T$ and Löwner-Heinz theorem for $\frac{n}{p} \in (0, 1)$. Therefore $T^mT^n \geq T^nT^n^*$ holds for $n = 1, 2, \cdots, m-1$.

Proof of (2). By (1) of Theorem 1' and Löwner-Heinz theorem for $\frac{p}{m} \in (0, 1]$ in case $n = m$, and by (2) of Theorem 1' and Löwner-Heinz theorem for $\frac{p}{p+1} \in (0, 1)$ in case $n = m+1, m+2, \cdots$, we have

$$(T^*T^n)^\frac{p}{n} \geq (T^*T)^p \geq (TT^*)^p \geq (T^nT^n^*)^\frac{p}{n}$$
since the second inequality holds by \( p \)-hyponormality of \( T \). Therefore \( (T^nT^*)^{\frac{p}{n}} \geq (T^nT^{n*})^{\frac{p}{n}} \) holds for \( n = m, m + 1, \ldots \). \( \square \)

### 3 Best possibilities of Theorem 1 and Corollary 2

Furuta and Yanagida [11] discussed the best possibilities of Theorem A.3 and Corollary A.2 on \( p \)-hyponormal operators for \( p \in (0, 1] \). In this section, more generally, we shall discuss the best possibilities of Theorem 1 and Corollary 2 on \( p \)-hyponormal operators for \( p > 0 \).

**Theorem 3.** Let \( n \) be a positive integer such that \( n \geq 2 \), \( p > 0 \) and \( \alpha > 1 \).

1. In case \( n < p + 1 \), the following assertions hold:

   (i) There exists a \( p \)-hyponormal operator \( T \) such that \( (T^nT^*)^\alpha \not\geq (T^*T)^{n\alpha} \).

   (ii) There exists a \( p \)-hyponormal operator \( T \) such that \( (TT^*)^{n\alpha} \not\geq (T^nT^{n*})^\alpha \).

2. In case \( n \geq p + 1 \), the following assertions hold:

   (i) There exists a \( p \)-hyponormal operator \( T \) such that \( (T^nT^*)^{\frac{(p+1)\alpha}{n}} \not\geq (T^*T)^{(p+1)\alpha} \).

   (ii) There exists a \( p \)-hyponormal operator \( T \) such that \( (TT^*)^{(p+1)\alpha} \not\geq (T^nT^{n*})^{\frac{(p+1)\alpha}{n}} \).

**Theorem 4.** Let \( n \) be a positive integer such that \( n \geq 2 \), \( p > 0 \) and \( \alpha > 1 \).

1. In case \( n < p \), there exists a \( p \)-hyponormal operator \( T \) such that \( (T^nT^*)^\alpha \not\geq (T^nT^{n*})^\alpha \).

2. In case \( n \geq p \), there exists a \( p \)-hyponormal operator \( T \) such that \( (T^nT^*)^{\frac{p+r}{q} \not\geq (T^nT^{n*})^{\frac{p+r}{q}} \).

Theorem 3 (resp. Theorem 4) asserts the best possibility of Theorem 1 (resp. Corollary 2). We need the following results to give proofs of Theorem 3 and Theorem 4.

**Theorem C.1 ([16][18]).** Let \( p > 0 \), \( q > 0 \), \( r > 0 \) and \( \delta > 0 \). If \( 0 < q < 1 \) or \( (\delta + r)q < p + r \), then the following assertions hold:

(i) There exist positive invertible operators \( A \) and \( B \) on \( \mathbb{R}^2 \) such that \( A^\delta \geq B^\delta \) and

\[
(B^\frac{\delta}{2}A^pB^\frac{\delta}{2})^{\frac{1}{q}} \not\geq B^\frac{p+r}{q}.
\]
(ii) There exist positive invertible operators $A$ and $B$ on $\mathbb{R}^2$ such that $A^\delta \geq B^\delta$ and
\[
A^{\frac{p+r}{q}} \not\geq (A^{\frac{p}{2}} B^{p} A^{\frac{r}{2}})^{\frac{1}{q}}.
\]

Lemma C.2 ([11]). For positive operators $A$ and $B$ on $H$, define the operator $T$ on $\bigoplus_{k=-\infty}^{\infty} H$ as follows:

\[
T = \begin{pmatrix}
\ddots & 0 & 0 & \cdots \\
0 & B^{\frac{1}{2}} & 0 & \cdots \\
B^{\frac{1}{2}} & 0 & \square & 0 & \cdots \\
A^{\frac{1}{2}} & 0 & \cdots & B^{\frac{1}{2}} & \cdots \ddots
\end{pmatrix},
\]

where $\square$ shows the place of the $(0,0)$ matrix element. Then the following assertion holds:

(i) $T$ is $p$-hyponormal for $p > 0$ if and only if $A^p \geq B^p$.

Furthermore, the following assertions hold for $\beta > 0$ and integers $n \geq 2$:

(ii) $(T^n T^*)^\frac{\beta}{n} \geq (T^* T)^\beta$ if and only if
\[
(B^{\frac{k}{2}} A^{n-k} B^{\frac{k}{2}})^{\frac{\beta}{n}} \geq \begin{pmatrix} 0 \\ B^{\frac{1}{2}} & 0 & \cdots & B^{\frac{1}{2}} & \cdots \ddots
\end{pmatrix}^{\beta} \text{ holds for } k = 1, 2, \ldots, n - 1.
\]

(iii) $(T T^*)^\beta \geq (T^n T^*)^\frac{\beta}{n}$ if and only if
\[
A^{\beta} \geq (A^{\frac{k}{2}} B^{n-k} A^{\frac{k}{2}})^{\frac{\beta}{n}} \text{ holds for } k = 1, 2, \ldots, n - 1.
\]

(iv) $(T^n T^*)^\frac{\beta}{n} \geq (T^n T^*)^\frac{\beta}{n}$ if and only if
\[
\begin{cases}
A^{\beta} \geq B^{\beta} \text{ holds and } \\
(B^{\frac{k}{2}} A^{n-k} B^{\frac{k}{2}})^{\frac{\beta}{n}} \geq B^{\beta} \text{ and } A^{\beta} \geq (A^{\frac{k}{2}} B^{n-k} A^{\frac{k}{2}})^{\frac{\beta}{n}} \text{ hold for } k = 1, 2, \ldots, n - 1.
\end{cases}
\]

Proof of Theorem 3. Let $n \geq 2$, $p > 0$ and $\alpha > 1$.

Proof of (1). Put $p_1 = n - 1 > 0$, $q_1 = \frac{1}{\alpha} \in (0, 1)$, $r_1 = 1 > 0$ and $\delta = p > 0$.

Proof of (i). By (i) of Theorem C.1, there exist positive operators $A$ and $B$ on $H$ such that $A^\delta \geq B^\delta$ and $(B^{\frac{p_1}{2}} A^{p_1} B^{\frac{r_1}{2}})^{\frac{1}{q_1}} \not\geq B^{\frac{p_1 + r_1}{q_1}}$, that is,
\[
A^{p_1} \geq B^{p_1}
\]
and
\[(B^{\frac{1}{2}}A^{n-1}B^{\frac{1}{2}})^{\alpha} \not\geq B^{n\alpha}. \] (3.6)

Define an operator $T$ on $\bigoplus_{k=-\infty}^{\infty} H$ as (3.1). Then $T$ is $p$-hyponormal by (3.5) and (i) of Lemma C.2, and $(T^{*}T)^{n\alpha} \not\geq (T^{*}T)^{n\alpha}$ by (ii) of Lemma C.2 since the case $k = 1$ of (3.2) does not hold for $\beta = n\alpha$ by (3.6).

Proof of (ii). By (ii) of Theorem C.1, there exist positive operators $A$ and $B$ on $H$ such that $A^\delta \geq B^\delta$ and $A^{\frac{p_1+r_1}{q_1}} \not\geq (A^{\frac{r_1}{2}}B^{p_1}A^{\frac{r_1}{2}})^{\frac{1}{q_1}}$, that is,
\[A^p \geq B^p \] (3.7)

and
\[A^{n\alpha} \not\geq (A^{\frac{1}{2}}B^{n-1}A^{\frac{1}{2}})^{\alpha}. \] (3.8)

Define an operator $T$ on $\bigoplus_{k=-\infty}^{\infty} H$ as (3.1). Then $T$ is $p$-hyponormal by (3.7) and (i) of Lemma C.2, and $(TT^{*})^{n\alpha} \not\geq (T^{*}T)^{n\alpha}$ by (iii) of Lemma C.2 since the case $k = 1$ of (3.3) does not hold for $\beta = n\alpha$ by (3.8).

Proof of (2). Put $p_1 = n - 1 > 0$, $q_1 = \frac{n}{(p+1)\alpha} > 0$, $r_1 = 1 > 0$ and $\delta = p > 0$, then we have $(\delta + r_1)q_1 = \frac{n}{\alpha} < n = p_1 + r_1$.

Proof of (i). By (i) of Theorem C.1, there exist positive operators $A$ and $B$ on $H$ such that $A^\delta \geq B^\delta$ and $(B^{\frac{r_2}{2}}A^{p_1}B^{\frac{r_1}{2}})^{\frac{1}{q_1}} \not\geq B^{(p+1)\alpha}$, that is,
\[A^p \geq B^p \] (3.9)

and
\[(B^{\frac{1}{2}}A^{n-1}B^{\frac{1}{2}})^{\frac{(p+1)\alpha}{n}} \not\geq B^{(p+1)\alpha}. \] (3.10)

Define an operator $T$ on $\bigoplus_{k=-\infty}^{\infty} H$ as (3.1). Then $T$ is $p$-hyponormal by (3.9) and (i) of Lemma C.2, and $(T^{n\alpha}T^{n\alpha})^{(p+1)\alpha} \not\geq (T^{*}T)^{(p+1)\alpha}$ by (ii) of Lemma C.2 since the case $k = 1$ of (3.2) does not hold for $\beta = (p + 1)\alpha$ by (3.10).

Proof of (ii). By (ii) of Theorem C.1, there exist positive operators $A$ and $B$ on $H$ such that $A^\delta \geq B^\delta$ and $A^{\frac{p_1+r_1}{q_1}} \not\geq (A^{\frac{r_1}{2}}B^{p_1}A^{\frac{r_1}{2}})^{\frac{1}{q_1}}$, that is,
\[A^{(p+1)\alpha} \not\geq (A^{\frac{1}{2}}B^{n-1}A^{\frac{1}{2}})^{\frac{(p+1)\alpha}{n}}. \] (3.11)
Define an operator $T$ on $\bigoplus_{k=-\infty}^{\infty} H$ as (3.1). Then $T$ is $p$-hyponormal by (3.11) and (i) of Lemma C.2, and $(TT^*)^{(p+1)\alpha} \not\geq (T^nT^*)^{(p+1)\frac{\alpha}{n}}$ by (iii) of Lemma C.2 since the case $k = 1$ of (3.3) does not hold for $\beta = (p + 1)\alpha$ by (3.12).

\[ \square \]

**Proof of Theorem 4.** Let $n \geq 2$, $p > 0$ and $\alpha > 1$.

**Proof of (1).** Put $p_1 = n - 1 > 0$, $q_1 = \frac{1}{\alpha} \in (0, 1)$, $r_1 = 1 > 0$ and $\delta = p > 0$. By (i) of Theorem C.1, there exist positive operators $A$ and $B$ on $H$ such that $A^\delta \geq B^\delta$ and $(B^{r_1} A^{p_1} B^{r_1})^{\frac{1}{q_1}} \not\geq B^{p_1 + r_1}$, that is,

\[ A^p \geq B^p \]  

(3.13)

and

\[ (B^{\frac{1}{2}} A^{n-1} B^{\frac{1}{2}})^{\alpha} \not\geq B^{n\alpha}. \]  

(3.14)

Define an operator $T$ on $\bigoplus_{k=-\infty}^{\infty} H$ as (3.1). Then $T$ is $p$-hyponormal by (3.13) and (i) of Lemma C.2, and $(T^n T^*)^{\alpha} \not\geq (T^n T^*)^{\alpha}$ by (iv) of Lemma C.2 since the case $k = 1$ of the second inequality of (3.4) does not hold for $\beta = n\alpha$ by (3.14).

**Proof of (2).** It is well known that there exist positive operators $A$ and $B$ on $H$ such that

\[ A^p \geq B^p \]  

(3.15)

and

\[ A^{p\alpha} \not\geq B^{p\alpha}. \]  

(3.16)

Define an operator $T$ on $\bigoplus_{k=-\infty}^{\infty} H$ as (3.1). Then $T$ is $p$-hyponormal by (3.15) and (i) of Lemma C.2, and $(T^n T^*)^{\alpha} \not\geq (T^n T^*)^{\alpha}$ by (iv) of Lemma C.2 since the first inequality of (3.4) does not hold for $\beta = p\alpha$ by (3.16).

\[ \square \]

**4 Concluding remarks**

**Remark 1.** An operator $T$ is said to be *log-hyponormal* if $T$ is invertible and $\log T^* T \geq \log TT^*$. It is easily obtained that every invertible $p$-hyponormal operator is log-hyponormal since $\log t$ is an operator monotone function, and Ando [3] showed that every log-hyponormal operator is paranormal. We remark that log-hyponormal can be regarded as 0-hyponormal since $(T^* T)^p \geq (TT^*)^p$ approaches $\log T^* T \geq \log TT^*$ as $p \to +0$.

As an extension of Theorem A.1, Yamazaki [17] obtained the following Theorem D.1 and Corollary D.2 on log-hyponormal operators.
Theorem D.1 ([17]). Let $T$ be a log-hyponormal operator. Then the following inequalities hold for all positive integer $n$:

1. $T^*T \leq (T^2T^{2})^{\frac{1}{2}} \leq \cdots \leq (T^nT^n)^{\frac{1}{n}}$.
2. $TT^* \geq (T^2T^2)^{\frac{1}{2}} \geq \cdots \geq (T^nT^n)^{\frac{1}{n}}$.

Corollary D.2 ([17]). If $T$ is a log-hyponormal operator, then $T^n$ is also log-hyponormal for any positive integer $n$.

The best possibilities of Theorem D.1 and Corollary D.2 are discussed in [12]. As a parallel result to Theorem D.1, Furuta and Yanagida [12] showed the following Theorem D.3 on $p$-hyponormal operators for $p \in (0, 1]$.

Theorem D.3 ([12]). Let $T$ be a $p$-hyponormal operator for $p \in (0, 1]$. Then the following inequalities hold for all positive integer $n$:

1. $(T^*T)^{p+1} \leq (T^2T^{2})^{\frac{p+1}{2}} \leq \cdots \leq (T^nT^n)^{\frac{p+1}{n}}$.
2. $(TT^*)^{p+1} \geq (T^2T^2)^{\frac{p+1}{2}} \geq \cdots \geq (T^nT^n)^{\frac{p+1}{n}}$.

In fact, Theorem D.3 in the case $p \rightarrow +0$ corresponds to Theorem D.1.

As a further extension of Theorem D.3, we obtain the following Theorem 5 on $p$-hyponormal operators for $p > 0$.

Theorem 5. For some positive integer $m$, let $T$ be a $p$-hyponormal operator for $m-1 < p \leq m$. Then the following inequalities hold for $n = m+1, m+2, \cdots$:

1. $(T^*T)^{p+1} \leq (T^{m+1}T^{m+1})^{\frac{p+1}{m+1}} \leq (T^{m+2}T^{m+2})^{\frac{p+1}{m+1}} \leq \cdots \leq (T^nT^n)^{\frac{p+1}{n}}$.
2. $(TT^*)^{p+1} \geq (T^{m+1}T^{m+1})^{\frac{p+1}{m+1}} \geq (T^{m+2}T^{m+2})^{\frac{p+1}{m+1}} \geq \cdots \geq (T^nT^n)^{\frac{p+1}{n}}$.

We remark that Theorem 5 yields Theorem D.3 by putting $m = 1$.

Remark 2. Recently, in [10], we introduced a new class of operators as follows: An operator $T$ belongs to class A if $|T^2| \geq |T|^2$. We call an operator $T$ "class A operator" briefly if $T$ belongs to class A. In [10], we showed that every log-hyponormal operator belongs to class A and every class A operator is paranormal. It turns out that these results contain another proof of Ando's result [3] which states that every log-hyponormal operator is paranormal. We remark that class A is defined by an operator inequality and paranormal is defined by a norm inequality, and their definitions appear to be similar forms.

We obtain the following Theorem 6 on class A.
Theorem 6. Let $T$ be an invertible and class $A$ operator. Then the following inequalities hold for all positive integer $n$:

(1) $|T|^2 \leq |T^2| \leq \cdots \leq |T^n|^\frac{2}{n}$, i.e., $T^* T \leq (T^2 T^2)^\frac{1}{2} \leq \cdots \leq (T^n T^n)^\frac{1}{n}$.

(2) $|T^*|^2 \geq |T^{2^*}| \geq \cdots \geq |T^{n^*}|^\frac{2}{n}$, i.e., $T T^* \geq (T^{2} T^{2*})^\frac{1}{2} \geq \cdots \geq (T^{n} T^{n^*})^\frac{1}{n}$.

Theorem 6 is an extension of Theorem D.1 since every log-hyponormal operator belongs to class $A$.

Related to Theorem 6, we have the following Proposition 7 on paranormal operators as a variant from the result in [7].

It is interesting to point out the contrast between Theorem 6 and Proposition 7.

Proposition 7. Let $T$ be a paranormal operator. Then

$$\|T x\| \leq \|T^2 x\|^\frac{1}{2} \leq \cdots \leq \|T^n x\|^\frac{1}{n}$$

hold for every unit vector $x \in H$ and all positive integer $n$.

References


[8] T. Furuta, $A \geq B \geq 0$ assures $(B^r A^p B^r)^{1/q} \geq B^{(p+2r)/q}$ for $r \geq 0$, $p \geq 0$, $q \geq 1$ with $(1+2r)q \geq p + 2r$, Proc. Amer. Math. Soc., 101 (1987), 85–88.


