A NOTE OF REAL PARTS OF SOME SEMI-HYponormal OPERATORS

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Let $\mathcal{H}$ be Hilbert space and $B(\mathcal{H})$ be set of all bounded linear operators on $\mathcal{H}$. Then for $T \in B(\mathcal{H})$

$$T: \text{semi-hyponormal} \iff |T| \geq |T^*|.$$  

About semi-hyponormal operators, we have following 3 problems:

(1) $\Re \sigma(T) = \sigma(\Re T)$ ?

(2) $\sigma_{\text{conv}}(T) = \overline{W(T)}$ ?

(3) $\|(T - z)^{-1}\| \leq \frac{1}{d(z, \sigma(T))}$ for every $z \notin \sigma(T)$ ?

We have 2 kinds of concrete examples of semi-hyponormal operators. D. Xia provides interesting examples (see [1],[5]). Let $\ell^2(Z)$ be the Hilbert space of all doubly-infinite sequences $a = \{a_k\}$ of complex numbers such that

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\[ ||a||^2 = \sum_{k=-\infty}^{\infty} |a_k|^2 < \infty, \] and let \( V \) be the bilateral shift: \((Va)_k = a_{k-1}\). Let \( K \) be a Hilbert space and let \( H \) be the Hilbert space of all doubly-infinite sequences \( x = \{x_k\} \) of elements of \( K \) such that \( ||x||^2 = \sum_{k=-\infty}^{\infty} ||x_k||^2 < \infty \).

Then we have \( H = \ell^2(\mathbb{Z}) \otimes K \). Let \( e_m = \{a_k\} \in \ell^2(\mathbb{Z}) \) such that \( a_m = 1 \) and 0's elsewhere. Every \( x = \{x_k\} \in H \) has the representation \( \sum_{k=-\infty}^{\infty} e_k \otimes x_k \).

Let \( \{A_k\} \) be a doubly-infinite sequence of positive operators on \( K \) such that \( \{||A_k||\} \) is bounded. We define bounded operators \( A \) and \( U \) on \( H \) by

\[ Ae_k \otimes x_k = e_k \otimes A_k x_k, \quad U e_k \otimes x_k = e_{k+1} \otimes x_k \quad (k = 0, \pm 1, \pm 2, \cdots), \]

respectively. Then \( U \) has the form \( V \otimes id_K \). Put \( T = UA \). Such an operator is called an operator valued bilateral weighted shift [3]. If positive operators \( \{A_k\} \) satisfy that \( A_{k+1} \geq A_k \) for every \( k \) and there exists \( j \) such that \( A_{j+1}^2 \not\geq A_j^2 \), then \( T \) is semi-hyponormal but not hyponormal.

Next second example is as follows: Let \( S \) on \( \ell^2(\mathbb{Z}) \) defined by \( S = V(P + I + \frac{1}{2}(V + V^*)) \), where \( P \) denotes the orthogonal projection from \( \ell^2(\mathbb{Z}) \) onto the subspace generated by \( \{e_0, e_1, e_2, \cdots\} \). Xia showed that \( S \) is semi-hyponormal but not hyponormal [8, Chapter 3, Corollary 1.4].

**2. Spectral properties.**

**Lemma 1.** Let \( T \) be an operator valued bilateral weighted shift. Then there exists a closed set \( F \) of positive real numbers such that

\[ \sigma(T) = \{z : |z| \in F\}. \]

**Proof.** Let \( c \in \mathbb{C} \) such that \(|c| = 1\). By [6, p. 52, Corollary 2] \( T \) and \( cT \) are unitarily equivalent. The proof follows from this property.

**Theorem 2.** Let \( T \) be an operator valued bilateral weighted shift such that \( r(T) = ||T|| \). Then

\[ \text{conv} \; \sigma(T) = \overline{W(T)} \quad (\text{i.e., } T \text{ is convexoid}). \]
\[ \sigma(\text{Re} \, T) = \text{Re} \, (\sigma(T)). \]

**Proof.** Let \( x \in \sigma(\text{Re} \, T) \). Suppose that \( x \notin \text{Re} \, \sigma(T) \). Let \( L \) be the line \( \text{Re} \, z = x \). Then \( L \) is disjoint from \( \sigma(T) \). Suppose that \( \sigma(T) \) is on the left side of \( L \). There exists \( \varepsilon > 0 \) such that \( \text{Re} \, \sigma(T) \leq x - \varepsilon \). For any complex number \( \lambda = |\lambda|e^{i\theta} \), we can choose \( z \in \sigma(T) \) such that \( z = \|T\|e^{i\theta} \) by Lemma 1. Since \( (\|T\| + |\lambda|)e^{i\theta} \in \sigma(T + \lambda I) \), we have
\[
\rho(T + \lambda I) \geq \|T\| + |\lambda| \quad (\geq \|T + \lambda I\|).
\]
Hence we have \( \rho(T + \lambda I) = \|T + \lambda I\| \), that is, \( T \) is a transaloid. Therefore by [3] or [5, Theorem 6.15.11] we have
\[
\text{conv} \, \sigma(T) = \overline{W(T)}.
\]
Thus
\[
\text{conv} \, \sigma(\text{Re} \, T) = \overline{W(\text{Re} \, T)} = \text{Re} \, \overline{W(T)} = \text{Re} \, \text{conv} \, \sigma(T) \leq x - \varepsilon.
\]
This implies that \( x \leq x - \varepsilon \), which is a contradiction. We proceed similarly in case \( \sigma(T) \) is on the right side. Therefore \( \sigma(\text{Re} \, T) \subseteq \text{Re} \, \sigma(T) \).

Let \( x \in \text{Re} \, \sigma(T) \). By Lemma 1, there exists \( z \in \sigma(T) \) such that \( \text{Re} \, z = x \) and \( |z| = \|T\| \). Since \( z \) is a boundary point of \( \sigma(T) \), there exists a sequence \( \{f_{n}\} \) of unit vectors such that \( \lim_{\substack{n \to \infty}} \|(T - zI)f_{n}\| = 0 \). By [5, Lemma 7.5.2], we have that \( \lim_{\substack{n \to \infty}} \|(T^{*} - i)\sigma_{n}\| = 0 \). Hence
\[
\lim_{\substack{n \to \infty}} \|(\text{Re} \, T - xI)f_{n}\| = 0,
\]
so that \( \text{Re} \, \sigma(T) \subseteq \sigma(\text{Re} \, T) \). Therefore, \( \text{Re} \, \sigma(T) = \sigma(\text{Re} \, T) \).

In general, it holds that if \( T \) is semi-hyponormal, then \( \rho(T) = \|T\| \). Hence we have

**Corollary 3.** Let \( T \) be a semi-hyponormal operator valued bilateral weighted shift. Then
\[
\text{conv} \, \sigma(T) = \overline{W(T)} \quad \text{and} \quad \sigma(\text{Re} \, T) = \text{Re} \, (\sigma(T)).
\]
Theorem 4. With the notations in the introduction, let $S = V(P + I + \frac{1}{2}(V + V^*))$. Then we have

$$\sigma(\text{Re } S) = \text{Re } \sigma(S).$$

Proof. It $F$ is a proper closed subset of $[0, 2\pi]$ such that $m([0, 2\pi]) = m(F)$. Since $[0, 2\pi] - F$ contains an open interval $(a, b)$, we have $m([0, 2\pi]) - m(F) \geq m((a, b)) > 0$. This is a contradiction. Hence there exists no proper closed set $F$ such that $m(F) = m([0, 2\pi])$. Applying [8, Chapter 4, Example 4.1] with $\alpha(\cdot) = 1$ and $\beta(\cdot) = 1 + \cos \theta$, we have that

$$\sigma(S) = \{ e^{i\theta}(1 + \cos \theta + k) : 0 \leq k \leq 1, 0 \leq \theta \leq 2\pi \}.$$  

Hence $\text{Re } \sigma(S) = \{(1 + k)\cos \theta + \cos^2 \theta : 0 \leq k \leq 1, 0 \leq \theta \leq 2\pi \} = [-1, 3]$. Since $S$ is semi-hyponormal, it holds that $\sigma_a(S) = \sigma_{na}(S)$. Hence we have

$$\text{Re } \sigma(S) \subseteq \sigma(\text{Re } S).$$

Next we will prove that $\sigma(\text{Re } S) \subseteq [-1, 3]$. First by the definition of $S$, we have $||\text{Re } S|| \leq ||S|| \leq 3$. Since $S$ is convexoid, we may only prove $(\text{Re } S) + I \geq 0$.

Since $\text{Re } S$ can be canonically represented by a matrix form with real components, for $\lambda \in \sigma(\text{Re } S)$ we choose a sequence $\{f_m\}$ of unit vectors in $l^2(\mathbb{Z})$ with real components such that $\lim_{m \to \infty} ||((\text{Re } S) - \lambda I)f_m|| = 0$. Since

$$2\text{Re } S = (V + V^*) + \frac{1}{2}(V^2 + V^{*2}) + (VP + PV^*) + VV^*,$$

we have, for $f = (\alpha_n)$ with all $\alpha_n \in \mathbb{R}$,

$$2(((\text{Re } S) + I)f, f) = ((V + V^*)f, f) + \frac{1}{2}((V^2 + V^{*2})f, f)$$

$$+((VP + PV^*)f, f) + (VV^*f, f) + 2(f, f)$$

$$= 2 \sum_{n=-\infty}^{\infty} \alpha_n \alpha_{n+1} + \sum_{n=-\infty}^{\infty} \alpha_n \alpha_{n+2} + 2 \sum_{n=0}^{\infty} \alpha_n \alpha_{n+1} + 3 \sum_{n=-\infty}^{\infty} \alpha_n^2$$

$$= \sum_{n=-\infty}^{\infty} \alpha_n \alpha_{n+2} + 4 \sum_{n=-\infty}^{\infty} \alpha_n \alpha_{n+1}$$
\[ -2 \sum_{n=-\infty}^{-1} \alpha_n \alpha_{n+1} + 3 \sum_{n=-\infty}^{\infty} \alpha_n^2. \]

If we can choose a sequence \( \{(a_n, b_n, c_n)\}_{n=-\infty}^{\infty} \) of triplets of positive numbers satisfying

\[
2(((\text{Re } S) + I)f, f) = \sum_{n=-\infty}^{\infty} (a_n a_n + b_n a_{n+1} + c_n a_{n+2})^2
\]

\[
= \sum_{n=-\infty}^{\infty} (a_{n+2}^2 + b_{n+1}^2 + c_n^2) \alpha_n^2 + 2 \sum_{n=-\infty}^{\infty} (a_n b_n + b_{n-1} c_{n-1}) \alpha_n \alpha_{n+1}
\]

\[+ 2 \sum_{n=-\infty}^{\infty} (a_n c_n) \alpha_n \alpha_{n+2}, \]

then we have \((\text{Re } S) + I \geq 0\) and we hence can finish the proof.

For \( n \geq -1 \), since

(i) \( a_{n+2}^2 + b_{n+1}^2 + c_n^2 = 3 \), (ii) \( 2(a_{n+1} b_{n+1} + b_n c_n) = 4 \) and (iii) \( 2a_n c_n = 1 \),

we define

\[
a_n = \frac{1}{\sqrt{2}}, \quad b_n = \sqrt{2} \quad \text{and} \quad c_n = \frac{1}{\sqrt{2}}. \]

For \( n \leq -2 \), since

(i) \( a_{n+2}^2 + b_{n+1}^2 + c_n^2 = 3 \), (ii) \( 2(a_{n+1} b_{n+1} + b_n c_n) = 2 \) and (iii) \( 2a_n c_n = 1 \),

inductively we define, in the following order:

\[
c_n = \sqrt{3 - a_{n+2}^2 - b_{n+1}^2}, \quad b_n = \frac{1 - a_{n+1} b_{n+1}}{c_n} \quad \text{and} \quad a_n = \frac{1}{2c_n}. \]

For a definition of \( c_n \), we need to check that \( 3 > a_{n+2}^2 + b_{n+1}^2 \). We calculate

\[
c_{-2} = \frac{1}{\sqrt{2}}, \quad b_{-2} = 0, \quad a_{-2} = \frac{1}{\sqrt{2}}
\]
\[
c_{-3} = \frac{\sqrt{5}}{2}, \quad b_{-3} = \frac{\sqrt{5}}{2}, \quad a_{-3} = \frac{1}{\sqrt{10}}
\]
\[
c_{-4} = \frac{\sqrt{21}}{10}, \quad b_{-4} = 4 \sqrt{\frac{2}{105}}, \quad a_{-4} = \frac{\sqrt{5}}{42}
\]
\[
c_{-5} = \sqrt{\frac{109}{42}}, \quad b_{-5} = 17 \sqrt{\frac{2}{2289}}, \quad a_{-5} = \sqrt{\frac{21}{218}}
\]
\[
c_{-6} = \sqrt{\frac{573}{218}}, \quad b_{-6} = 92 \sqrt{\frac{2}{62457}} \quad \text{and} \quad a_{-6} = \sqrt{\frac{109}{1146}}. \]
Then we have that

$$1.61 \leq c_{-5}, c_{-6} \leq 1.64, \quad 0.50 \leq b_{-5}, b_{-6} \leq 0.53, \quad 0.30 \leq a_{-5}, a_{-6} \leq 0.32,$$

$$1.64 \leq \sqrt{3 - 0.53^2 - 0.32^2} \leq c_{-7} \leq \sqrt{3 - 0.50^2 - 0.30^2} \leq 1.64,$$

$$0.50 \leq \frac{1 - 0.53 \times 0.32}{1.64} \leq b_{-7} \leq \frac{1 - 0.50 \times 0.30}{1.61} \leq 0.53$$

and

$$0.30 \leq \frac{1}{2 \cdot 1.64} \leq a_{-7} \leq \frac{1}{2 \cdot 1.61} \leq 0.32.$$

Thus we can define $c_n, b_n$ and $a_n$ for $n \leq -8$. This completes the proof.

By a similar argument in Theorem 2, we have that $\text{Im} \sigma(T) = \sigma(\text{Im} \ T)$ for $T$ of Theorem 2. In the proof of Theorem 4 we regarded $\text{Re} \ S$ as an infinite matrix with real components.

**References**


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