<table>
<thead>
<tr>
<th>Title</th>
<th>A NOTE OF REAL PARTS OF SOME SEMI-HYPONORMAL OPERATORS (Operator Inequalities and Related Area)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Author(s)</td>
<td>Cho, Muneo; Huruya, Tadasi; Kim, Young Ok; Lee, Jun Ik</td>
</tr>
<tr>
<td>Citation</td>
<td>数理解析研究所講究録 (2000), 1144: 85-91</td>
</tr>
<tr>
<td>Issue Date</td>
<td>2000-04</td>
</tr>
<tr>
<td>URL</td>
<td><a href="http://hdl.handle.net/2433/63913">http://hdl.handle.net/2433/63913</a></td>
</tr>
<tr>
<td>Type</td>
<td>Departmental Bulletin Paper</td>
</tr>
<tr>
<td>Textversion</td>
<td>publisher</td>
</tr>
</tbody>
</table>

Kyoto University
A NOTE OF REAL PARTS OF SOME SEMI-HYPONORMAL OPERATORS

Let $\mathcal{H}$ be Hilbert space and $B(\mathcal{H})$ be set of all bounded linear operators on $\mathcal{H}$. Then for $T \in B(\mathcal{H})$

\[ T: \text{semi-hyponormal} \iff |T| \geq |T^*|. \]

About semi-hyponormal operators, we have following 3 problems:

(1) \hspace{1cm} \text{Re } \sigma(T) = \sigma(\text{Re } T) \ ?

(2) \hspace{1cm} \text{conv } \sigma(T) = \overline{W(T)} \ ?

(3) \hspace{1cm} \|(T - z)^{-1}\| \leq \frac{1}{d(z, \sigma(T))} \text{ for every } z \notin \sigma(T) \ ?

We have 2 kinds of concrete examples of semi-hyponormal operators. D. Xia provides interesting examples (see [1],[5]). Let $\ell^2(\mathbb{Z})$ be the Hilbert space of all doubly-infinite sequences $a = \{a_k\}$ of complex numbers such that

This research was partially supported by Grant-in-Aid for Scientific Research (No.09640229)
$$||a||^2 = \sum_{k=-\infty}^{\infty} |a_k|^2 < \infty,$$
and let $V$ be the bilateral shift: $(Va)_k = a_{k-1}$. Let $\mathcal{K}$ be a Hilbert space and let $\mathcal{H}$ be the Hilbert space of all doubly-infinite sequences $x = \{x_k\}$ of elements of $\mathcal{K}$ such that $||x||^2 = \sum_{k=-\infty}^{\infty} ||x_k||^2 < \infty$.

Then we have $\mathcal{H} = \ell^2(\mathbb{Z}) \otimes \mathcal{K}$. Let $e_m = \{a_k\} \in \ell^2(\mathbb{Z})$ such that $a_m = 1$ and 0's elsewhere. Every $x = \{x_k\} \in \mathcal{H}$ has the representation $\sum_{k=-\infty}^{\infty} e_k \otimes x_k$.

Let $\{A_k\}$ be a doubly-infinite sequence of positive operators on $\mathcal{K}$ such that $\{||A_k||\}$ is bounded. We define bounded operators $A$ and $U$ on $\mathcal{H}$ by

$$Ae_k \otimes x_k = e_k \otimes A_k x_k, \quad Ue_k \otimes x_k = e_{k+1} \otimes x_k \ (k = 0, \pm 1, \pm 2, \cdots),$$
respectively. Then $U$ has the form $V \otimes id$. Put $T = UA$. Such an operator is called an operator valued bilateral weighted shift [3]. If positive operators $\{A_k\}$ satisfy that $A_{k+1} \geq A_k$ for every $k$ and there exists $j$ such that $A_{j+1}^2 \not\geq A_j^2$, then $T$ is semi-hyponormal but not hyponormal.

Next second example is as follows: Let $S$ on $\ell^2(\mathbb{Z})$ defined by $S = V(P + I + \frac{1}{2}(V + V^*))$, where $P$ denotes the orthogonal projection from $\ell^2(\mathbb{Z})$ onto the subspace generated by $\{e_0, e_1, e_2, \cdots\}$. Xia showed that $S$ is semi-hyponormal but not hyponormal [8, Chapter 3, Corollary 1.4].

2. Spectral properties.

Lemma 1. Let $T$ be an operator valued bilateral weighted shift. Then there exists a closed set $F$ of positive real numbers such that

$$\sigma(T) = \{z : |z| \in F\}.$$ 

Proof. Let $c \in \mathbb{C}$ such that $|c| = 1$. By [6, p. 52, Corollary 2] $T$ and $cT$ are unitarily equivalent. The proof follows from this property.

Theorem 2. Let $T$ be an operator valued bilateral weighted shift such that $r(T) = ||T||$. Then

$$\text{conv } \sigma(T) = \overline{W(T)} \quad (i.e., \ T \text{ is convexoid})$$.
and

$$\sigma(\text{Re} \ T) = \text{Re} (\sigma(T)).$$

**Proof.** Let $$x \in \sigma(\text{Re} \ T)$$. Suppose that $$x \notin \text{Re} \sigma(T)$$. Let $$L$$ be the line $$\text{Re} \ z = x$$. Then $$L$$ is disjoint from $$\sigma(T)$$. Suppose that $$\sigma(T)$$ is on the left side of $$L$$. There exists $$\varepsilon > 0$$ such that $$\text{Re} \sigma(T) \leq x - \varepsilon$$. For any complex number $$\lambda = |\lambda|e^{i\theta}$$, we can choose $$z \in \sigma(T)$$ such that $$z = ||T||e^{i\theta}$$ by Lemma 1. Since $$(||T|| + |\lambda|)e^{i\theta} \in \sigma(T + \lambda I)$$, we have

$$r(T + \lambda I) \geq ||T|| + |\lambda| \ (\geq ||T + \lambda I||).$$

Hence we have $$r(T + \lambda I) = ||T + \lambda I||$$, that is, $$T$$ is a transaloid. Therefore by [3] or [5, Theorem 6.15.11] we have

$$\text{conv} \ \sigma(T) = \overline{W(T)}.$$

Thus

$$\text{conv} \ \sigma(\text{Re} \ T) = \overline{W(\text{Re} \ T)} = \text{Re} \ \overline{W(T)} = \text{Re} \ \text{conv} \ \sigma(T) \leq x - \varepsilon.$$  

This implies that $$x \leq x - \varepsilon$$, which is a contradiction. We proceed similarly in case $$\sigma(T)$$ is on the right side. Therefore $$\sigma(\text{Re} \ T) \subseteq \text{Re} \ \sigma(T)$$.

Let $$x \in \text{Re} \ \sigma(T)$$. By Lemma 1, there exists $$z \in \sigma(T)$$ such that $$\text{Re} \ z = x$$ and $$|z| = ||T||$$. Since $$z$$ is a boundary point of $$\sigma(T)$$, there exists a sequence $$\{f_n\}$$ of unit vectors such that $$\lim \limits_{n \to \infty} ||(T - zI)f_n|| = 0$$. By [5, Lemma 7.5.2], we have that $$\lim \limits_{n \to \infty} ||(T^* - \overline{z}I)f_n|| = 0$$. Hence

$$\lim \limits_{n \to \infty} ||(\text{Re} \ T - xI)f_n|| = 0,$$

so that $$\text{Re} \ \sigma(T) \subseteq \sigma(\text{Re} \ T)$$. Therefore, $$\text{Re} \ \sigma(T) = \sigma(\text{Re} \ T)$$.

In general, it holds that if $$T$$ is semi-hyponormal, then $$r(T) = ||T||$$. Hence we have

**Corollary 3.** Let $$T$$ be a semi-hyponormal operator valued bilateral weighted shift. Then

$$\text{conv} \ \sigma(T) = \overline{W(T)} \text{ and } \sigma(\text{Re} T) = \text{Re}(\sigma(T)).$$
Theorem 4. With the notations in the introduction, let \( S = V(P + I + \frac{1}{2}(V + V^*)) \). Then we have \( \sigma(\text{Re} \, S) = \text{Re} \sigma(S) \).

Proof. It \( F \) is a proper closed subset of \([0, 2\pi]\) such that \( m([0, 2\pi]) = m(F) \). Since \([0, 2\pi] - F \) contains an open interval \((a, b)\), we have \( m([0, 2\pi]) - m(F) \geq m((a, b)) > 0 \). This is a contradiction. Hence there exists no proper closed set \( F \) such that \( m(F) = m([0, 2\pi]) \). Applying [8, Chapter 4, Example 4.1] with \( \alpha(\cdot) = 1 \) and \( \beta(\cdot) = 1 + \cos \theta \), we have that

\[
\sigma(S) = \{ e^{i\theta}(1 + \cos \theta + k) : 0 \leq k \leq 1, \ 0 \leq \theta \leq 2\pi \}.
\]

Hence \( \text{Re} \sigma(S) = \{ (1 + k)\cos \theta + \cos^2 \theta : 0 \leq k \leq 1, 0 \leq \theta \leq 2\pi \} = [-1, 3] \). Since \( S \) is semi-hyponormal, it holds that \( \sigma_a(S) = \sigma_{na}(S) \). Hence we have

\[
\text{Re} \sigma(S) \subseteq \sigma(\text{Re} \, S).
\]

Next we will prove that \( \sigma(\text{Re} \, S) \subseteq [-1, 3] \). First by the definition of \( S \), we have \( ||\text{Re} \, S|| \leq ||S|| \leq 3 \). Since \( \text{Re} \, S \) is convexoid, we may only prove \( (\text{Re} \, S) + I \geq 0 \).

Since \( \text{Re} \, S \) can be canonically represented by a matrix form with real components, for \( \lambda \in \sigma(\text{Re} \, S) \) we choose a sequence \( \{f_m\} \) of unit vectors in \( l^2(\mathbb{Z}) \) with real components such that \( \lim_{m \to \infty} ||((\text{Re} \, S) - \lambda I)f_m|| = 0 \).

Since \( 2\text{Re} \, S = (V + V^*) + \frac{1}{2}(V^2 + V^{*2}) + (VP + PV^*) + VV^* \), we have, for \( f = (\alpha_n) \) with all \( \alpha_n \in \mathbb{R} \),

\[
2((\text{Re} \, S) + I)f, f) = ((V + V^*)f, f) + \frac{1}{2}((V^2 + V^{*2})f, f)
+ ((VP + PV^*)f, f) + (VV^*f, f) + 2(f, f)
= 2 \sum_{n=-\infty}^{\infty} \alpha_n \alpha_{n+1} + \sum_{n=-\infty}^{\infty} \alpha_n \alpha_{n+2} + 2 \sum_{n=0}^{\infty} \alpha_n \alpha_{n+1} + 3 \sum_{n=-\infty}^{\infty} \alpha_n^2
= \sum_{n=-\infty}^{\infty} \alpha_n \alpha_{n+2} + 4 \sum_{n=-\infty}^{\infty} \alpha_n \alpha_{n+1}
\]
$$- 2 \sum_{n=-\infty}^{1} \alpha_n \alpha_{n+1} + 3 \sum_{n=-\infty}^{\infty} \alpha_n^2.$$ 

If we can choose a sequence $\{(a_n, b_n, c_n)\}_{n=-\infty}^{\infty}$ of triplets of positive numbers satisfying

$$2(((\operatorname{Re} S) + I)f, f) = \sum_{n=-\infty}^{\infty} (a_n \alpha_n + b_n \alpha_{n+1} + c_n \alpha_{n+2})^2$$

$$= \sum_{n=-\infty}^{\infty} (a_{n+2}^2 + b_{n+1}^2 + c_n^2) \alpha_n^2 + 2 \sum_{n=-\infty}^{\infty} (a_n b_n + b_{n-1} c_{n-1}) \alpha_n \alpha_{n+1}$$

$$+ 2 \sum_{n=-\infty}^{\infty} (a_n c_n) \alpha_n \alpha_{n+2},$$

then we have $(\operatorname{Re} S) + I \geq 0$ and we hence can finish the proof.

For $n \geq -1$, since

(i) $a_{n+2}^2 + b_{n+1}^2 + c_n^2 = 3$,  
(ii) $2(a_{n+1} b_{n+1} + b_n c_n) = 4$ and  
(iii) $2a_n c_n = 1$,

we define

$$a_n = \frac{1}{\sqrt{2}}, \quad b_n = \sqrt{2} \quad \text{and} \quad c_n = \frac{1}{\sqrt{2}}.$$ 

For $n \leq -2$, since

(i) $a_{n+2}^2 + b_{n+1}^2 + c_n^2 = 3$,  
(ii) $2(a_{n+1} b_{n+1} + b_n c_n) = 2$ and  
(iii) $2a_n c_n = 1$,

inductively we define, in the following order:

$$c_n = \sqrt{3 - a_{n+2}^2 - b_{n+1}^2}, \quad b_n = \frac{1 - a_{n+1} b_{n+1}}{c_n} \quad \text{and} \quad a_n = \frac{1}{2c_n}.$$ 

For a definition of $c_n$, we need to check that $3 > a_{n+2}^2 + b_{n+1}^2$. We calculate

$$c_{-2} = \frac{1}{\sqrt{2}}, \quad b_{-2} = 0, \quad a_{-2} = \frac{1}{\sqrt{2}};$$

$$c_{-3} = \sqrt{\frac{5}{2}}, \quad b_{-3} = \sqrt{\frac{5}{2}}, \quad a_{-3} = \sqrt{\frac{1}{10}};$$

$$c_{-4} = \sqrt{\frac{21}{10}}, \quad b_{-4} = 4\sqrt{\frac{2}{105}}, \quad a_{-4} = \sqrt{\frac{5}{42}};$$

$$c_{-5} = \sqrt{\frac{109}{42}}, \quad b_{-5} = 17\sqrt{\frac{2}{2289}}, \quad a_{-5} = \sqrt{\frac{21}{218}};$$

$$c_{-6} = \sqrt{\frac{573}{218}}, \quad b_{-6} = 92\sqrt{\frac{2}{62457}} \quad \text{and} \quad a_{-6} = \sqrt{\frac{109}{1146}}.$$
Then we have that
\[ 1.61 \leq c_{-5}, c_{-6} \leq 1.64, \quad 0.50 \leq b_{-5}, b_{-6} \leq 0.53, \quad 0.30 \leq a_{-5}, a_{-6} \leq 0.32, \]
\[ 1.64 \leq \sqrt{3 - 0.53^2 - 0.32^2} \leq c_{-7} \leq \sqrt{3 - 0.50^2 - 0.30^2} \leq 1.64, \]
\[ 0.50 \leq \frac{1 - 0.53 \times 0.32}{1.64} \leq b_{-7} \leq \frac{1 - 0.50 \times 0.30}{1.61} \leq 0.53 \]
and
\[ 0.30 \leq \frac{1}{2 \cdot 1.64} \leq a_{-7} \leq \frac{1}{2 \cdot 1.61} \leq 0.32. \]
Thus we can define \( c_n, b_n \) and \( a_n \) for \( n \leq -8 \). This completes the proof.

By a similar argument in Theorem 2, we have that \( \text{Im} \sigma(T) = \sigma(\text{Im} T) \) for \( T \) of Theorem 2. In the proof of Theorem 4 we regarded \( \text{Re} S \) as an infinite matrix with real components.

References


Muneo Chō  
Department of Mathematics, Kanagawa University, Yokohama 221-8686, Japan

Tadashi Huruya  
Faculty of Education and Human Sciences, Niigata University, Niigata 950-2181, Japan

Young Ok Kim and Jun Ik Lee  
Department of Mathematics, Sungkyunkwan University, Suwon 440-764, Korea