A NOTE OF REAL PARTS OF SOME SEMI-HYPONORMAL OPERATORS

Let $\mathcal{H}$ be Hilbert space and $B(\mathcal{H})$ be set of all bounded linear operators on $\mathcal{H}$. Then for $T \in B(\mathcal{H})$

\[ T: \text{semi-hyponormal} \iff |T| \geq |T^*|. \]

About semi-hyponormal operators, we have following 3 problems:

(1) \[ \text{Re} \sigma(T) = \sigma(\text{Re} T)? \]

(2) \[ \text{conv} \sigma(T) = \overline{W(T)}? \]

(3) \[ \| (T - z)^{-1} \| \leq \frac{1}{d(z, \sigma(T))} \text{ for every } z \notin \sigma(T)? \]

We have 2 kinds of concrete examples of semi-hyponormal operators. D. Xia provides interesting examples (see [1],[5]). Let $\ell^2(\mathbb{Z})$ be the Hilbert space of all doubly-infinite sequences $a = \{a_k\}$ of complex numbers such that

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\[ \|a\|^2 = \sum_{k=-\infty}^{\infty} |a_k|^2 < \infty, \] and let \( V \) be the bilateral shift: \( (Va)_k = a_{k-1} \). Let \( \mathcal{K} \) be a Hilbert space and let \( \mathcal{H} \) be the Hilbert space of all doubly-infinite sequences \( x = \{x_k\} \) of elements of \( \mathcal{K} \) such that \( \|x\|^2 = \sum_{k=-\infty}^{\infty} \|x_k\|^2 < \infty \). Then we have \( \mathcal{H} = \ell^2(\mathbb{Z}) \otimes \mathcal{K} \).

Let \( e_m = \{a_k\} \in \ell^2(\mathbb{Z}) \) such that \( a_m = 1 \) and 0's elsewhere. Every \( x = \{x_k\} \in \mathcal{H} \) has the representation \( \sum_{k=-\infty}^{\infty} e_k \otimes x_k \).

Let \( \{A_k\} \) be a doubly-infinite sequence of positive operators on \( \mathcal{K} \) such that \( \{\|A_k\|\} \) is bounded. We define bounded operators \( A \) and \( U \) on \( \mathcal{H} \) by

\[ Ae_k \otimes x_k = e_k \otimes A_k x_k, \quad Ue_k \otimes x_k = e_{k+1} \otimes x_k \quad (k = 0, \pm 1, \pm 2, \cdots), \]

respectively. Then \( U \) has the form \( V \otimes \text{id}_\mathcal{K} \). Put \( T = UA \). Such an operator is called an operator valued bilateral weighted shift [3]. If positive operators \( \{A_k\} \) satisfy that \( A_{k+1} \geq A_k \) for every \( k \) and there exists \( j \) such that \( A_{j+1}^2 \not\geq A_j^2 \), then \( T \) is semi-hyponormal but not hyponormal.

Next second example is as follows: Let \( S \) on \( \ell^2(\mathbb{Z}) \) defined by \( S = V(P + I + \frac{1}{2}(V + V^*)) \), where \( P \) denotes the orthogonal projection from \( \ell^2(\mathbb{Z}) \) onto the subspace generated by \( \{e_0, e_1, e_2, \cdots\} \). Xia showed that \( S \) is semi-hyponormal but not hyponormal [8, Chapter 3, Corollary 1.4].

2. Spectral properties.

**Lemma 1.** Let \( T \) be an operator valued bilateral weighted shift. Then there exists a closed set \( F \) of positive real numbers such that

\[ \sigma(T) = \{z : |z| \in F\}. \]

**Proof.** Let \( c \in \mathbb{C} \) such that \( |c| = 1 \). By [6, p. 52, Corollary 2] \( T \) and \( cT \) are unitarily equivalent. The proof follows from this property.

**Theorem 2.** Let \( T \) be an operator valued bilateral weighted shift such that \( r(T) = \|T\| \). Then

\[ \text{conv} \sigma(T) = \overline{W(T)} \quad (\text{i.e., } T \text{ is convexoid}) \]
and

$$
\sigma(\Re T) = \Re (\sigma(T)).
$$

**Proof.** Let $x \in \sigma(\Re T)$. Suppose that $x \notin \Re \sigma(T)$. Let $L$ be the line $\Re z = x$. Then $L$ is disjoint from $\sigma(T)$. Suppose that $\sigma(T)$ is on the left side of $L$. There exists $\varepsilon > 0$ such that $\Re \sigma(T) \leq x - \varepsilon$. For any complex number $\lambda = |\lambda|e^{i\theta}$, we can choose $z \in \sigma(T)$ such that $z = ||T||e^{i\theta}$ by Lemma 1. Since $||T|| + |\lambda|e^{i\theta} \in \sigma(T + \lambda I)$, we have

$$
r(T + \lambda I) \geq ||T|| + |\lambda| \ (\geq ||T + \lambda I||).
$$

Hence we have $r(T + \lambda I) = ||T + \lambda I||$, that is, $T$ is a transaloid. Therefore by [3] or [5, Theorem 6.15.11] we have

$$
\text{conv } \sigma(T) = \overline{W(T)}.
$$

Thus

$$
\text{conv } \sigma(\Re T) = \overline{W(\Re T)} = \Re \overline{W(T)} = \Re \text{conv } \sigma(T) \leq x - \varepsilon.
$$

This implies that $x \leq x - \varepsilon$, which is a contradiction. We proceed similarly in case $\sigma(T)$ is on the right side. Therefore $\sigma(\Re T) \subseteq \Re \sigma(T)$.

Let $x \in \Re \sigma(T)$. By Lemma 1, there exists $z \in \sigma(T)$ such that $\Re z = x$ and $|z| = ||T||$. Since $z$ is a boundary point of $\sigma(T)$, there exists a sequence $\{f_n\}$ of unit vectors such that $\lim_{n \to \infty} ||(T - zI)f_n|| = 0$. By [5, Lemma 7.5.2], we have that $\lim_{n \to \infty} ||(T^* - \overline{z}I)f_n|| = 0$. Hence

$$
\lim_{n \to \infty} ||(\Re T - xI)f_n|| = 0,
$$

so that $\Re \sigma(T) \subseteq \sigma(\Re T)$. Therefore, $\Re \sigma(T) = \sigma(\Re T)$.

In general, it holds that if $T$ is semi-hyponormal, then $r(T) = ||T||$. Hence we have

**Corollary 3.** Let $T$ be a semi-hyponormal operator valued bilateral weighted shift. Then

$$
\text{conv } \sigma(T) = \overline{W(T)} \text{ and } \sigma(\Re T) = \Re(\sigma(T)).
$$
Theorem 4. With the notations in the introduction, let \( S = V(P + I + \frac{1}{2}(V + V^*)) \). Then we have

\[
\sigma(\text{Re} \, S) = \text{Re} \, \sigma(S).
\]

Proof. It \( F \) is a proper closed subset of \([0, 2\pi]\) such that \( m([0, 2\pi]) = m(F) \). Since \([0, 2\pi] \setminus F\) contains an open interval \((a, b)\), we have \( m([0, 2\pi]) - m(F) \geq m((a, b)) > 0 \). This is a contradiction. Hence there exist no proper closed set \( F \) such that \( m(F) = m([0, 2\pi]) \). Applying [8, Chapter 4, Example 4.1] with \( \alpha(\cdot) = 1 \) and \( \beta(\cdot) = 1 + \cos \theta \), we have that

\[
\sigma(S) = \{ e^{i\theta}(1 + \cos \theta + k) : 0 \leq k \leq 1, \ 0 \leq \theta \leq 2\pi \}.
\]

Hence \( \text{Re} \, \sigma(S) = \{ (1 + k)\cos \theta + \cos^2 \theta : 0 \leq k \leq 1, 0 \leq \theta \leq 2\pi \} = [-1, 3] \).

Since \( S \) is semi-hyponormal, it holds that \( \sigma_a(S) = \sigma_n(S) \). Hence we have

\[
\text{Re} \, \sigma(S) \subseteq \sigma(\text{Re} \, S).
\]

Next we will prove that \( \sigma(\text{Re} \, S) \subseteq [-1, 3] \). First by the definition of \( S \), we have \( ||\text{Re} \, S|| \leq ||S|| \leq 3 \). Since \( \text{Re} \, S \) is convexoid, we may only prove \( (\text{Re} \, S) + I \geq 0 \).

Since \( \text{Re} \, S \) can be canonically represented by a matrix form with real components, for \( \lambda \in \sigma(\text{Re} \, S) \) we choose a sequence \( \{f_m\} \) of unit vectors in \( l^2(\mathbb{Z}) \) with real components such that

\[
\lim_{m \to \infty} ||(\text{Re} \, S - \lambda I) f_m|| = 0.
\]

Since \( 2\text{Re} \, S = (V + V^*) + \frac{1}{2}(V^2 + V^*2) + (VP + PV^*) + VV^* \), we have, for \( f = (\alpha_n) \) with all \( \alpha_n \in \mathbb{R} \),

\[
2(\text{Re} \, S + I)f, f = ((V + V^*)f, f) + \frac{1}{2}((V^2 + V^*2)f, f) + ((VP + PV^*)f, f) + (VV^*f, f) + 2(f, f)
\]

\[
= 2 \sum_{n=-\infty}^{\infty} \alpha_n \alpha_{n+1} + \sum_{n=-\infty}^{\infty} \alpha_n \alpha_{n+2} + 3 \sum_{n=-\infty}^{\infty} \alpha_n^2
\]

\[
= \sum_{n=-\infty}^{\infty} \alpha_n \alpha_{n+2} + 4 \sum_{n=-\infty}^{\infty} \alpha_n \alpha_{n+1}
\]
\(- 2 \sum_{n=-\infty}^{-1} \alpha_{n}\alpha_{n+1} + 3 \sum_{n=-\infty}^{\infty} \alpha_{n}^{2}\).

If we can choose a sequence \(\{(a_{n}, b_{n}, c_{n})\}_{n=-\infty}^{\infty}\) of triplets of positive numbers satisfying

\[
2(((\text{Re } S) + I)f, f) = \sum_{n=-\infty}^{\infty} (a_{n}\alpha_{n} + b_{n}\alpha_{n+1} + c_{n}\alpha_{n+2})^{2}
\]

\[
= \sum_{n=-\infty}^{\infty} (a_{n+2}^{2} + b_{n+1}^{2} + c_{n}^{2})\alpha_{n}^{2} + 2 \sum_{n=-\infty}^{\infty} (a_{n}b_{n} + b_{n-1}c_{n-1})\alpha_{n}\alpha_{n+1}
\]

\[
+ 2 \sum_{n=-\infty}^{\infty} (a_{n}c_{n})\alpha_{n}\alpha_{n+2},
\]

then we have \((\text{Re } S) + I \geq 0\) and we hence can finish the proof.

For \(n \geq -1\), since

(i) \(a_{n+2}^{2} + b_{n+1}^{2} + c_{n}^{2} = 3\),
(ii) \(2(a_{n+1}b_{n+1} + b_{n}c_{n}) = 4\) and
(iii) \(2a_{n}c_{n} = 1\),

we define

\[
a_{n} = \frac{1}{\sqrt{2}}, \quad b_{n} = \sqrt{2} \quad \text{and} \quad c_{n} = \frac{1}{\sqrt{2}}.
\]

For \(n \leq -2\), since

(i) \(a_{n+2}^{2} + b_{n+1}^{2} + c_{n}^{2} = 3\),
(ii) \(2(a_{n+1}b_{n+1} + b_{n}c_{n}) = 2\) and
(iii) \(2a_{n}c_{n} = 1\),

inductively we define, in the following order:

\[
c_{n} = \sqrt{3 - a_{n+2}^{2} - b_{n+1}^{2}}, \quad b_{n} = \frac{1 - a_{n+1}b_{n+1}}{c_{n}} \quad \text{and} \quad a_{n} = \frac{1}{2c_{n}}.
\]

For a definition of \(c_{n}\), we need to check that \(3 > a_{n+2}^{2} + b_{n+1}^{2}\). We calculate:

\[
\begin{align*}
c_{-2} &= \frac{1}{\sqrt{2}}, & b_{-2} &= 0, & a_{-2} &= \frac{1}{\sqrt{2}} \\
c_{-3} &= \sqrt{\frac{5}{2}}, & b_{-3} &= \frac{1}{\sqrt{2}}, & a_{-3} &= \frac{1}{10} \\
c_{-4} &= \sqrt{\frac{21}{10}}, & b_{-4} &= \frac{4}{\sqrt{229}}, & a_{-4} &= \frac{5}{42} \\
c_{-5} &= \sqrt{\frac{219}{30}}, & b_{-5} &= 17\sqrt{\frac{2}{225}}, & a_{-5} &= \sqrt{\frac{21}{218}} \\
c_{-6} &= \sqrt{\frac{573}{218}}, & b_{-6} &= 92\sqrt{\frac{2}{62457}}, \quad \text{and} \quad a_{-6} &= \sqrt{\frac{109}{1146}}.
\end{align*}
\]
Then we have that

\[ 1.61 \leq c_{-5}, c_{-6} \leq 1.64, \quad 0.50 \leq b_{-5}, b_{-6} \leq 0.53, \quad 0.30 \leq a_{-5}, a_{-6} \leq 0.32, \]

\[ 1.64 \leq \sqrt{3 - 0.53^2 - 0.32^2} \leq c_{-7} \leq \sqrt{3 - 0.50^2 - 0.30^2} \leq 1.64, \]

\[ 0.50 \leq \frac{1 - 0.53 \times 0.32}{1.64} \leq b_{-7} \leq \frac{1 - 0.50 \times 0.30}{1.61} \leq 0.53 \]

and

\[ 0.30 \leq \frac{1}{2 \cdot 1.64} \leq a_{-7} \leq \frac{1}{2 \cdot 1.61} \leq 0.32. \]

Thus we can define \( c_n, b_n \) and \( a_n \) for \( n \leq -8 \). This completes the proof.

By a similar argument in Theorem 2, we have that \( \text{Im} \sigma(T) = \sigma(\text{Im} T) \) for \( T \) of Theorem 2. In the proof of Theorem 4 we regarded \( \text{Re} S \) as an infinite matrix with real components.

References


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