A NOTE OF REAL PARTS OF SOME SEMI-HYPONORMAL OPERATORS

Let $\mathcal{H}$ be Hilbert space and $B(\mathcal{H})$ be set of all bounded linear operators on $\mathcal{H}$. Then for $T \in B(\mathcal{H})$

$$T: \text{semi-hyponormal} \iff |T| \geq |T^*|.$$ 

About semi-hyponormal operators, we have following 3 problems:

1. $\Re \sigma(T) = \sigma(\Re T)$
2. $\operatorname{conv} \sigma(T) = \overline{W(T)}$
3. $\| (T-z)^{-1} \| \leq \frac{1}{d(z, \sigma(T))}$ for every $z \notin \sigma(T)$

We have 2 kinds of concrete examples of semi-hyponormal operators. D. Xia provides interesting examples (see [1],[5]). Let $\ell^2(\mathbb{Z})$ be the Hilbert space of all doubly-infinite sequences $a = \{a_k\}$ of complex numbers such that

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This research was partially supported by Grant-in-Aid for Scientific Research (No.09640229)
\[ \|a\|^2 = \sum_{k=-\infty}^{\infty} |a_k|^2 < \infty, \]
and let \( V \) be the bilateral shift: \( (Va)_k = a_{k-1} \). Let \( \mathcal{K} \) be a Hilbert space and let \( \mathcal{H} \) be the Hilbert space of all doubly-infinite sequences \( x = \{x_k\} \) of elements of \( \mathcal{K} \) such that \( \|x\|^2 = \sum_{k=-\infty}^{\infty} \|x_k\|^2 < \infty \).

Then we have \( \mathcal{H} = \ell^2(\mathbb{Z}) \otimes \mathcal{K} \).

Let \( \mathcal{K} \) be a Hilbert space and \( \mathcal{H} \) be the Hilbert space of all doubly-infinite sequences \( x = \{x_k\} \) of elements of \( \mathcal{K} \) such that \( \|x\|^2 = \sum_{k=-\infty}^{\infty} \|x_k\|^2 < \infty \).

2. Spectral properties.

**Lemma 1.** Let \( T \) be an operator valued bilateral weighted shift. Then there exists a closed set \( F \) of positive real numbers such that

\[ \sigma(T) = \{z : |z| \in F\}. \]

**Proof.** Let \( c \in \mathbb{C} \) such that \( |c| = 1 \). By [6, p. 52, Corollary 2] \( T \) and \( cT \) are unitarily equivalent. The proof follows from this property.

**Theorem 2.** Let \( T \) be an operator valued bilateral weighted shift such that \( r(T) = \|T\| \). Then

\[ \text{conv } \sigma(T) = \overline{W(T)} \quad (i.e., \, T \text{ is convexoid.}) \]
and
\[ \sigma(\text{Re } T) = \text{Re } (\sigma(T)). \]

**Proof.** Let \( x \in \sigma(\text{Re } T) \). Suppose that \( x \notin \text{Re } \sigma(T) \). Let \( L \) be the line \( \text{Re } z = x \). Then \( L \) is disjoint from \( \sigma(T) \). Suppose that \( \sigma(T) \) is on the left side of \( L \). There exists \( \varepsilon > 0 \) such that \( \text{Re } \sigma(T) \leq x - \varepsilon \). For any complex number \( \lambda = |\lambda|e^{i\theta} \), we can choose \( z \in \sigma(T) \) such that \( z = ||T||e^{i\theta} \) by Lemma 1. Since \( (||T|| + |\lambda|)e^{i\theta} \in \sigma(T + \lambda I) \), we have

\[ r(T + \lambda I) \geq ||T|| + |\lambda| \quad (\geq ||T + \lambda I||). \]

Hence we have \( r(T + \lambda I) = ||T + \lambda I|| \), that is, \( T \) is a transaloid. Therefore by [3] or [5, Theorem 6.15.11] we have

\[ \text{conv } \sigma(T) = \overline{W(T)}. \]

Thus

\[ \text{conv } \sigma(\text{Re } T) = \overline{W(\text{Re } T)} = \text{Re } \overline{W(T)} = \text{Re } \text{conv } \sigma(T) \leq x - \varepsilon. \]

This implies that \( x \leq x - \varepsilon \), which is a contradiction. We proceed similarly in case \( \sigma(T) \) is on the right side. Therefore \( \sigma(\text{Re } T) \subseteq \text{Re } \sigma(T) \).

Let \( x \in \text{Re } \sigma(T) \). By Lemma 1, there exists \( z \in \sigma(T) \) such that \( \text{Re } z = x \) and \( |z| = ||T|| \). Since \( z \) is a boundary point of \( \sigma(T) \), there exists a sequence \( \{f_n\} \) of unit vectors such that \( \lim_{n \to \infty} ||(T - zI)f_n|| = 0 \). By [5, Lemma 7.5.2], we have that \( \lim_{n \to \infty} ||(T^* - \overline{z}I)f_n|| = 0 \). Hence

\[ \lim_{n \to \infty} ||(\text{Re } T - xI)f_n|| = 0, \]

so that \( \text{Re } \sigma(T) \subseteq \sigma(\text{Re } T) \). Therefore, \( \text{Re } \sigma(T) = \sigma(\text{Re } T) \).

In general, it holds that if \( T \) is semi-hyponormal, then \( r(T) = ||T|| \). Hence we have

**Corollary 3.** Let \( T \) be a semi-hyponormal operator valued bilateral weighted shift. Then

\[ \text{conv } \sigma(T) = \overline{W(T)} \text{ and } \sigma(\text{Re } T) = \text{Re } (\sigma(T)). \]
**Theorem 4.** With the notations in the introduction, let \( S = V(P + I + \frac{1}{2}(V + V^*)) \). Then we have

\[
\sigma(\text{Re } S) = \text{Re } \sigma(S).
\]

**Proof.** It \( F \) is a proper closed subset of \([0, 2\pi]\) such that \( m([0, 2\pi]) = m(F) \). Since \([0, 2\pi] - F\) contains an open interval \((a, b)\), we have \( m([0, 2\pi]) - m(F) \geq m((a, b)) > 0 \). This is a contradiction. Hence there exits no proper closed set \( F \) such that \( m(F) = m([0, 2\pi]) \). Applying [8, Chapter 4, Example 4.1] with \( \alpha(\cdot) = 1 \) and \( \beta(\cdot) = 1 + \cos \theta \), we have that

\[
\sigma(S) = \{e^{i\theta} (1 + \cos \theta + k) : 0 \leq k \leq 1, \ 0 \leq \theta \leq 2\pi \}.
\]

Hence \( \text{Re } \sigma(S) = \{(1 + k)\cos \theta + \cos^2 \theta : 0 \leq k \leq 1, 0 \leq \theta \leq 2\pi \} = [-1, 3]. \) Since \( S \) is semi-hyponormal, it holds that \( \sigma_{a}(S) = \sigma_{na}(S) \). Hence we have

\[
\text{Re } \sigma(S) \subseteq \sigma(\text{Re } S).
\]

Next we will prove that \( \sigma(\text{Re } S) \subseteq [-1, 3] \). First by the definition of \( S \), we have \( ||\text{Re } S|| \leq ||S|| \leq 3 \). Since \( \text{Re } S \) is convexoid, we may only prove \( (\text{Re } S) + I \geq 0 \).

Since \( \text{Re } S \) can be canonically represented by a matrix form with real components, for \( \lambda \in \sigma(\text{Re } S) \) we choose a sequence \( \{f_{m}\} \) of unit vectors in \( \ell^2(\mathbb{Z}) \) with real components such that

\[
\lim_{m \to \infty} ||((\text{Re } S) - \lambda I)f_{m}|| = 0.
\]

Since

\[
2\text{Re } S = (V + V^*) + \frac{1}{2}(V^2 + V^{*2}) + (VP + PV^*) + VV^*,
\]

we have, for \( f = (\alpha_{n}) \) with all \( \alpha_{n} \in \mathbb{R} \),

\[
2(((\text{Re } S) + I)f, f) = ((V + V^*)f, f) + \frac{1}{2}((V^2 + V^{*2})f, f)
\]

\[
+((VP + PV^*)f, f) + (VV^*f, f) + 2(f, f)
\]

\[
= 2 \sum_{n=-\infty}^{\infty} \alpha_{n}\alpha_{n+1} + \sum_{n=-\infty}^{\infty} \alpha_{n}\alpha_{n+2} + 2 \sum_{n=0}^{\infty} \alpha_{n}\alpha_{n+1} + 3 \sum_{n=-\infty}^{\infty} \alpha_{n}^2
\]

\[
= \sum_{n=-\infty}^{\infty} \alpha_{n}\alpha_{n+2} + 4 \sum_{n=-\infty}^{\infty} \alpha_{n}\alpha_{n+1}
\]
$- 2 \sum_{n=-\infty}^{-1} \alpha_n\alpha_{n+1} + 3 \sum_{n=-\infty}^{\infty} \alpha_n^2$.

If we can choose a sequence \( \{(a_n, b_n, c_n)\}_{n=-\infty}^{\infty} \) of triplets of positive numbers satisfying

\[
2(((\text{Re } S) + I)f, f) = \sum_{n=-\infty}^{\infty} (a_n\alpha_n + b_n\alpha_{n+1} + c_n\alpha_{n+2})^2
\]

\[
= \sum_{n=-\infty}^{\infty} (a_{n+2}^2 + b_{n+1}^2 + c_n^2)\alpha_n^2 + 2 \sum_{n=-\infty}^{\infty} (a_nb_n + b_{n-1}c_{n-1})\alpha_n\alpha_{n+1}
\]

\[+ 2 \sum_{n=-\infty}^{\infty} (a_nc_n)\alpha_n\alpha_{n+2},
\]

then we have \((\text{Re } S) + I \geq 0\) and we hence can finish the proof.

For \( n \geq -1 \), since

(i) \( a_{n+2}^2 + b_{n+1}^2 + c_n^2 = 3 \), (ii) \( 2(a_{n+1}b_{n+1} + b_nc_n) = 4 \) and (iii) \( 2a_nc_n = 1 \), we define

\[
a_n = \frac{1}{\sqrt{2}}, \quad b_n = \sqrt{2} \quad \text{and} \quad c_n = \frac{1}{\sqrt{2}}.
\]

For \( n \leq -2 \), since

(i) \( a_{n+2}^2 + b_{n+1}^2 + c_n^2 = 3 \), (ii) \( 2(a_{n+1}b_{n+1} + b_nc_n) = 2 \) and (iii) \( 2a_nc_n = 1 \), inductively we define, in the following order:

\[
c_n = \sqrt{3 - a_{n+2}^2 - b_{n+1}^2}, \quad b_n = \frac{1 - a_{n+1}b_{n+1}}{c_n} \quad \text{and} \quad a_n = \frac{1}{2c_n}.
\]

For a definition of \( c_n \), we need to check that \( 3 > a_{n+2}^2 + b_{n+1}^2 \). We calculate

\[
c_{-2} = \frac{1}{\sqrt{2}}, \quad b_{-2} = 0, \quad a_{-2} = \frac{1}{\sqrt{2}}
\]
\[
c_{-3} = \sqrt{\frac{5}{2}}, \quad b_{-3} = \sqrt{\frac{5}{2}}, \quad a_{-3} = \frac{1}{\sqrt{10}}
\]
\[
c_{-4} = \sqrt{\frac{21}{10}}, \quad b_{-4} = 4\sqrt{\frac{2}{105}}, \quad a_{-4} = \frac{\sqrt{5}}{42}
\]
\[
c_{-5} = \sqrt{\frac{109}{42}}, \quad b_{-5} = 17\sqrt{\frac{2}{2289}}, \quad a_{-5} = \sqrt{\frac{21}{218}}
\]
\[
c_{-6} = \sqrt{\frac{573}{218}}, \quad b_{-6} = 92\sqrt{\frac{2}{62457}}, \quad a_{-6} = \sqrt{\frac{109}{1146}}.
\]
Then we have that

\[1.61 \leq c_{-5}, c_{-6} \leq 1.64, \quad 0.50 \leq b_{-5}, b_{-6} \leq 0.53, \quad 0.30 \leq a_{-5}, a_{-6} \leq 0.32,\]

\[1.64 \leq \sqrt{3 - 0.53^2 - 0.32^2} \leq c_{-7} \leq \sqrt{3 - 0.50^2 - 0.30^2} \leq 1.64,\]

\[0.50 \leq \frac{1 - 0.53 \times 0.32}{1.64} \leq b_{-7} \leq \frac{1 - 0.50 \times 0.30}{1.61} \leq 0.53\]

and

\[0.30 \leq \frac{1}{2 \cdot 1.64} \leq a_{-7} \leq \frac{1}{2 \cdot 1.61} \leq 0.32.\]

Thus we can define \(c_n, b_n\) and \(a_n\) for \(n \leq -8\). This completes the proof.

By a similar argument in Theorem 2, we have that \(\text{Im } \sigma(T) = \sigma(\text{Im } T)\) for \(T\) of Theorem 2. In the proof of Theorem 4 we regarded \(\text{Re } S\) as an infinite matrix with real components.

**References**


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