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ON SPECTRAL PROPERTIES OF LOG-HYПONORMAL OPERATORS

Abstract

In this paper we consider spectral mapping theorem about two kinds of functional transformations for log-hyponormal operators and the continuity of the spectrum for log-hyponormal operators.

Introduction.

Let $\mathcal{H}$ be a complex Hilbert space and let $B(\mathcal{H})$ denote the set of all bounded linear operators on $\mathcal{H}$. For $A \in B(\mathcal{H})$, we denote the spectrum, the point spectrum, the residual spectrum and the approximate point spectrum of $A$ by $\sigma(A)$, $\sigma_p(A)$, $\sigma_r(A)$ and $\sigma_a(A)$, respectively. For the study of spectral theory of operators, spectral mapping theorems are important. In this paper we consider spectral mapping theorems about two kinds of functional transformations for log-hyponormal operators. It is familiar that if $A$ is normal then for every polynomial $f(\lambda, \lambda^*)$ one has $\sigma(f(A)) = f(\sigma(A)) = \{f(\lambda, \lambda^*); \lambda \in \sigma(A)\}$. In particular, we called the equality $\sigma(\text{Re} (A)) = \text{Re} (\sigma(A))$ with the polynomial $f(\lambda + \lambda^*) := \frac{1}{2}(\lambda + \lambda^*) = \text{Re} (\lambda)$ for any operator $A$ the “projective” property.

The projective property for semi-normal operators was shown by C. Putnam [11] and the projective property for Toeplitz operators was shown by S. Berberian [2]. We will show the subprojective property for $p$-hyponormal or log-hyponormal operators. On the other hand, in [14], D Xia studied the following functional transformation $\varphi_{\{\xi, \psi\}}(T) = \xi(U)\psi(|T|)$ for a semi-hyponormal operator $T = U|T|$. And in [6], M. Itoh extended this result to $p$-hyponormal operators. Recently, M. Chó and B. P. Duggal [4] gave an elementary proof of
Itoh's result for invertible operator cases and generalized this result. We will extend this result for log-hyponormal operator.

On the other hand, in [8] it was shown that the spectrum $\sigma$ is continuous on the set of $p$-hyponormal operators. We also show that this is still true for log-hyponormal operators.

An operator $A$ is called $p$-hyponormal if $(A^*A)^p - (AA^*)^p \geq 0$ for some $p \in (0, \infty)$. If $p = 1$, $A$ is hyponormal and if $p = \frac{1}{2}$, $A$ is semi-hyponormal. By the consequence of L"{o}wener's inequality [10] if $A$ is $p$-hyponormal for some $p \in (0, \infty)$, then $A$ is also $q$-hyponormal for every $q \in (0, p]$. Thus we assume, without loss of generality, that $p \in (0, \frac{1}{2})$. Let $\mathcal{H}(p)$ denote the class of $p$-hyponormal operators. An operator $T$ is called log-hyponormal if $T$ is invertible and satisfies $\log((T^*T) \geq \log((TT^*)$. Since $\log : (0, \infty) \rightarrow (-\infty, \infty)$ is monotone function, every invertible $p$-hyponormal operator is log-hyponormal. But there exists a log-hyponormal operator which is not $p$-hyponormal (cf. [12, Example 12]).

An operator $A \in B(\mathcal{H})$ has a unique polar decomposition $A = U|A|$, where $|A| = (AA^*)^{\frac{1}{2}}$ and $U$ is a partial isometry with the initial space the closure of the range of $|A|$ and the final space the closure of the range of $A$. In particular, if $A = U|A|$ is log-hyponormal, then the operator $U$ is unitary. Associated with $A$ there is a related operator $\tilde{A} = |A|^\frac{1}{2}U|A|^\frac{1}{2}$, we call it the Aluthge transform of $A$. Aluthge transform has been used as a useful tool for study of $p$-hyponormal operators.

The followings are basic properties for $\tilde{A}$.

(i) If $A = U|A|$ be $p$-hyponormal $(0 < p < \frac{1}{2})$, then the operator $\tilde{A} = |A|^\frac{1}{2}U|A|^\frac{1}{2}$ is $(p + \frac{1}{2})$ hyponormal (cf. [1, Theorem 2]).

(ii) If $A \in B(\mathcal{H})$ be a log-hyponormal operator with a polar decomposition $A = U|A|$, then $\tilde{A} = |A|^\frac{1}{2}U|A|^\frac{1}{2}$ is semi-hyponormal (cf. [12, Theorem 4]).

Form the fact above, the second Aluthge transform of a $p$-hyponormal operator or log-hyponormal operator is hyponormal.

**Theorem A** For every $A \in B(\mathcal{H})$ and its Aluthge transform $\tilde{T} = |A|^\frac{1}{2}U|A|^\frac{1}{2}$, it holds that

$$\omega(A) = \omega(\tilde{A})$$

where $\omega = \sigma, \sigma_a$ or $\sigma_p$.

**Proof.** It is known from [9, Theorem 1.3].
1. Functional transformations for log-hyponormal operators.

First, we will show the "subprojective" property for the spectra of $p$-hyponormal operators and log-hyponormal operators. For a operator $T$, a point $z$ is in the normal approximate point spectrum $\sigma_{na}(T)$ of $T$ if there exists a sequence $\{x_n\}$ of unit vectors such that

$$(T - z)x_n \to 0 \quad \text{and} \quad (T - z)^*x_n \to 0 \quad \text{as} \quad n \to \infty.$$ 

We begin with the following lemma. Proof is easy. So we omit it.

**Lemma 1.1.** If $T \in B(H)$ and $\sigma_a(T) = \sigma_{na}(T)$, then

$$\text{Re} (\sigma(T)) \subset \sigma(\text{Re} T) \quad \text{and} \quad \text{Im} (\sigma(T)) \subset \sigma(\text{Im} T). \quad (1.1.1)$$

**Corollary 1.2.** Let $T$ be $p$-hyponormal or log-hyponormal. Then (1.1.1) holds.

**Proof.** Since $\sigma_a(T) = \sigma_{na}(T)$ for a $p$-hyponormal or a log-hyponormal operator $T$. This follows from Lemma 1.1.

**Theorem 1.3.** Let $T = U|T| = H + iK$ be $p$-hyponormal or log-hyponormal and $\hat{T}$ be the second Aluthge transform of $T$. Let $\hat{T} = \hat{H} + i\hat{K}$ be the Cartesian decomposition of $\hat{T}$. Then

$$\sigma(\hat{H}) \subset \sigma(H) \quad \text{and} \quad \sigma(\hat{K}) \subset \sigma(K).$$

**Proof.** By Theorem A,

$$\sigma(T) = \sigma(\hat{T}) \implies \text{Re} (\sigma(T)) = \text{Re} (\sigma(\hat{T})), \quad \text{Im} (\sigma(T)) = \text{Im} (\sigma(\hat{T})).$$

Since $\hat{T}$ is hyponormal, $\text{Re} (\sigma(\hat{T}) = \sigma(\text{Re} \hat{T})$ and $\text{Im} (\sigma(\hat{T}) = \sigma(\text{Im} \hat{T})$. Thus

$$\sigma(\text{Re} \hat{T}) \subset \sigma(\text{Re} T) \quad \text{and} \quad \sigma(\text{Im} \hat{T}) \subset \sigma(\text{Im} T). \quad \square$$

**Corollary 1.4.** Let $T$ be log-hyponormal. If $T$ has a compact real (imaginary) part, then $T$ is normal.
Proof. Since, by Theorem 1.3, $\text{meas}(\sigma(\hat{H})) = 0$, $\hat{T}$ is normal. And since $T$ is normal if and only if $\hat{T}$ is normal. Thus $T$ is normal.

Let $E$ be a bounded closed subset of all real numbers $\mathbb{R}$, and $M(E) = \{\psi : \psi \text{ is a bounded real Baire function on } E\}$. Let $M_0(E) = \{\psi \in M(E) : \psi(x) \geq 0 \text{ for all } x \in E \text{ and } \psi(0) = 0\}$. Let $J(E) = \{\psi : \psi \text{ is a strictly monotone increasing continuous function on } E\}$ and $J_0(E) = M_0(E) \cap J(E)$. Let $S(E) = \{\psi \in M(E) : K_{\psi} \geq 0\}$, where $K_{\psi}$ is the singular integral operator defined on $L^2(E)$ by

$$(K_{\psi}f)(x) = s - \lim_{\epsilon \to 0^+} \frac{1}{2\pi} \int_E \frac{\psi(x) - \psi(y)}{x - (y + i\epsilon)} f(y) dy.$$  

If $E$ is a closed subset of the unit circle $T$, let $M_0(E) = \{\xi : \xi \text{ is a complex Baire function on } E\}$, $J_0(E) = \{\xi : \xi \text{ is a direction preserving homomorphism on } E\}$ and $S_0(E) = \{\xi : \xi \in M_0(E) \text{ and } K_{\xi} \geq 0\}$, where $K_{\xi}$ is the singular integral operator defined on $L^2(E)$ by

$$(K_{\xi}f)(e^{i\theta}) = s - \lim_{\epsilon \to 0^+} \frac{1}{2\pi} \int_E \frac{1 - \xi(e^{i\theta})\overline{\xi(e^{i\eta})}}{1 - e^{i\theta i\eta}e^{-}(1-\epsilon)} f(e^{i\eta}) d\eta.$$  

For functions $f$ and $g$, we denote the functional transformation $F_{[f,g]}(T) = f(U)\exp(g(\log|T|))$ for a log-hyponormal operator $T = U|T|$ and $F_{[f,g]}(re^{i\theta}) = f(re^{i\theta})\exp(g(\log r))$ in the complex plane.

**Lemma 1.5.** Let $T \in B(\mathcal{H})$ be a semi-hyponormal operator with operator decomposition $T = U|T|$. Then $Ue^{|T|}$ is log-hyponormal and

$$
\sigma_a(Ue^{|T|}) = \{e^r e^{i\theta} : re^{i\theta} \in \sigma_a(T)\};
$$

$$
\sigma_r(Ue^{|T|}) = \{e^r e^{i\theta} : re^{i\theta} \in \sigma_r(T)\};
$$

$$
\sigma(Ue^{|T|}) = \{e^r e^{i\theta} : re^{i\theta} \in \sigma(T)\}.
$$

**Proof.** Proof is from [13, Lemmas 5 and 6].

**Theorem 1.6.** Let $T = U|T|$ be log-hyponormal and $\log|T| \geq 0$. Suppose that $f \in J_0(\sigma(U)) \cap S_0(\sigma(U))$ and $g \in J_0(\sigma(\log|T|)) \cap S_0(\sigma(\log|T|))$ if $\sigma(U) \neq T$ and $g \in J_0([0, ||\log|T||]) \cap S_0([0, ||\log|T||])$ if $\sigma(U) = T$. Then $F_{[f,g]}(T)$ is log-hyponormal and $F_{[f,g]}(\sigma_w(T)) = \sigma_w(F_{[f,g]}(T))$, where $\sigma_w = \sigma, \sigma_a \text{ or } \sigma_r$. 

\[ \square \]
Proof. Let $T = U|T|$ be log-hyponormal, then $S = U \log |T|$ is semi-hyponormal and $\sigma_w(S) = \{(\log r)e^{i\theta} : re^{i\theta} \in \sigma_w(T)\}$. From Theorem VI, 3.1 of [14], $f(U)g(\log |T|)$ is also semi-hyponormal. Thus $\sigma_w(f(U)g(\log |T|)) = \{f(e^{i\theta})g(\log r) : (\log r)e^{i\theta} \in \sigma_w(U \log |T|)\}$. Moreover, from Lemma 1.5 we can see that $F_{[f,g]}(\tau) = f(U)\exp(g(\log |T|))$ is log-hyponormal. Thus $\sigma_w(F_{[f,g]}(T)) = \sigma_w(f(U)\exp(g(\log |T|)))$ is semi-hyponormal. Thus

$$\sigma_w(F_{[f,g]}(T)) = \{e^{g(\log r)}f(e^{i\theta}) : (\log r)e^{i\theta} \in \sigma_w(U \log |T|), re^{i\theta} \in \sigma_w(T)\}.$$ 

2. Continuity of $\sigma$ on the set of all log-hyponormal operators.

In [8], it was shown that the spectrum $\sigma$ is continuous on the set of all $p$-hyponormal operators. In this section we show that this is still true for log-hyponormal operators. To do this we recall that $T \in B(\mathcal{H})$ is said to be bounded below if there exists $k > 0$ for which $\|x\| \leq k\|Tx\|$ for each $x \in \mathcal{H}$. For $A \in B(\mathcal{H})$, $\gamma(A)$ denote the reduced minimum modulus, $\gamma(A) = \inf_{x \in \mathcal{H} \setminus \{0\}} \|Ax\|$ for $\dist(x, \ker A) = \frac{\|Ax\|}{\|x\|}$. Let $y \in \mathcal{H}$ and $\|y\| = 1$. Then there exist $x$ and $x_n$ in $\mathcal{H}$ (for $n \in \mathbb{Z}^+$) such that $y = |T|x$ and $y = |T_n|x_n$. Since $\gamma(S)$ is the supremum of all real number $\gamma$ such that $\gamma\|x\| \leq \|Sx\|$, we have

$$\|x\| \leq \frac{1}{\gamma(|T|)} \|T|x\| = \frac{1}{\gamma(T)} \|y\| = \frac{1}{\gamma(T)} < 2/\alpha.$$
Similarly, $\|x_n\| < 2/\alpha$ for all $n \in \mathbb{Z}^+$. Therefore
\[
\|U_n y - U y\| = \|U_n |T_n| x_n - U |T| x\| \leq \|U_n |T_n| x_n - U_n |T_n| x\| + \|U_n |T_n| x - U |T| x\|.
\]
But
\[
\|U_n |T_n| x - U |T| x\| \leq \|T_n - T\| \|x\| < \frac{2\|T_n - T\|}{\alpha} \rightarrow 0 \quad \text{as} \quad n \rightarrow \infty.
\]
We now claim that $\|x_n - x\| \rightarrow 0$ as $n \rightarrow \infty$. If it is not so, then there exist $\delta > 0$ and a sequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $\|x_{n_k} - x\| > \delta$ for all $k$. Hence
\[
\|T|(x_{n_k} - x)\| = \|T|x_{n_k} - |T_{n_k}|x_{n_k}\| \leq \|T| - |T_{n_k}|\||x_{n_k}\| < \frac{2}{\alpha} \|T| - |T_{n_k}|\| \rightarrow 0
\]
as $n \rightarrow \infty$. This implies that $|T|$ is not bounded below. It is a contradiction. Therefore, we have
\[
\|U_n |T_n| x_n - U_n |T_n| x\| \leq \|T_n\| \|x_n - x\| \rightarrow 0 \quad \text{as} \quad n \rightarrow \infty.
\]

Now we have:

**Theorem 2.2.** The spectrum $\sigma$ is continuous on the set of all log-hyponormal operators.

**Proof.** Suppose that $T = U|T|$ and $T_n = U_n |T_n|$ for $n \in \mathbb{Z}^+$ are log-hyponormal operators such that $T_n$ converges to $T$. Since $T$ is invertible it follows from Lemma 2.1 that $U_n$ converges to $U$, so that
\[
\tilde{T}_n = |T_n|^{\frac{1}{2}} U_n |T_n|^{\frac{1}{2}} \rightarrow \tilde{T} = |T|^{\frac{1}{2}} U |T|^{\frac{1}{2}} \quad \text{as} \quad n \rightarrow \infty.
\]
Since $\tilde{T} = |T|^{\frac{1}{2}} U |T|^{\frac{1}{2}}$ is semi-hyponormal and the spectrum is continuous on the set of all $p$-hyponormal operators, we have
\[
\sigma(T_n) = \sigma(\tilde{T}_n) \rightarrow \sigma(\tilde{T}) = \sigma(T).
\]

For an operator $A \in B(\mathcal{H})$, $z$ is in the approximate defect spectrum $\sigma_{\delta}(A)$ if there exists a sequence $\{x_n\}$ of unit vectors in $\mathcal{H}$ such that $\lim_{n \rightarrow \infty} \|(A - z)^* x_n\| = 0$. Then we have
Theorem 2.3. Let $T$ be a log-hyponormal operator. Then

$$\sigma(T) = \sigma_{\delta}(T).$$

Proof. By Lemma 3 of [13], we have

$$\sigma_a(T) \subset \sigma_{\delta}(T).$$

Therefore,

$$\sigma(T) = \sigma_{\delta}(T).$$

We conclude with:

Corollary 2.4. The approximate defect spectrum $\sigma_{\delta}$ is continuous on the set of all log-hyponormal operators.

References


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