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ON SPECTRAL PROPERTIES OF LOG-HYPONORMAL OPERATORS

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Abstract

In this paper we consider spectral mapping theorem about two kinds of functional transformations for log-hyponormal operators and the continuity of the spectrum for log-hyponormal operators.

Introduction.

Let $\mathcal{H}$ be a complex Hilbert space and let $B(\mathcal{H})$ denote the set of all bounded linear operators on $\mathcal{H}$. For $A \in B(\mathcal{H})$, we denote the spectrum, the point spectrum, the residual spectrum and the approximate point spectrum of A by $\sigma(A)$, $\sigma_p(A)$, $\sigma_r(A)$ and $\sigma_a(A)$, respectively. For the study of spectral theory of operators, spectral mapping theorems are important. In this paper we consider spectral mapping theorems about two kinds of functional transformations for log-hyponormal operators. It is familiar that if A is normal then for every polynormial $f(\lambda, \lambda^*)$ one has $\sigma(f(A)) = f(\sigma(A)) = \{f(\lambda, \lambda^*); \lambda \in \sigma(A)\}$. In particular, we called the equality $\sigma(\text{Re }(A)) = \text{Re }(\sigma(A))$ with the polynomial $f(\lambda + \lambda^*) := \frac{1}{2}(\lambda + \lambda^*) = \text{Re }(\lambda)$ for any operator A the" projective" property.

The projective property for semi-normal operators was shown by C. Putnam [11] and the projective property for Toeplitz operators was shown by S. Berberian [2]. We will show the subprojective property for p-hyponormal or log-hyponormal operators. On the other hand, in [14], D Xia studied the following functional transformation $\varphi_{\{\xi,\psi\}}(T) = \xi(U)\psi(|T|)$ for a semi-hyponormal operator $T = U|T|$. And in [6], M. Itoh extended this result to p-hyponormal operators. Recently, M. Chō and B. P. Duggal [4] gave an elementary proof of

Itoh's result for invertible operator cases and generalized this result. We will extend this result for log-hyponormal operator.

On the other hand, in [8] it was shown that the spectrum σ is continuous on the set of p-hyponormal operators. We also show that this is still true for log-hyponormal operators.

An operator A is called *p-hyponormal* if $(A^*A)^p - (AA^*)^p \geq 0$ for some $p \in (0, \infty)$. If $p = 1$, A is *hyponormal* and if $p = \frac{1}{2}$, A is *semi-hyponormal*. By the consequence of Löwener's inequality [10] if A is p-hyponomal for some $p \in (0, \infty)$, then A is also q-hyponormal for every $q \in (0, p]$. Thus we assume, without loss of generality, that $p \in (0, \frac{1}{2})$. Let $\mathcal{H}(p)$ denote the class of p-hyponormal operators . An operator T is called *log-hyponormal* if T is invertible and satisfies log $(T^*T) \geq$ log (TT^*). Since log : $(0, \infty) \longrightarrow (-\infty, \infty)$ is monotone function, every invertible p-hyponormal operator is log-hyponormal. But there exists a log-hyponormal operator which is not p-hyponormal (cf. [12, Example 12]).

An operator $A \in B(\mathcal{H})$ has a unique polar decomposition $A = U|A|$, where $|A| = (AA^*)^{\frac{1}{2}}$ and U is a partial isometry with the initial space the closure of the range of $|A|$ and the final space the closure of the range of A. In particular, if $A = U|A|$ is log-hyponormal, then the operator U is unitary. Associated with A there is a related operator $\tilde{A} = |A|^{\frac{1}{2}}U|A|^{\frac{1}{2}}$, we call it the Aluthge transform of A. Aluthge transform has been used as a useful tool for study of p-hyponormal operators.

The followings are basic properties for $\tilde{A}$.

(i) If $A = U|A|$ be p-hyponormal $(0 < p < \frac{1}{2})$, then the operator $\tilde{A} = |A|^{\frac{1}{2}}U|A|^{\frac{1}{2}}$ is $(p + \frac{1}{2})$ hyponormal (cf. [1, Theorem 2]).

(ii) If $A \in B(\mathcal{H})$ be a log-hyponormal operator with a polar decomposition $A = U|A|$, then $\tilde{A} = |A|^{\frac{1}{2}}U|A|^{\frac{1}{2}}$ is semi-hyponormal (cf. [12, Theorem 4]).

Form the fact above, the second Aluthge transform of a p-hyponormal operator or log-hyponormal operator is hyponormal.

THEOREM A For every $A \in B(\mathcal{H})$ and its Aluthge transform $\tilde{T} = |A|^{\frac{1}{2}}U|A|^{\frac{1}{2}}$, it holds that

$$\omega(A) = \omega(\tilde{A})$$

where $\omega = \sigma, \sigma_a$ or σ_p.

Proof. It is known from [9, Theorem 1.3].

1. Functional transformations for log-hyponormal operators.

First, we will show the "subprojective" property for the spectra of p-hyponormal operators and log-hyponormal operators. For a operator T, a point z is in the normal approximate point spectrum $\sigma_{na}(T)$ of T if there exists a sequence $\{x_n\}$ of unit vectors such that

$$(T-z)x_n \to 0 \quad \text{and} \quad (T-z)^* x_n \to 0 \quad \text{as} \quad n \to \infty.$$

We begin with the following lemma. Proof is easy. So we omit it.

LEMMA 1.1. *If $T \in B(\mathcal{H})$ and $\sigma_a(T) = \sigma_{na}(T)$, then*

$$\text{Re } (\sigma(T)) \subset \sigma(\text{Re } T) \quad \text{and} \quad \text{Im } (\sigma(T)) \subset \sigma(\text{Im } T). \tag{1.1.1}$$

COROLLARY 1.2. *Let T be p-hyponormal or log-hyponormal. Then* (1.1.1) *holds.*

Proof. Since $\sigma_a(T) = \sigma_{na}(T)$ for a p-hyponormal or a log-hyponormal operator T. This follows from Lemma 1.1 □

THEOREM 1.3. *Let $T = U|T| = H + iK$ be p-hyponormal or log-hyponormal and $\hat{T}$ be the second Aluthge transform of T. Let $\hat{T} = \hat{H} + i\hat{K}$ be the Cartesian decomposition of $\hat{T}$. Then*

$$\sigma(\hat{H}) \subset \sigma(H) \quad \text{and} \quad \sigma(\hat{K}) \subset \sigma(K).$$

Proof. By Theorem A,

$$\sigma(T) = \sigma(\hat{T}) \Longrightarrow \text{Re } (\sigma(T)) = \text{Re } (\sigma(\hat{T})), \quad \text{Im } (\sigma(T)) = \text{Im } (\sigma(\hat{T})).$$

Since $\hat{T}$ is hyponormal, Re $(\sigma(\hat{T}) = \sigma(\text{Re } \hat{T})$ and Im $(\sigma(\hat{T}) = \sigma(\text{Im } \hat{T})$. Thus

$$\sigma(\text{Re } \hat{T}) \subset \sigma(\text{Re } T) \quad \text{and} \quad \sigma(\text{Im } \hat{T}) \subset \sigma(\text{Im } T).$$

□

COROLLARY 1.4. *Let T be log-hyponormal. If T has a compact real* (*imaginary*) *part, then T is normal.*

Proof. Since, by Theorem 1.3, $meas(\sigma(\hat{H})) = 0$, $\hat{T}$ is normal. And since T is normal if and only if $\hat{T}$ is normal. Thus T is normal.

□

Let E be a bouded closed subset of all real numbers $\mathbf{R}$, and $\mathrm{M(E)} = \{\psi : \psi$ is a bounded real Baire function on E$\}$. Let $\mathrm{M_0(E)} = \{\psi \in \mathrm{M(E)} : \psi(x) \geq 0$ for all $x \in \mathrm{E}$ and $\psi(0) = 0\}$. Let $\mathcal{J}(\mathrm{E}) = \{\psi : \psi$ is a strictly monotone increasing continuous function on E$\}$ and $\mathcal{J}_0(\mathrm{E}) = \mathrm{M_0(E)} \cap \mathcal{J}(\mathrm{E})$. Let $\mathcal{S}(\mathrm{E}) = \{\psi \in \mathrm{M(E)} : K_\psi \geq 0\}$, where K_ψ is the singular integral operator defined on $L^2(\mathrm{E})$ by

$$(K_\psi f)(x) = \mathrm{s} - \lim_{\epsilon \to 0+} \frac{1}{2\pi} \int_{\mathrm{E}} \frac{\psi(x) - \psi(y)}{x - (y + i\epsilon)} f(y) dy.$$

If E is a closed subset of the unit circle $\mathbf{T}$, let $\mathrm{M_0(E)} = \{\xi : \xi$ is a complex Baire function on $\mathrm{E} \to \mathbf{T}\}$, $\mathcal{J}_0(\mathrm{E}) = \{\xi : \xi$ is a direction preserving homomorphism on E$\}$ and $\mathcal{S}_0(\mathrm{E}) = \{\xi : \xi \in \mathrm{M_0(E)}$ and $K_\xi \geq 0\}$, where K_ξ is the singular integral operator defined on $L^2(\mathrm{E})$ by

$$(K_\xi f)(e^{i\theta}) = \mathrm{s} - \lim_{\epsilon \to 0+} \frac{1}{2\pi} \int_{\mathrm{E}} \frac{1 - \xi(e^{i\theta})\overline{\xi(e^{i\eta})}}{1 - e^{i\theta} e^{-i\eta}(1-\epsilon)} f(e^{i\eta}) d\eta.$$

For functions f and g, we denote the functional transformation $F_{[f,g]}(T) = f(U)\exp(g(\log|T|))$ for a log-hyponormal operator $T = U|T|$ and $F_{[f,g]}(re^{i\theta}) = f(e^{i\theta})\exp(g(\log r))$ in the complex plane.

LEMMA 1.5. *Let $T \in B(\mathcal{H})$ be a semi-hyponormal operator with operator decomposition $T = U|T|$. Then $Ue^{|T|}$ is log-hyponormal and*

$$\sigma_a(Ue^{|T|}) = \{e^r e^{i\theta} : re^{i\theta} \in \sigma_a(T)\};$$

$$\sigma_r(Ue^{|T|}) = \{e^r e^{i\theta} : re^{i\theta} \in \sigma_r(T)\};$$

$$\sigma(Ue^{|T|}) = \{e^r e^{i\theta} : re^{i\theta} \in \sigma(T)\}.$$

Proof. Proof is from [13, Lemmas 5 and 6].

□

THEOREM 1.6. *Let $T = U|T|$ be log-hyponormal and $\log|T| \geq 0$. Suppose that $f \in \mathcal{J}_0(\sigma(U)) \cap \mathcal{S}_0(\sigma(U))$ and $g \in \mathcal{J}_0(\sigma(\log|T|)) \cap \mathcal{S}_0(\sigma(\log|T|))$ if $\sigma(U) \neq \mathbf{T}$ and $g \in \mathcal{J}_0([0, \|\log|T|\|]) \cap \mathcal{S}_0([0, \|\log|T|\|])$ if $\sigma(U) = \mathbf{T}$. Then $F_{[f,g]}(T)$ is log-hyponormal and $F_{[f,g]}(\sigma_w(T)) = \sigma_w(F_{[f,g]}(T))$, where $\sigma_w = \sigma, \sigma_a$ or σ_r.*

Proof. Let $T = U|T|$ be log-hyponormal, then $S = U\log|T|$ is semi-hyponormal and $\sigma_w(S) = \{(\log r)e^{i\theta} : re^{i\theta} \in \sigma_w(T)\}$. From Theorem VI, 3.1 of [14], $f(U)g(\log|T|)$ is also semi-hyponormal. Thus $\sigma_w(f(U)g(\log|T|)) = \{f(e^{i\theta})g(\log r) : (\log r)e^{i\theta} \in \sigma_w(U\log|T|)\}$. Moreover, from Lemma 1.5 we can see that

$$F_{[f,g]}(T) = f(U)\exp(g(\log|T|))$$

is log-hyponormal. Thus

$$\begin{aligned} \sigma_w(F_{[f,g]}(T)) &= \sigma_w(f(U)\exp(g(\log|T|)) \\ &= \{e^{g(\log r)}f(e^{i\theta}) : f(e^{i\theta})g(\log r) \in \sigma_w(f(U)g(\log|T|), \\ &\quad (\log r)e^{i\theta} \in \sigma_w(U\log|T|)\} \\ &= \{e^{g(\log r)}f(e^{i\theta}) : (\log r)e^{i\theta} \in \sigma_w(U\log|T|), re^{i\theta} \in \sigma_w(T)\} \\ &= \{e^{g(\log r)}f(e^{i\theta}) : re^{i\theta} \in \sigma_w(T)\} \\ &= F_{[f,g]}(\sigma_w(T)). \end{aligned}$$

□

2. Continuity of σ on the set of all log-hyponormal operators.

In [8], it was shown that the spectrum σ is continuous on the set of all p-hyponormal operators. In this section we show that this is still true for log-hyponormal operators. To do this we recall that $T \in B(\mathcal{H})$ is said to be *bounded below* if there exists $k > 0$ for which $\|x\| \le k\|Tx\|$ for each $x \in \mathcal{H}$. For $A \in B(\mathcal{H})$, $\gamma(A)$ denote the *reduced minimum modulus*,$\gamma(A) = inf_{x\in\mathcal{H}}\frac{\|Ax\|}{dist(x,KerA)}$, where $\frac{0}{0}$ is defined to be ∞. Before proving the main theorem we establish the following :

LEMMA 2.1. *Let $T = U|T|$ and $T_n = U_n|T_n| \in B(\mathcal{H})$ for $n \in Z^+$. If T is bounded below and T_n converges to T, then U_n converges to U.*

Proof. Since T is bounded below, we have that if $\gamma(\cdot)$ denote the reduced minimum modulus, then $\gamma(T) = \alpha > 0$ and T is a continuity point of γ (cf. [7, Theorem 4.3]). Hence, without loss of generality, we may assume that $\gamma(T_n) > \varepsilon/2$ for all n. Since the set of bounded below operators is an open set, it follows that for sufficiently large n, T_n's are bounded below and hence $|T|$ and $|T_n|$ are invertible (cf. [5, Theorem 8.6.4]). Let $y \in \mathcal{H}$ and $\|y\| = 1$. Then there exist x and x_n in $\mathcal{H}$ ($n \in Z^+$) such that $y = |T|x$ and $y = |T_n|x_n$. Since $\gamma(S)$ is the supremum of all real number γ such that $\gamma\|x\| \le \|Sx\|$, we have

$$\|x\| \le \frac{1}{\gamma(|T|)} \parallel |T|x \parallel = \frac{1}{\gamma(T)}\|y\| = \frac{1}{\gamma(T)} < 2/\alpha.$$

Similarly, $\|x_n\| < 2/\alpha$ for all $n \in Z^+$. Therefore

$$\|U_n y - Uy\| = \|U_n|T_n|x_n - U|T|x\| \leq \|U_n|T_n|x_n - U_n|T_n|x\| + \|U_n|T_n|x - U|T|x\|.$$

But

$$\|U_n|T_n|x - U|T|x\| \leq \|T_n - T\|\|x\| < \frac{2\|T_n - T\|}{\alpha} \longrightarrow 0 \quad \text{as} \quad n \to \infty.$$

We now claim that $\|x_n - x\| \to 0$ as $n \to \infty$. If it is not so, then there exist $\delta > 0$ and a sequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $\|x_{n_k} - x\| > \delta$ for all k. Hence

$$\||T|(x_{n_k} - x)\| = \||T|x_{n_k} - |T_{n_k}|x_{n_k}\| \leq \||T| - |T_{n_k}|\|\|x_{n_k}\| < \frac{2}{\alpha}\||T| - |T_{n_k}|\| \to 0$$

as $n \to \infty$. This implies that $|T|$ is not bounded below. It is a contradiction. Therefore, we have

$$\|U_n|T_n|x_n - U_n|T_n|x\| \leq \|T_n\|\|x_n - x\| \to 0 \quad \text{as} \quad n \to \infty.$$

□

Now we have :

THEOREM 2.2. *The spectrum σ is continuous on the set of all log-hyponormal operators.*

Proof. Suppose that $T = U|T|$ and $T_n = U_n|T_n|$ for $n \in Z^+$ are log-hyponormal operators such that T_n converges to T. Since T is invertible it follows from Lemma 2.1 that U_n converges to U, so that

$$\tilde{T}_n = |T_n|^{\frac{1}{2}} U_n |T_n|^{\frac{1}{2}} \longrightarrow \tilde{T} = |T|^{\frac{1}{2}} U |T|^{\frac{1}{2}} \quad \text{as} \quad n \to \infty.$$

Since $\tilde{T} = |T|^{\frac{1}{2}} U |T|^{\frac{1}{2}}$ is semi-hyponormal and the spectrum is continuous on the set of all p-hyponormal operators, we have

$$\sigma(T_n) = \sigma(\tilde{T}_n) \longrightarrow \sigma(\tilde{T}) = \sigma(T).$$

□

For an operator $A \in B(\mathcal{H})$, z is in the approximate defect spectrum $\sigma_\delta(A)$ if there exists a sequence $\{x_n\}$ of unit vectors in $\mathcal{H}$ such that $\lim_{n\to\infty} \|(A-z)^* x_n\| = 0$. Then we have

THEOREM 2.3. *Let T be a log-hyponormal operator. Then*

$$\sigma(T) = \sigma_\delta(T).$$

Proof. By Lemma 3 of [13], we have

$$\sigma_a(T) \subset \sigma_\delta(T).$$

Therefore,

$$\sigma(T) = \sigma_\delta(T).$$

□

We conclude with :

COROLLARY 2.4. *The approximate defect spectrum σ_δ is continuous on the set of all log-hyponormal operators.*

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