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AN APPLICATION OF GRAND FURUTA INEQUALITY TO KANTOROVICH TYPE INEQUALITIES

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ABSTRACT. As an application of the grand Furuta inequality, we shall show characterizations of usual order and chaotic order associated with operator equation and Kantorovich type order preserving operator inequalities by using essentially the same idea of T.Furuta. Also, we present a Kantorovich type inequality which is a parallel result with Yamazaki and Yanagida's one.

1. Introduction. This note is based on a joint work [15] with T.Furuta and [17].

In what follows, a capital letter means a bounded linear operator on a complex Hilbert space H. An operator T is said to be positive (in symbol: $T \geq 0$) if $(Tx, x) \geq 0$ for all $x \in H$. Also an operator T is strictly positive (in symbol: $T > 0$) if T is positive and invertible. We recall the celebrated Kantorovich inequality: If a positive operator A on a Hilbert space H satisfies $M \geq A \geq m > 0$, then

$$(A^{-1}x, x) \leq \frac{(M+m)^2}{4Mm}(Ax, x)^{-1}$$

for every unit vector $x \in H$. The number $\frac{(M+m)^2}{4Mm}$ is called the Kantorovich constant.

The Löwner-Heinz theorem asserts that $A \geq B \geq 0$ ensures $A^p \geq B^p$ $(0 \leq p \leq 1)$. However $A \geq B$ does not always ensure $A^2 \geq B^2$ in general. As an application of the Kantorovich inequality, Fujii, Izumino, Nakamoto and the author [5] showed that t^2 is order preserving in the following sense: If $A \geq B \geq 0$ and $M \geq A \geq m > 0$, then

$$\frac{(M+m)^2}{4Mm}A^2 \geq B^2.$$

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Related to this, Furuta [13] showed the following order preserving operator inequality:

Theorem A. *If* $A \geq B \geq 0$ *and* $M \geq A \geq m > 0$, *then*

$$\left(\frac{M}{m}\right)^{p-1} A^p \geq K_+(m, M, p) A^p \geq B^p \qquad \textit{holds for all } p \geq 1,$$

where

$$K_+(m, M, p) = \frac{(p-1)^{p-1}}{p^p} \frac{(M^p - m^p)^p}{(M-m)(mM^p - Mm^p)^{p-1}}.$$

The order between positive invertible operators A and B defined by $\log A \geq \log B$ is said to be chaotic order $A \gg B$ in [4] which is a weaker order than usual order $A \geq B$. In [22], Yamazaki and Yanagida showed the following chaotic order version of Theorem A:

Theorem B. *If* $\log A \geq \log B$ *and* $M \geq A \geq m > 0$, *then*

$$\left(\frac{M}{m}\right)^{p} A^p \geq K_+(m, M, p+1) A^p \geq B^p \qquad \textit{holds for all } p > 0.$$

In fact, $\log A \geq \log B$ does not always ensure $A \geq B$ in general. However, by Theorem B, it follows that $\log A \geq \log B$ implies $\frac{(M+m)^2}{4Mm} A \geq B$.

Moreover, Yamazaki and Yanagida gave a new characterization of chaotic order by means of the Kantorovich constant.

Theorem C. *Let* A *and* B *be invertible positive operators and* $M \geq A \geq m > 0$. *Then the following properties are mutually equivalent:*

(I) $$A \gg B \qquad (i.e., \log A \geq \log B).$$

(II) $$\frac{(M^p + m^p)^2}{4M^p m^p} A^p \geq B^p \qquad \textit{holds for all } p \geq 0.$$

In this paper, as an application of the grand Furuta inequality, we shall show characterizations of usual order and chaotic order associated with operator equation and Kantorovich type order preserving operator inequalities which interpolates Theorem A and Theorem B by using essentially the same idea of [12]. Also, we present a Kantorovich type inequality which is a parallel result with Theorem C.

2. Kantorovich type operator inequalities. Firstly we shall show the following characterization of chaotic order associated with operator equation.

Theorem 1. *Let A and B be invertible positive operators. Then the following properties are mutually equivalent:*

(I) $A \gg B$ (i.e., $\log A \geq \log B$).

(II) *For each $\alpha \in [0,1]$, $p \geq 0$ and $u \geq 0$, there exists the unique invertible positive contraction T such that*

$$(A^{\frac{\alpha u}{2}} B^p A^{\frac{\alpha u}{2}})^s = TA^{(p+\alpha u)s}T$$

holds for any $s \geq 1$ and $(p+\alpha u)s \geq (1-\alpha)u$.

(III) *For each $\alpha \in [0,1]$ and $p \geq u \geq 0$, there exists the unique invertible positive contraction T such that*

$$(A^{\frac{\alpha u}{2}} B^p A^{\frac{\alpha u}{2}})^s = TA^{(p+\alpha u)s}T$$

holds for any $s \geq 1$.

(IV) *For each $p \geq 0$, there exists the unique invertible positive contraction T such that*

$$B^p = TA^pT.$$

As an application of Theorem 1, we obtain the following extension of Theorem C on a Kantorovich type characterization of chaotic order.

Theorem 2. *Let A and B be invertible positive operators and $M \geq A \geq m > 0$. Then the following properties are mutually equivalent:*

(I) $A \gg B$ (i.e., $\log A \geq \log B$).

(II) *For each $\alpha \in [0,1]$, $p \geq 0$ and $u \geq 0$,*

$$\frac{(M^{(p+\alpha u)s} + m^{(p+\alpha u)s})^2}{4M^{(p+\alpha u)s} m^{(p+\alpha u)s}} A^{(p+\alpha u)s} \geq \left(A^{\frac{\alpha u}{2}} B^p A^{\frac{\alpha u}{2}}\right)^s$$

holds for any $s \geq 1$ and $(p+\alpha u)s \geq (1-\alpha)u$.

(III) *For each* $\alpha \in [0,1]$ *and* $p \geq u \geq 0$,

$$\frac{(M^{(p+\alpha u)s} + m^{(p+\alpha u)s})^2}{4M^{(p+\alpha u)s} m^{(p+\alpha u)s}} A^{(p+\alpha u)s} \geq \left(A^{\frac{\alpha u}{2}} B^p A^{\frac{\alpha u}{2}}\right)^s$$

holds for any $s \geq 1$.

(IV) $$\frac{(M^p + m^p)^2}{4M^p m^p} A^p \geq B^p \qquad \text{holds for all } p \geq 0.$$

Next, we shall show the following characterizations of usual order associated with operator equation.

Theorem 3. *Let* A *and* B *be positive invertible operators. Then the following assertions are mutually equivalent:*

(I) $A \geq B$.

(II) *For each* $t \in [0,1]$, $p \geq 1$ *and* $s \geq 1$ *such that* $(p-t)s \geq t$, *there exists a unique invertible positive contraction* T *such that*

$$TA^{(p-t)s}T = \left(A^{-t/2} B^p A^{-t/2}\right)^s.$$

(III) *For all* $p \geq 2$, *there exists a unique invertible positive contraction* T *such that*

$$TA^{p-1}T = A^{-1/2} B^p A^{-1/2}.$$

As an application of Theorem 3, we obtain the following Kantorovich type order preserving operator inequality:

Theorem 4. *Let* A *and* B *be positive and invertible operators on a Hilbert space* H *satisfying* $M \geq A \geq m > 0$. *Then the following assertions are mutually equivalent:*

(I) $A \geq B$.

(II) *For each* $t \in [0,1]$,

$$\frac{(M^{(p-t)s} + m^{(p-t)s})^2}{4M^{(p-t)s} m^{(p-t)s}} A^{(p-t)s} \geq \left(A^{-\frac{t}{2}} B^p A^{-\frac{t}{2}}\right)^s$$

holds for any $p \geq 1$ *and* $s \geq 1$ *such that* $(p-t)s \geq t$.

(III)
$$\left(\frac{(M^{(p-1)s}+m^{(p-1)s})^2}{4M^{(p-1)s}m^{(p-1)s}}\right)^{\frac{1}{s}} A^p \geq B^p$$
holds for any $s \geq 1$ *and* $p \geq \frac{1}{s}+1$.

(IV) $\left(\frac{M}{m}\right)^{p-1} A^p \geq B^p$ *holds for all* $p \geq 1$.

By Theorem 4, we have the following corollary which is a parallel result with Theorem C associated with usual order.

Corollary 5. *If* $A \geq B \geq 0$ *and* $M \geq A \geq m > 0$, *then*
$$\frac{(M^{p-1}+m^{p-1})^2}{4m^{p-1}M^{p-1}} A^p \geq B^p \qquad \text{holds for all } p \geq 2.$$

Let A and B be positive invertible operators on a Hilbert space H. We consider an order $A^\delta \geq B^\delta$ for $\delta \in (0,1]$ which interpolates usual order $A \geq B$ and chaotic order $A \gg B$ continuously. The following theorem is easily obtained by Theorem 4.

Theorem 6. *Let* A *and* B *be positive and invertible operators on a Hilbert space* H *satisfying* $A^\delta \geq B^\delta$ *for* $\delta \in (0,1]$ *and* $M \geq A \geq m > 0$, *then*
$$\left(\frac{(M^{(p-\delta)s}+m^{(p-\delta)s})^2}{4m^{(p-\delta)s}M^{(p-\delta)s}}\right)^{\frac{1}{s}} A^p \geq B^p \qquad \text{holds for all } s \geq 1 \text{ and } p \geq (\frac{1}{s}+1)\delta.$$

Remark 1. Theorem 6 interpolates Theorem A and Theorem B by means of the Kantorovich constant. Let A and B be positive invertible operators and $M \geq A \geq m > 0$. Then the following assertions holds:

(i) $A \geq B$ implies $\left(\frac{M}{m}\right)^{p-1} A^p \geq B^p$ for all $p \geq 1$.

(ii) $A^\delta \geq B^\delta$ implies $\left(\frac{(M^{(p-\delta)s}+m^{(p-\delta)s})^2}{4m^{(p-\delta)s}M^{(p-\delta)s}}\right)^{\frac{1}{s}} A^p \geq B^p$ for all $s \geq 1$ and $p \geq (\frac{1}{s}+1)\delta$.

(iii) $\log A \geq \log B$ implies $\left(\frac{M}{m}\right)^p A^p \geq B^p$ for all $p > 0$.

It follows that the Kantorovich constant of (ii) interpolates the scalar of (i) and (iii) continuously. In fact, if we put $\delta = 1$ and $s \to +\infty$ in (ii), then we have (*i*), also if we put $\delta \to 0$ and $s \to +\infty$ in (ii), then we have (iii).

Moreover, Theorem 6 interpolates Theorem C and Corollary 5 by means of the Kantorovich constant:

(i) $A \geq B$ implies $\dfrac{(M^{p-1}+m^{p-1})^2}{4m^{p-1}M^{p-1}}A^p \geq B^p$ for all $p \geq 2$.

(ii) $A^\delta \geq B^\delta$ imples $\left(\dfrac{(M^{(p-\delta)s}+m^{(p-\delta)s})^2}{4m^{(p-\delta)s}M^{(p-\delta)s}}\right)^{\frac{1}{s}} A^p \geq B^p$ for all $s \geq 1$ and $p \geq (\frac{1}{s}+1)\delta$.

(iii) $\log A \geq \log B$ implies $\dfrac{(M^p+m^p)^2}{4m^pM^p}A^p \geq B^p$ for all $p > 0$.

The Kantorovich constant of (ii) interpolates the scalar of (i) and (iii). In fact, if we put $\delta = 1$ and $s = 1$ in (ii), then we have (i), also if we put $s = 1$ and $\delta \to 0$ in (ii), then we have (iii).

3. Proof of the results. Related to the extension of the Löwner-Heinz theorem, Furuta established the following ingenious order preserving operator inequality which is called the Furuta inequality.

Theorem F (Furuta inequality)([8]). If $A \geq B \geq 0$, then for each $r \geq 0$,

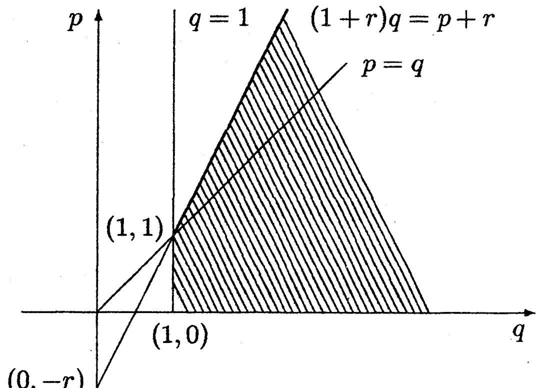
(i) $\left(B^{\frac{r}{2}}A^pB^{\frac{r}{2}}\right)^{\frac{1}{q}} \geq \left(B^{\frac{r}{2}}B^pB^{\frac{r}{2}}\right)^{\frac{1}{q}}$

and

(ii) $\left(A^{\frac{r}{2}}A^pA^{\frac{r}{2}}\right)^{\frac{1}{q}} \geq \left(A^{\frac{r}{2}}B^pA^{\frac{r}{2}}\right)^{\frac{1}{q}}$

hold for $p \geq 0$ and $q \geq 1$ with

$$(1+r)q \geq p+r.$$



Figure

Alternative proofs of Theorem F have been given in [3], [16], and one-page proof in [9]. The domain drawn for p, q and r in Figure is the best possible one [18] for Theorem F.

As a corollary of [11, Theorem 1.1], Furuta established the following grand Furuta inequality which interpolates Theorem F itself and an inequality equivalent to main theorem of log majorization by Ando-Hiai [2].

Theorem G (The grand Furuta inequality) ([11]). *If* $A \geq B \geq 0$ *and* A *is* *invertible, then for each* $t \in [0,1]$,

$$\{A^{\frac{r}{2}}(A^{-\frac{t}{2}}A^pA^{-\frac{t}{2}})^sA^{\frac{r}{2}}\}^{\frac{1}{q}} \geq \{A^{\frac{r}{2}}(A^{-\frac{t}{2}}B^pA^{-\frac{t}{2}})^sA^{\frac{r}{2}}\}^{\frac{1}{q}}$$

holds for any $s \geq 0$, $p \geq 0$, $q \geq 1$ *and* $r \geq t$ *with* $(s-1)(p-1) \geq 0$ *and* $(1-t+r)q \geq (p-t)s+r$.

An alternative proof of Theorem G in [6] and one-page proof in [14] and the best possibility of Theorem G is shown in [19], and two very simple proofs of the best possibility of Theorem G are in [21] and [7].

We need the following lemmas in order to give proofs of our results.

Lemma 7. *Let* T *be a nonsingular positive operator. If* $XTX = YTY$ *holds for some* $X \geq 0$ *and* $Y \geq 0$, *then* $X = Y$.

Proof. If $XTX = YTY$ holds for some $X, Y \geq 0$, then we have $(T^{\frac{1}{2}}XT^{\frac{1}{2}})^2 = (T^{\frac{1}{2}}YT^{\frac{1}{2}})^2$, so that $T^{\frac{1}{2}}XT^{\frac{1}{2}} = T^{\frac{1}{2}}YT^{\frac{1}{2}}$ holds and the nonsingularity of T ensures $X = Y$.

Lemma 8. *If* A *is a positive operator such that* $M \geq A \geq m > 0$ *and* B *is a positive contraction, then*

$$\frac{(M+m)^2}{4Mm}A \geq BAB.$$

Proof. By the Kantorovich inequality, we have $(ABx, Bx)(A^{-1}Bx, Bx) \leq K\|Bx\|^4$ for any unit vector $x \in H$, where $K = \frac{(M+m)^2}{4Mm}$. Hence it follows that

$$\begin{aligned}
(ABx, Bx)(A^{-1}Bx, Bx) &\leq K(B^2x, x)^2 \\
&\leq K(Bx, x)^2 \qquad \text{by } I \geq B \geq 0 \\
&= K(A^{-\frac{1}{2}}Bx, A^{\frac{1}{2}}x)^2 \\
&\leq K(A^{-1}Bx, Bx)(Ax, x),
\end{aligned}$$

so the proof is complete.

Remark 2. (1) In Lemma 8, one might conjecture the following $(*)$

(*) $\quad A \geq BAB$ holds for any positive operator A and any positive contraction B

instead of $\frac{(M+m)^2}{4Mm}A \geq BAB$. But we can give a counterexample to this conjecture as follows. Take A and B as follows:

$$A = \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix} \quad \text{and} \quad B = \frac{1}{2}\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}.$$

Then $A \geq 0$ and $I \geq B \geq 0$, but we have

$$A - BAB = \begin{pmatrix} \frac{5}{4} & -\frac{3}{4} \\ -\frac{3}{4} & \frac{1}{4} \end{pmatrix} \not\geq 0.$$

(2) Moreover, one might conjecture the following (**)

(**) $\frac{(M+m)^2}{4Mm}A \geq B^*AB$ holds for any positive operator A and any contraction B instead of the positive contractivity of B. But we can give a counterexample to this conjecture as follows. Take A and B as follows:

$$A = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}.$$

Then $A \geq 0$ and $I \geq B^*B$, but we have

$$\frac{(M+m)^2}{4Mm}A - B^*AB = \begin{pmatrix} \frac{9}{2} & \frac{9}{4} \\ \frac{9}{4} & \frac{1}{4} \end{pmatrix} \not\geq 0.$$

The following characterization of chaotic order is shown in [4] and [10].

Theorem D. *Let A and B be invertible positive operators. Then the following properties are mutually equivalent:*

(I) $\quad A \gg B \quad$ (i.e., $\log A \geq \log B$).

(II) $\quad A^p \geq \left(A^{\frac{p}{2}}B^pA^{\frac{p}{2}}\right)^{\frac{1}{2}} \quad$ *holds for all $p \geq 0$.*

(III) $\quad A^u \geq \left(A^{\frac{u}{2}}B^pA^{\frac{u}{2}}\right)^{\frac{u}{p+u}} \quad$ *holds for all $p \geq 0$ and $u \geq 0$.*

(I) $\Longleftrightarrow$(II) is shown in [1]. Recently a simple and excellent proof of (I) $\Longrightarrow$(III) is shown in [20] by only applying Theorem F. Here we cite the following simplifed implication since (III) $\Longrightarrow$(II) is trivial.

Simplified proof of (II) $\Longrightarrow$(I) of Theorem D. (II) yields

$$\frac{A^p-1}{p} \geq \frac{(A^{\frac{p}{2}}B^pA^{\frac{p}{2}})^{\frac{1}{2}}-I}{p}$$
$$= \left(\frac{A^{\frac{p}{2}}(B^p-I)A^{\frac{p}{2}}}{p}+\frac{A^p-I}{p}\right)\{(A^{\frac{p}{2}}B^pA^{\frac{p}{2}})^{\frac{1}{2}}+I\}^{-1}$$

and tending $p \downarrow 0$, so we have $\log A \geq \frac{1}{2}(\log B + \log A)$, that is , $\log A \geq \log B$.

Lemma 9. *If $M > m > 0$, then*

$$\lim_{s\to+\infty}\left(\frac{(M^s+m^s)^2}{4m^sM^s}\right)^{\frac{1}{s}} = \frac{M}{m}.$$

Proof. Put $x = \frac{M}{m} > 1$, then it follows from L'Hospital's theorem that

$$\lim_{s\to+\infty}\frac{\log(1+x^s)^2}{s} = \lim_{s\to+\infty}\frac{2x^s\log x}{1+x^s} = \log x^2.$$

Therefore we have

$$\lim_{s\to+\infty}\left(\frac{(M^s+m^s)^2}{4m^sM^s}\right)^{\frac{1}{s}} = \lim_{s\to+\infty}\left(\frac{(1+x^s)^2}{4x^s}\right)^{\frac{1}{s}} = \lim_{s\to+\infty}\left(\frac{(1+x^s)^{\frac{2}{s}}}{4^{1/s}x}\right) = x = \frac{M}{m}.$$

Now, we start with the proofs of our theorems.

Proof of Theorem 1.

(I) $\Longrightarrow$(II). For each $p \geq 0$ and $u \geq 0$, put $A_1 = A^u$ and $B_1 = (A^{\frac{u}{2}}B^pA^{\frac{u}{2}})^{\frac{u}{p+u}}$ in (III) of Theorem D. Then we have $A_1 \geq B_1 \geq 0$. By Theorem A, it follows that for each $t \in [0,1]$,

$$A_1^{\frac{(p_1-t)s+r}{q}} \geq \{A_1^{\frac{r}{2}}(A_1^{-\frac{t}{2}}B_1^{p_1}A_1^{-\frac{t}{2}})^sA_1^{\frac{r}{2}}\}^{\frac{1}{q}} \tag{1}$$

holds for any $s \geq 1$, $p_1 \geq 1$, $q \geq 1$, and the following conditions (2) and (3)

$$r \geq t, \tag{2}$$

$$(1-t+r)q \geq (p_1-t)s+r. \tag{3}$$

Put $p_1 = \frac{p+u}{u} \geq 1$ in case $u > 0$, $q = 2$, $r = (p_1-t)s$ and also put $\alpha = 1-t$ in (2) and (3). Then (3) is satisfied, so the only required condition (2) is equivalent to the following

$$(p+\alpha u)s \geq (1-\alpha)u. \tag{4}$$

Therefore, (1) implies that for each $\alpha \in [0,1]$, $p \geq 0$ and $u \geq 0$,

$$I \geq A^{-\frac{(p+\alpha u)s}{2}}\{A^{\frac{(p+\alpha u)s}{2}}(A^{\frac{\alpha u}{2}}B^pA^{\frac{\alpha u}{2}})^sA^{\frac{(p+\alpha u)s}{2}}\}^{\frac{1}{2}}A^{-\frac{(p+\alpha u)s}{2}} \tag{5}$$

holds for $s \geq 1$ and the condition (4). Let T be defined by the right hand side of (5). Then it turns out that T is an invertible positive contraction by (5), so that we have

$$A^{\frac{(p+\alpha u)s}{2}}TA^{\frac{(p+\alpha u)s}{2}} = \{A^{\frac{(p+\alpha u)s}{2}}(A^{\frac{\alpha u}{2}}B^pA^{\frac{\alpha u}{2}})^sA^{\frac{(p+\alpha u)s}{2}}\}^{\frac{1}{2}}. \tag{6}$$

Taking square both sides of (6), we obtain

$$A^{\frac{(p+\alpha u)s}{2}}TA^{(p+\alpha u)s}TA^{\frac{(p+\alpha u)s}{2}} = A^{\frac{(p+\alpha u)s}{2}}(A^{\frac{\alpha u}{2}}B^pA^{\frac{\alpha u}{2}})^sA^{\frac{(p+\alpha u)s}{2}}.$$

That is, we have the following equation

$$TA^{(p+\alpha u)s}T = \left(A^{\frac{\alpha u}{2}}B^pA^{\frac{\alpha u}{2}}\right)^s \tag{7}$$

holds for $s \geq 1$ and $(p+\alpha u)s \geq (1-\alpha)u$ in case $u > 0$. Next we check (7) in case $u = 0$. In fact (II) of Theorem D ensures $I \geq T = A^{\frac{-p}{2}}\left(A^{\frac{p}{2}}B^pA^{\frac{p}{2}}\right)^{\frac{1}{2}}A^{\frac{-p}{2}}$ for all $p \geq 0$, so $TA^{ps}T = B^{ps}$ holds for $p \geq 0$, $s \geq 1$ and this equation is just (7) in case $u = 0$. The uniqueness of T in (7) follows by Lemma 7.

(II)$\Longrightarrow$(III). Put $p \geq u \geq 0$ in (II). Then the required condition $(p+\alpha u)s \geq (1-\alpha)u$ is satisfied, so we have (III).

(III)$\Longrightarrow$(IV). Put $u = 0$ or $\alpha = 0$ and $s = 1$ in (III).

(IV)$\Longrightarrow$(I). Assume (IV). Then we have

$$\left(A^{\frac{p}{2}}TA^{\frac{p}{2}}\right)^2 = A^{\frac{p}{2}}TA^pTA^{\frac{p}{2}} = A^{\frac{p}{2}}B^pA^{\frac{p}{2}} \qquad \text{by (IV).}$$

By raising each sides to power $\frac{1}{2}$, it follows from Löwner-Heinz inequality that

$$A^p \geq A^{\frac{p}{2}}TA^{\frac{p}{2}} \geq (A^{\frac{p}{2}}B^pA^{\frac{p}{2}})^{\frac{1}{2}}, \tag{8}$$

and the first inequality holds since $I \geq T \geq 0$ and we have (I) by Thereom D.

Whence the proof of Theorem 1 is complete.

Proof of Theorem 2.

(I)$\Longrightarrow$(II). The hypothesis $M \geq A \geq m > 0$ ensures $M^{(p+\alpha u)s} \geq A^{(p+\alpha u)s} \geq m^{(p+\alpha u)s} > 0$ for the hypothesis on α, p, u and s, so the proof is complete by (II) of Theorem 1 and Lemma 8.

(II)$\Longrightarrow$(III). Put $p \geq u \geq 0$ in (II). Then the required condition $(p+\alpha u)s \geq (1-\alpha)u$ is satisfied, so we have (III).

(III)$\Longrightarrow$(IV). We have only to put $u = 0$ or $\alpha = 0$ and $s = 1$ in (III).

(IV)$\Longrightarrow$(I) is shown by Theorem C.

Whence the proof of Theorem 2 is complete.

Proof of Theorem 3.

(I) $\Longrightarrow$(II). Since $A \geq B \geq 0$ and $A > 0$, if we put $q = 2$ in the grand Furuta inequality, then for $p \geq 1$, $s \geq 1$ and $t \in (0,1]$

$$A^{\frac{(p-t)s+r}{2}} \geq \{A^{\frac{r}{2}}\left(A^{-\frac{t}{2}}B^pA^{-\frac{t}{2}}\right)^s A^{\frac{r}{2}}\}^{\frac{1}{2}} \tag{9}$$

holds under the following conditions (10) and (11)

$$r \geq t, \tag{10}$$

$$2(1-t+r) \geq (p-t)s+r. \tag{11}$$

If we moreover put $r = (p-t)s$, then (11) is satisfied and (10) is equivalent to the following

$$(p-t)s \geq t. \tag{12}$$

Therefore, (9) implies that for $t \in (0,1]$, $p \geq 1$ and $s \geq 1$

$$I \geq A^{\frac{-(p-t)s}{2}}\{A^{\frac{(p-t)s}{2}}\left(A^{-\frac{t}{2}}B^pA^{-\frac{t}{2}}\right)^s A^{\frac{(p-t)s}{2}}\}^{\frac{1}{2}}A^{\frac{-(p-t)s}{2}} \tag{13}$$

holds for the condition (12). Let T be defined by the right hand side of (13). Then it turns out that T is an invertible positive contraction by (13), so that we have

$$A^{\frac{(p-t)s}{2}}TA^{\frac{(p-t)s}{2}} = \{A^{\frac{r}{2}}\left(A^{-\frac{t}{2}}B^pA^{-\frac{t}{2}}\right)^s A^{\frac{r}{2}}\}^{\frac{1}{2}}.$$

Taking square both sides, we obtain

$$A^{\frac{(p-t)s}{2}}TA^{(p-t)s}TA^{\frac{(p-t)s}{2}} = A^{\frac{(p-t)s}{2}}\left(A^{-\frac{t}{2}}B^{p}A^{-\frac{t}{2}}\right)^{s}A^{\frac{(p-t)s}{2}}.$$

That is, we have the following equation

$$TA^{(p-t)s}T = \left(A^{-t/2}B^{p}A^{-t/2}\right)^{s}.$$

(II) $\Longrightarrow$(III). Put $t = 1$ and $s = 1$ in (II).

(III) $\Longrightarrow$(I). If we put $p = 2$ in (III), then we have

$$TAT = A^{-1/2}B^{2}A^{-1/2},$$

so that it follows that

$$(A^{1/2}TA^{1/2})^{2} = A^{1/2}TATA^{1/2} = B^{2}.$$

By raising each sides to power $\frac{1}{2}$, it follows that

$$A \geq A^{1/2}TA^{1/2} = B,$$

and the first inequality holds since $I \geq T \geq 0$.

Whence the proof of Theorem 3 is complete.

Proof of Theorem 4.

(I) $\Longrightarrow$(II). The hypothesis $M \geq A \geq m > 0$ ensures $M^{(p-t)s} \geq A^{(p-t)s} \geq m^{(p-t)s} > 0$ for the hypothesis on t, p and s, so the proof is complete by (II) of Theorem 3 and Lemma 8.

(II) $\Longrightarrow$(III). If we put $t = 1$ in (II), then we have (III) by the Löwner-Heinz theorem.

(III) $\Longrightarrow$(IV). If we put $s \to \infty$, then we have (IV) by Lemma 9.

(IV) $\Longrightarrow$(I). If we put $p = 1$, then we have (I).

Proof of Corollary 5. Put $s = 1$ in (III) of Theorem 4.

Proof of Theorem 6. Put $A_1 = A^\delta$ and $B_1 = B^\delta$, then $A_1 \geq B_1 \geq 0$ and $M^\delta \geq A^\delta \geq m^\delta$. By applying (III) of Theorem 4 to A_1 and B_1, it follows that

$$\left(\frac{(M^{\delta(p_1-1)s}+m^{\delta(p_1-1)s})^2}{4m^{\delta(p_1-1)s}M^{\delta(p_1-1)s}}\right)^{\frac{1}{s}} A_1^{p_1} \geq B_1^{p_1} \qquad \text{holds for } p_1 \geq \frac{1}{s}+1.$$

Put $p_1 = \frac{p}{\delta} \geq \frac{1}{s}+1$, then we have the desired inequality

$$\left(\frac{(M^{(p-\delta)s}+m^{(p-\delta)s})^2}{4m^{(p-\delta)s}M^{(p-\delta)s}}\right)^{\frac{1}{s}} A^{p} \geq B^{p} \qquad \text{holds for all } s \geq 1 \text{ and } p \geq (\frac{1}{s}+1)\delta.$$

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