Continuous and Discrete Fourier Coefficients of Equi-distant Piecewise Linear Continuous Periodic Functions: Application to Mathematical Analysis of An FEM-CSM Combined Method for 2D Exterior Laplace Problems (Numerical Solution of Partial Differential Equations and Related Topics)

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Continuous and Discrete Fourier Coefficients of Equi-distant Piecewise Linear Continuous Periodic Functions

- Application to Mathematical Analysis of An FEM-CSM Combined Method for 2D Exterior Laplace Problems -

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Abstract

The author has investigated an FEM-CSM combined method for 2D exterior Laplace problems during these years ([2], [3]). Here the abbreviation of CSM is employed for the charge simulation method (See [1]). In the mathematical analysis for the method, especially in the proof of an a priori error estimate for the approximate solutions obtained by the method, a relation between continuous and discrete Fourier coefficients of equi-distant piecewise linear continuous 2\pi-periodic function plays a key role. In this paper, the relation is introduced with illustrative examples of application to the mathematical analysis mentioned above.

1. Relation between continuous and discrete Fourier coefficients for equi-distant piecewise linear continuous 2\pi-periodic functions

Let \( f(\theta) \) be a complex valued continuous \( 2\pi \)-periodic function of \( \theta \). For \( n \in \mathbb{Z} \), a continuous Fourier coefficient \( f_n \) of the function \( f(\theta) \) is defined through

\[
 f_n = \frac{1}{2\pi} \int_{0}^{2\pi} f(\theta) e^{-in\theta} d\theta.
\]

Fix a positive integer \( N \). Set

\[
 \theta_1 = \frac{2\pi}{N}, \quad \theta_j = j\theta_1 \quad \text{for} \quad j \in \mathbb{Z}.
\]

For \( n \in \mathbb{Z} \), a discrete Fourier coefficient \( f_n^{(N)} \) of the function \( f(\theta) \) is defined through

\[
 f_n^{(N)} = \frac{1}{N} \sum_{j=0}^{N-1} f(\theta_j) e^{-in\theta_j}.
\]
It is to be noted that we have for any continuous $2\pi$-periodic function $f(\theta)$,

\begin{equation}
(1) \quad f_{n+Nr}^{(N)} = f_n^{(N)}, \quad n \in \mathbb{Z}, \quad r \in \mathbb{Z} - \{0\}.
\end{equation}

Let $\hat{w}(\theta)$ be the reference roof function defined through

\begin{equation}
\hat{w}(\theta) = \begin{cases} 
1 - |\theta| : & |\theta| \leq 1, \\
0 : & |\theta| \geq 1.
\end{cases}
\end{equation}

For any $j \in \mathbb{Z}$, define a piecewise linear basis function $w_j^{(N)}(\theta)$ through the following formula:

\begin{equation}
w_j^{(N)}(\theta) = \hat{w}\left(\frac{\theta - \theta_j}{\theta_1}\right), \quad -\infty < \theta < \infty.
\end{equation}

A complex valued function $f(\theta)$ is said to be an **equi-distant piecewise linear continuous $2\pi$-periodic function** (with $N$ nodal points) in this paper if $f(\theta)$ is represented as

\begin{equation}
f(\theta) = \sum_{j=0}^{N} f(\theta_j)w_j^{(N)}(\theta), \quad 0 \leq \theta \leq 2\pi,
\end{equation}

with

\begin{equation}
f(2\pi) = f(0).
\end{equation}

Introduce a function $\alpha(\theta)$ through the formula:

\begin{equation}
\alpha(\theta) = \frac{2(1 - \cos \theta)}{\theta^2} \quad \text{for} \quad \theta \neq 0, \quad \text{with} \quad \alpha(0) = 1.
\end{equation}

**Theorem 1** We have the following relation for any **equi-distant piecewise linear continuous $2\pi$-periodic function** (with $N$ nodal points) $f(\theta)$,

\begin{equation}
f_n = \alpha(\theta_n)f_n^{(N)}, \quad n \in \mathbb{Z}.
\end{equation}

**Proof** A straightforward calculus leads the relation. \(\square\)

**Corollary** We have the following identity for any **equi-distant piecewise linear continuous $2\pi$-periodic function** (with $N$ nodal points) $f(\theta)$,

\begin{equation}
f_{n+Nr} = \left(\frac{n}{n+Nr}\right)^2 f_n, \quad n \in \mathbb{Z}, \quad r \in \mathbb{Z} - \{0\}.
\end{equation}

**Proof** Since we have

\begin{equation}
\alpha(\theta_{n+Nr}) = \left(\frac{n}{n+Nr}\right)^2 \alpha(\theta_n), \quad n \in \mathbb{Z}, \quad r \in \mathbb{Z} - \{0\},
\end{equation}

Theorem 1 together with Equality (1) implies Equality (3). \(\square\)

2. Boundary bilinear forms of Steklov type for exterior Laplace problems and its CSM-approximation forms
Let \( D_a \) be the interior of the disc with radius \( a \) being the origin as its center, and let \( \Gamma_a \) be the boundary of \( D_a \). Let \( \Omega_e = (D_a \cup \Gamma_a)^{C} \), which is said to be the exterior domain. We use the notation \( \mathbf{r} = r(\theta) \) for the point in the plane corresponding to the complex number \( re^{i\theta} \) with \( r = |\mathbf{r}| \) being the origin as its center, and \( \mathbf{a} = a(\theta) \), and \( \vec{\rho} = \vec{\rho}(\theta) \), corresponding to \( ae^{i\theta} \) with \( a = |\mathbf{a}| \), and \( \rho e^{i\theta} \) with \( \rho = |\vec{\rho}| \), respectively.

For functions \( u(a(\theta)) \) and \( v(a(\theta)) \) of \( H^{1/2}(\Gamma_a) \), let us introduce the boundary bilinear form of Steklov type for exterior Laplace problem through the following formula:

\[
 b(u, v) = 2\pi \sum_{n=-\infty}^{\infty} |n| f_n \overline{g_n},
\]

where \( f_n \), and \( g_n \), are continuous Fourier coefficients of \( u(a(\theta)) \), and \( v(a(\theta)) \), respectively.

It is to be noted that the following fact:

If \( u(a(\theta)) \) is the boundary value on \( \Gamma_a \) of the function \( u(\mathbf{r}) \) satisfying the following boundary value problem (E):

\[
 \begin{cases}
 -\Delta u = 0 & \text{in } \Omega_e, \\
 u = \varphi & \text{on } \Gamma_a, \\
 \sup_{\Omega_e} |u| < \infty,
\end{cases}
\]

with

\[ \varphi = u(a(\theta)), \]

then

\[
 b(u, v) = -\int_{\Gamma_a} \frac{\partial u}{\partial r} v d\Gamma.
\]

The CSM approximate form for \( b(u, v) \) of the first type, which is denoted by \( b^{(N)}(u, v) \), is represented through the following formula (6):

\[
 b^{(N)}(u, v) = -\int_{\Gamma_a} \frac{\partial u^{(N)}}{\partial r} v^{(N)} d\Gamma,
\]

where \( u^{(N)}(\mathbf{r}) \) is a CSM-approximate solution for \( u(\mathbf{r}) \) satisfying (E) with \( \varphi = u(a(\theta)) \). Namely \( u^{(N)}(\mathbf{r}) \) is determined through the following problem (E\(^{(N)}\)):

\[
 \begin{cases}
 u^{(N)}(\mathbf{r}) = \sum_{j=0}^{N-1} q_j G_j(\mathbf{r}) + q_N, \\
 u^{(N)}(a_j) = u(a_j), \quad 0 \leq j \leq N - 1, \\
 \sum_{j=0}^{N-1} q_j = 0,
\end{cases}
\]

where

\[ a_j = a(\theta_j), \quad \vec{\rho}_j = \vec{\rho}(\theta_j) \quad \text{with} \quad 0 < \rho < a, \]
\[ G_j(r) = E(r - \bar{\rho}_j) - E(r), \quad E(r) = -\frac{1}{2\pi} \log r. \]

Problem \((E^{(N)}) \) is to find \( N + 1 \) unknowns \( q_j, \ 0 \leq j \leq N \), and it is uniquely solvable for any fixed \( \rho \in (0, a) \).

The CSM approximate form for \( b(u, v) \) of the second type, which is denoted by \( \overline{b}^{(N)}(u, v) \), is represented through the following formula \((7)\):

\[ \overline{b}^{(N)}(u, v) = -\frac{2\pi a}{N} \sum_{j=0}^{N-1} \frac{\partial u^{(N)}(a_j)}{\partial r} v^{(N)}(a_j), \]

which is the quadrature formula for \( b^{(N)}(u, v) \) with the use of trapezoidal rule.

We use the following notations:

\[ b(v) = b(v, v)^{1/2}, \quad b^{(N)}(v) = b(v, v)^{1/2}, \quad \overline{b}^{(N)}(v) = \overline{b}(N)(v, v)^{1/2}. \]

Denote the totality of equi-distant piecewise linear continuous \( 2\pi \)-periodic functions (with \( N \) nodal points) \( v(a(\theta)) \) by \( V_N \):

\[ V_N = \{v(a(\theta)) = \sum_{j=0}^{N} v(a_j) w^{(j)}(N) \theta\}. \]

Let

\[ N(\gamma) = \frac{\log 2}{-\log \gamma} \quad \text{with} \quad \gamma = \frac{\rho}{a}. \]

**Theorem 2** We have the following inequalities for any \( v \in V_N \).

\[ \frac{1}{4\sqrt{1 + 2\zeta(3)}} b(v) \leq b^{(N)}(v) \leq \frac{\pi^2}{2} b(v) \]

provided that \( N \geq N(\gamma) \), where

\[ \zeta(3) = \sum_{r=1}^{\infty} \frac{1}{r^3}. \]

**Theorem 3** For \( u, v \in V_N \), we have

\[ |b^{(N)}(u, v) - \overline{b}^{(N)}(u, v)| \leq 8\gamma^{2N} b^{(N)}(u) b^{(N)}(v) \]

provided that \( N \geq N(\gamma) \).

3. Proof of Theorem 2
For a fixed positive integer $N$, introduce sets of integers $\mathcal{N}_r$ through
\[ \mathcal{N}_r = \{ n : -\frac{N}{2} \leq n - Nr < \frac{N}{2}, \ n \neq Nr \} \]
with
\[ r = 0, \pm 1, \pm 2, \cdots. \]

For any integer $n \in [1, N-1]$, define a function $s_n^{(N)}(\gamma)$ of $\gamma \in (0,1)$, numbers $\Lambda_n^{(N)}$ and $\overline{\Lambda}_n^{(N)}$ as follows.
\[ s_n^{(N)}(\gamma) = \int_{0}^{\gamma} \frac{x^{n-1} + x^N - n - 1}{1 - x^N} dx, \]
\[ \Lambda_n^{(N)} = \frac{s_n^{(N)}(\gamma)^2}{\{s_n^{(N)}(\gamma)\}^2}, \quad \overline{\Lambda}_n^{(N)} = \frac{\gamma \frac{d}{d\gamma} s_n^{(N)}(\gamma)}{s_n^{(N)}(\gamma)}. \]

We admit the validity of the following Proposition 1 without proof.

**Proposition 1** For $u, v \in V_N$, we have
\[ b^{(N)}(u, v) = 2\pi \sum_{n \in \mathcal{N}_0} \Lambda_n^{(N)} f_n^{(N)} g_n^{(N)} \]
and
\[ \overline{b}^{(N)}(u, v) = 2\pi \sum_{n \in \mathcal{N}_0} \overline{\Lambda}_n^{(N)} f_n^{(N)} g_n^{(N)}, \]
where $f_n^{(N)}$ and $g_n^{(N)}$ are discrete Fourier coefficients of $u(a(\theta))$ and $v(a(\theta))$, respectively.

Using the representation of $\Lambda_n^{(N)}$, we obtain

**Proposition 2** If $N \geq N(\gamma)$, then
\[ \frac{n}{16} \leq \Lambda_n^{(N)} \leq 4n, \quad 1 \leq n \leq \frac{N}{2}. \]

An elemental calculus leads

**Proposition 3** It holds
\[ \frac{4}{\pi^2} \leq \alpha(\theta) \leq 1, \quad -\pi \leq \theta \leq \pi. \]

**Proposition 4** For $v \in V_N$, we have
\[ \frac{1}{16} \left\{ 2\pi \sum_{n \in \mathcal{N}_0} |n||g_n|^2 \right\} \leq b^{(N)}(v, v) \leq \frac{\pi^4}{4} \left\{ 2\pi \sum_{n \in \mathcal{N}_0} |n||g_n|^2 \right\} \]
provided that $N \geq N(\gamma)$.

Proof Due to Theorem 1 and Proposition 1, we have

$$b^{(N)}(v, v) = 2\pi \sum_{n \in N_0} \Lambda^{(N)}_{|n|} \frac{1}{|\alpha(\theta_n)|^2} |g_n|^2.$$ 

Propositions 2 and 3 imply the conclusion of Proposition 4. □

**Proposition 5** For $v \in V_N$, we have

$$\left\{ 2\pi \sum_{n \in N_0} |n| |g_n|^2 \right\} \leq b(v, v) \leq (1 + 2\zeta(3)) \left\{ 2\pi \sum_{n \in N_0} |n| |g_n|^2 \right\}.$$ 

Proof Due to Corollary of Theorem 1, we have

$$\frac{1}{2\pi} b(v, v) = \sum_{r \in \mathbb{Z}} \sum_{n \in N_0} |n| |g_n|^2.$$ 

For $r \in \mathbb{Z} - \{0\}$, we have

$$\left| \frac{n}{n + Nr} \right| \leq \frac{1}{|r|}, \quad n \in N_0.$$ 

Therefore

$$b(v, v) \leq \left( 1 + 2 \sum_{r=1}^{\infty} \frac{1}{r^3} \right) \left\{ 2\pi \sum_{n \in N_0} |n| |g_n|^2 \right\}.$$ 

Hence the second inequality of the conclusion is valid, while the first one is trivial by definition of $b(u, v)$. □

Propositions 4 and 5 complete the proof of Theorem 2.

4. Proof of Theorem 3

**Proposition 6** For an integer $n \in [1, N - 1]$, define $B_n$ through the following formula:

$$B_n = \sum_{p \in \mathbb{Z}} \sum_{q \in \mathbb{Z}} \left\{ \gamma^{n+Np} \frac{\gamma^{n+Nq}}{|n + Nq|} \right\}.$$ 

Then we have

$$s_n^{(N)}(\gamma^2) \leq B_n \leq (1 + 8\gamma^2)s_n^{(N)}(\gamma^2)$$ 

provided that $N \geq N(\gamma)$.

Proof A lengthy but straightforward calculus leads the conclusion. □

**Proposition 7** For $N \geq N(\gamma)$, we have

$$\Lambda_n^{(N)} \leq \Lambda_n^{(N)} \leq (1 + 8\gamma^2)\Lambda_n^{(N)}.$$
Proof Let

\[ \Gamma_n = s_n^{(N)}(\gamma). \]

Then we have

\[ \Lambda_n^{(N)} = \frac{s_n^{(N)}(\gamma^2)}{\Gamma_n^2}, \]

and

\[ \overline{\Lambda}_n^{(N)} = \frac{B_n}{\Gamma_n^2}. \]

Hence Proposition 6 implies the conclusion. \(\square\)

The proof of Theorem 3 is now straightforward. In fact, we have

\[ b^{(N)}(u, v) - \overline{b}^{(N)}(u, v) = 2\pi \sum_{n \in N_0} (\Lambda_n^{(N)} - |n|\overline{\Lambda}_n^{(N)}) f_n^{(N)} g_n^{(N)}. \]

Hence it holds

\[ |b^{(N)}(u, v) - \overline{b}^{(N)}(u, v)| \leq 2\pi \left\{ \sum_{n \in N_0} |\Lambda_n^{(N)} - \overline{\Lambda}_n^{(N)}||f_n^{(N)}|^2 \right\}^{1/2} \times \left\{ \sum_{n \in N_0} |\Lambda_n^{(N)} - \overline{\Lambda}_n^{(N)}||g_n^{(N)}|^2 \right\}^{1/2}. \]

Let \( N \geq N(\gamma). \) Proposition 7 implies

\[ 0 \leq |\Lambda_n^{(N)} - \overline{\Lambda}_n^{(N)}| \leq 8\gamma^2\Lambda_n^{(N)}, \quad n \in N_0. \]

Therefore we get

\[ |b^{(N)}(u, v) - \overline{b}^{(N)}(u, v)| \leq 8\gamma^2 \times \left\{ 2\pi \sum_{n \in N_0} \Lambda_n^{(N)} |f_n^{(N)}|^2 \right\}^{1/2} \times \left\{ 2\pi \sum_{n \in N_0} \Lambda_n^{(N)} |g_n^{(N)}|^2 \right\}^{1/2}, \]

provided that \( N \geq N(\gamma). \) Due to Proposition 1 we have the conclusion of Theorem 3.

5. Application to mathematical analysis of an FEM-CSM combined method for exterior Laplace problems

Fix a simply connected bounded domain \( \mathcal{O} \) in the plane. Assume that the boundary \( \mathcal{C} \) of \( \mathcal{O} \) is sufficiently smooth. The exterior domain of \( \mathcal{C} \) is denoted by \( \Omega. \)

Fix a function \( f \in L^2(\Omega). \) Assume that the support of \( f, \text{supp}(f), \) is compact.

Choose a so large that the open disc \( D_a \) may contain the union \( \mathcal{O} \cup \text{supp}(f) \) in its interior.

As a model problem the following Poisson equation (E) is employed.

\[
\begin{cases}
-\Delta u = f & \text{in } \Omega, \\
u = 0 & \text{on } \mathcal{C}, \\
\sup_{|r| > a} |u| < \infty.
\end{cases}
\]
The intersection of the domain $\Omega$ and the disc $D_a$ is said to be the interior domain, denoted by $\Omega_i$:

$$\Omega_i = \Omega \cap D_a.$$ 

Consider the Dirichlet inner product $a(u,v)$ for $u,v \in H^1(\Omega_i)$:

$$a(u,v) = \int_{\Omega_i} \text{grad}u \cdot \text{grad}v \, d\Omega.$$ 

Since the trace $\gamma_a v$ on $\Gamma_a$ is an element of $H^{1/2}(\Gamma_a)$ for any $v \in H^1(\Omega_i)$, the boundary bilinear form of Steklov type $b(u,v)$ is well defined for $u,v \in H^1(\Omega_i)$. Therefore we can define a continuous symmetric bilinear form:

$$t(u,v) = a(u,v) + b(u,v)$$

for $u,v \in H^1(\Omega_i)$.

Let $F(v)$ be a continuous linear functional on $H^1(\Omega_i)$ defined through the following formula:

$$F(v) = \int_{\Omega_i} fv \, d\Omega.$$ 

A function space $V$ is defined as follows:

$$V = \{v \in H^1(\Omega_i) : v = 0 \text{ on } C \}. $$

Using these notations, the following weak formulation problem $(\Pi)$ is defined.

$$(\Pi) \quad \begin{align*}
  t(u,v) &= F(v), \quad v \in V, \\
  u &\in V.
\end{align*}$$

Admitting the equivalence between the equation $(E)$ and the problem $(\Pi)$, we consider the problem $(\Pi)$ and its approximate ones.

Fix a positive number $\rho$ so as to satisfy $0 < \rho < a$. For a fixed positive integer $N$, set the points $\bar{\rho}_j, a_j, 0 \leq j \leq N - 1$, as is defined in Section 2.

A family of finite dimensional subspaces of $V$:

$$\{V_N : N = N_0, N_0 + 1, \ldots \}$$

is supposed to have the following properties:

$$(V_N - 1) \quad V_N \subset C(\overline{\Omega_i}).$$

$$(V_N - 2) \quad \begin{align*}
  \{ &\text{For any } v \in V_N, \ v(a(\theta)) \text{ is an equi–distant piecewise linear} \\
  &\text{continuous } 2\pi–\text{periodic function with respect to } \theta. \}
\end{align*}$$

$$(V_N - 3) \quad \min_{v \in V_N} a(v - v_N) \leq \frac{C}{N} \|v\|_{H^2(\Omega_i)}, \quad v \in V \cap H^2(\Omega_i). $$
In the property \((V_N - 3)\), \(C\) is a constant independent of \(N\) and \(v\), and
\[
a(v) = a(v, v)^{1/2}, \quad v \in V.
\]

For \(u, v \in H^1(\Omega_i) \cap C(\Omega_i)\), we define bilinear forms \(t^{(N)}(u, v)\) and \(\overline{t}^{(N)}(u, v)\) as follows.
\[
t^{(N)}(u, v) = a(u, v) + b^{(N)}(u, v),
\]
and
\[
\overline{t}^{(N)}(u, v) = a(u, v) + \overline{b}^{(N)}(u, v).
\]

Now two approximate problems (\(\Pi^{(N)}\)) and (\(\overline{\Pi}^{(N)}\)) are stated as follows.

(\(\Pi^{(N)}\))
\[
\begin{cases}
t^{(N)}(u_N, v) = F(v), & v \in V_N, \\
u_N \in V_N.
\end{cases}
\]

(\(\overline{\Pi}^{(N)}\))
\[
\begin{cases}
\overline{t}^{(N)}(\overline{u}_N, v) = F(v), & v \in V_N, \\
\overline{u}_N \in V_N.
\end{cases}
\]

With the aide of Theorems 2 and 3 and other necessary discussions, we can show the following error estimate.

**Theorem 4** For a constant \(C\), we have the following estimate.
\[
\frac{||u - u_N||_{H^1(\Omega_i)}}{||u - \overline{u}_N||_{H^1(\Omega_i)}} \leq \frac{C}{N} ||u||_{H^2(\Omega_i)}. 
\]

In the above, the constant \(C\) is independent of the solution \(u\) of \((\Pi)\) and \(N\).

**References**

