<table>
<thead>
<tr>
<th>項目</th>
<th>内容</th>
</tr>
</thead>
<tbody>
<tr>
<td>タイトル</td>
<td>计算的線形微分方程的手法と対応する問題の解法と関連する課題</td>
</tr>
<tr>
<td>著者</td>
<td>佐々木 マナブ</td>
</tr>
<tr>
<td>引用</td>
<td>数理解析研究所講究録 (2000), 1145: 204-210</td>
</tr>
<tr>
<td>発行年月</td>
<td>2000-04</td>
</tr>
<tr>
<td>URL</td>
<td><a href="http://hdl.handle.net/2433/63932">http://hdl.handle.net/2433/63932</a></td>
</tr>
<tr>
<td>タイプ</td>
<td>学術論文</td>
</tr>
<tr>
<td>テキストバージョン</td>
<td>出版社</td>
</tr>
</tbody>
</table>

京都大学 学術情報リポジトリ
KURENAI
Kyoto University Research Information Repository
Planar Cubic Spirals

Manabu Sakai (酒井 官) *
Department of Mathematics, Faculty of Science
Univ. of Kagoshima, Kagoshima 890-0065 Japan

Abstract

Spiral segments have several advantages of containing neither inflection points, singularities nor curvature extrema. The object of this note is to obtain sufficient conditions for cubic spiral segments with help of Mathematica.

1 Introduction

Spiral segments have several advantages of containing neither inflection points, singularities nor curvature extrema. They are used to join (i) a straight line to a circle, (ii) two circles with a broken back $C$, (iii) two circles with an $S$, (iv) two non-parallel straight lines and also (v) two circles with one circle inside the other ([4]). In this paper, all spirals have zero curvature at their beginning points, nonnegative curvature at their ending points, and are parameterized on the interval $[0, 1]$. Walton & Meek have obtained a cubic spiral condition by choosing the second control point of a cubic Bézier midway between the first and third control points when the quintic numerator of the derivative of the curvature reduces to a special quartic polynomial composed of even powers, i.e., they successfully relaxed the requirement of a curvature extremum at the ending fourth control point([4]). In addition, the spiral region has been given for the first or fourth control point. Farouki has explored generalization by moving the second control point along the line segment joining the first and third control points([1]).

The object of this paper is to expand their regions for the cubic spiral with help of Mathematica (A system for Mathematics by Computer). In Section 3, we examine a

*e-mail:msakai@sci.kagoshima-u.ac.jp
nonnegativity condition for a quintic numerator of the derivative of the curvature on $[0, 1]$ which is to be reduced to $[0, \infty)$. In Section 4, we determine the position of the end vertex of a cubic spiral with the other one fixed.

2 Preliminary results

For a parametric cubic curve segment $z(t) = (x(t), y(t)), 0 \leq t \leq 1$ with control vertices $P_i, 0 \leq i \leq 3$:

$$z(t) = P_0(1-t)^3 + 3P_1(1-t)^2t + 3P_2(1-t)t^2 + P_3t^3,$$
(2.1)

its curvature $\kappa(t)$ is given by

$$\kappa(t) = (z' \times z'')(t)/\|z'(t)\|^3, \quad 0 \leq t \leq 1$$
(2.2)

where $\|\bullet\|$ means the Euclidean norm. Since the cubic curve segment has eight parameters, we require it to satisfy the following six conditions:

(i) $z(0) = (0,0)$,  
(ii) $z'(0) \parallel (1,0)$,  
(iii) $\kappa(0) = 0$

(iv) $\kappa(1) = 1$,  
(v) $z'(1) \parallel (\cos \theta, \sin \theta)$

The above four conditions (i)-(iii) mean that the segment is tangent to the the positive part of the $x$-axis at the origin (the starting point $P_0$ of the segment), and the remaining three ones (iv)-(v) mean that the segment meets the circle centered $(x(1)-\sin \theta, y(1)+\cos \theta)$ (the terminal point $P_3$ of the segment) with a radius 1. For analysis, we rewrite $z(t)$ of the form:

$$x(t) = a_0 + a_1 t + a_2 t^2 + a_3 t^3$$
(2.4)

$$y(t) = b_0 + b_1 t + b_2 t^2 + b_3 t^3$$

with

$$a_0 = 0, a_3 = \frac{-a_1 - 2a_2 + \sqrt{2(a_1 + a_2)\sin \theta \cos \theta}}{3}$$
(2.5)

$$b_0 = 0, b_1 = 0, b_2 = 0, b_3 = \frac{(a_1 + 2a_2 + 3a_3)\tan \theta}{3}$$
Introduce two parameters \( p, d \) as
\[
a_1 = p, \quad a_2 = d^2/2 - p
\]
(2.6)
to obtain
\[
x(t) = pt + \left(\frac{d^2}{2} - p\right)t^2 + \frac{p - d^2 + d \cos \theta \sqrt{\sin \theta}}{3} t^3
\]
(2.7)
y(t) = \frac{d \sqrt{\sin^3 \theta}}{3} t^3

Then, we have
\[
\kappa(t) = \frac{dt(2p + d^2t - 2pt) \sqrt{\sin^3 \theta}}{\{(p + d^2t - 2pt - d^2t^2 + pt^2 + dt^2 \cos \theta \sqrt{\sin \theta})^2 + d^2t^4 \sin^3 \theta\}^{3/2}}
\]
(2.8)
First, differentiate \( \kappa(t) \) with respect to \( t \) and next let \( t = 1/(1+s), 0 \leq s \leq \infty \) to obtain
\[
\kappa'(t) = \left\{d(1+s)^5 \sin \theta \sqrt{\sin \theta} \sum_{i=0}^{5} u_i s^i / \left(\sum_{i=0}^{4} v_i s^i\right)^{5/2}\right\}
\]
(2.9)
with
\[
u_5 = 2p^3, \quad u_4 = 10p^3, \quad u_3 = dp(-3d^3 + 20dp - 8p \cos \theta \sqrt{\sin \theta})
\]
(2.10)
\[
u_2 = d(-d^5 + 13d^3p - 16d^2p \cos \theta \sqrt{\sin \theta} + 8p^2 \cos \theta \sqrt{\sin \theta})
\]
\[
u_1 = d^2(3d^4 - 5d^3 \cos \theta \sqrt{\sin \theta} + 8dp \cos \theta \sqrt{\sin \theta} - 10p \sin \theta)
\]
\[
u_0 = d^2(3d^3 \cos \theta - 4d^2 \sqrt{\sin \theta} - 2p \sqrt{\sin \theta} \sqrt{\sin \theta});
\]
\[
u_4 = p^2, \quad v_3 = 2d^2p, \quad v_2 = d^4 + 2dp \cos \theta \sqrt{\sin \theta}
\]
\[
u_1 = 2d^3 \cos \theta \sqrt{\sin \theta}, \quad v_0 = d^2 \sin \theta
\]
(2.11)
Being difficult to obtain a necessary and sufficient condition for the above quintic numerator of (2.9) to have no zero \( \in (0, 1) \), we consider a sufficient condition, i.e., \( u_i \geq 0, 0 \leq i \leq 5 \) which gives the following Lemma. Note that functions \( f_i(d, \theta), i = 1, 2, 3 \) are obtained by solving \( u_i = 0 \) for \( p \).

**Lemma 2.1** The cubic segment \( z(t)(= (x(t), y(t)) \) of (2.7) is a spiral if
\[
\text{Max}\{f_1(d, \theta), f_2(d, \theta), f_3(d, \theta)\} \leq p \leq f_0(d, \theta)
\]
(2.12)
where
\[
\begin{align*}
  f_0(d, \theta) &= \frac{3d^3 \cos \theta - 4d^2 \sqrt{\sin \theta}}{2\sqrt{\sin \theta}} \\
  f_1(d, \theta) &= \frac{-(3d^4 - 5d^3 \cos \theta \sqrt{\sin \theta})}{2(4d \cos \theta \sqrt{\sin \theta} - 5\sin \theta)} \\
  f_2(d, \theta) &= \frac{d^2(-13d + 16 \cos \theta \sqrt{\sin \theta} + \sqrt{169d^2 - 384d \cos \theta \sqrt{\sin \theta} + 256 \cos^2 \theta \sin \theta})}{16 \cos \theta \sqrt{\sin \theta}} \\
  f_3(d, \theta) &= \frac{3d^3}{4(5d^2 - 2 \cos \theta \sqrt{\sin \theta})}
\end{align*}
\]

(2.13)

In order to determine \( f_4(d, \theta) = \max\{f_1(d, \theta), f_2(d, \theta), f_3(d, \theta)\} \), let \( d_0 = 4\sqrt{\sin \theta} / (3 \cos \theta) \) and \( d_3 = 2(11 + 3\sqrt{5}) \cos \theta \sqrt{\sin \theta} / 19 \). Then, we have

**Case 1:** \( \cos 2\theta \leq (152 - 57\sqrt{5})/57 \) \( \iff \theta \leq 0.562821... \)

\[
f_4(d, \theta) = f_3(d, \theta), \quad d \in [d_0, \infty)
\]

(2.14)

**Case 2:** \( (152 - 57\sqrt{5})/57 \leq \cos 2\theta \leq (51 + \sqrt{105})/96 \) \( \iff 0.439456... \leq \theta \leq 0.562820... \)

\[
f_4(d, \theta) = \begin{cases} 
  f_2(d, \theta), & d \in [d_0, d_3] \\
  f_3(d, \theta), & d \in [d_3, \infty)
\end{cases}
\]

(2.15)

Note that \( f_4(d, \theta) \) intersects \( f_3(d, \theta) \) or \( f_2(d, \theta) \) for \( (152 - 57\sqrt{5})/57 \leq \cos 2\theta \leq (28 - 11\sqrt{5})/6 \) or \( (28 - 11\sqrt{5})/6 \leq \cos 2\theta \leq (51 + \sqrt{105})/96 \).

**Case 3:** \( (51 + \sqrt{105})/96 \leq \cos 2\theta \) \( \iff 0 \leq \theta \leq 0.439456... \)

\[
f_4(d, \theta) = \begin{cases} 
  f_1(d, \theta), & d \in [d_0, d_2] \\
  f_2(d, \theta), & d \in [d_2, d_3] \\
  f_3(d, \theta), & d \in [d_3, \infty)
\end{cases}
\]

(2.16)

where \( d_2 \) is a unique positive root of a cubic equation:

\[
48d^3 \cos \theta - 6d^2(23 + 10 \cos 2\theta) \sqrt{\sin \theta} + (119 \sin 2\theta + 11 \sin 4\theta) d = 0
\]

(2.17)

Note that \( f_4(d, \theta) \) intersects \( f_2(d, \theta) \) or \( f_1(d, \theta) \) for \( (51 + \sqrt{105})/96 \leq \cos 2\theta \leq 13/14 \) or \( 13/14 \leq \cos 2\theta \). Consequently, we have a (sufficient) spiral condition:

**Theorem 2.1** The cubic segment \( z(t) = (x(t), y(t)) \) of (2.7) is a spiral if

(i) for \( \cos 2\theta \leq (28 - 11\sqrt{5})/6 \), \( f_5(d, \theta) \leq p \leq f_0(d, \theta) \) \( (d \geq d_4) \)

(ii) for \( (28 - 11\sqrt{5})/6 \leq \cos 2\theta \leq 13/14 \), \( f_2(d, \theta) \leq p \leq f_0(d, \theta) \) \( (d \geq d_5) \)

(iii) for \( 13/14 \leq \cos 2\theta \), \( f_1(d, \theta) \leq p \leq f_0(d, \theta) \) \( (d \geq d_6) \)
Note that $f_i(d, \theta), i = 3, 2, 1$ intersect $f_0(d, \theta)$ at $d = d_{7-i}$ where

\[ d_4 = \frac{(49 + 6 \cos 2\theta) \sqrt{\sin \theta} + \sqrt{1645 \sin \theta - 195 \sin 3\theta + 9 \sin 5\theta}}{60 \cos \theta} \]
\[ d_5 = \frac{9(7 + 4 \cos 2\theta) \sqrt{\sin \theta} + \sqrt{3(175 \sin \theta - 56 \sin 3\theta + 12 \sin 5\theta)}}{3(22 \cos \theta + 3 \cos 3\theta)} \]
\[ d_6 = \frac{18 \cos \theta \sqrt{\sin \theta} + \sqrt{3(-13 \sin \theta + 7 \sin 3\theta)}}{9 + 6 \cos 2\theta} \]  

(2.19)

Here we remark that the sufficient condition $a_2 = 0$ proposed by Walton & Meek [3] is equivalent to

\[ p = d^2/2, \quad p \leq f_0(d, \theta) \]  

(2.20)

![Figure 1. Spiral condition on $p, d$ for $\theta = 0.6$](image)

3 Main results

The above spiral condition (2.12) determines a spiral region for $P_3$ (the end control point of the Bézier curve(2.1)).

For $(m, n) = P_3$, we have

\[ d = \frac{3n}{\sin \theta \sqrt{\sin \theta}}, \quad p = \frac{3(2m \sin^3 \theta - 2n \sin^2 \theta \cos \theta - 3n^2)}{2 \sin^3 \theta} \]  

(3.1)
Sufficient condition 1 (Walton & Meek [4]):
Equation (2.20) gives
\[ m = \frac{n \cos \theta}{\sin \theta} + \frac{3n^2}{\sin^3 \theta}, \quad n \geq \frac{5 \sin^2 \theta}{9 \cos \theta} \] (3.2)
In Figure 2, the parametrized boundary of the above region is given by
\[ m = \frac{5 \sin \theta(2 \cos^2 \theta + 5)}{27 \cos 2\theta}, \quad n = \frac{5 \sin^2 \theta}{9 \cos \theta} \] (3.3)
where its implicit form is given by
\[ 6561m^4 - 2187n^2(32 + 27n^2)m^2 - n^2(243n^2 + 320)^2 = 0 \] (3.4)
Theorem 2.1 gives a larger spiral region:

Sufficient condition 2:
First, Theorem 2.1 requires \( f_0(d, p) \geq p \) which is equivalent to
\[ 27n^3 \cos \theta - 9n^2 \sin^2 \theta + 2n \cos \theta \sin^4 \theta \geq 2m \sin^5 \theta \] (3.5)
(i) for \( \cos 2\theta \leq (28 - 11\sqrt{5})/6 \); in addition to (3.5)
\[ m \geq \frac{n(117n^2 + 48n \cos \theta \sin^2 \theta - 8 \cos^2 \theta \sin^4 \theta)}{4(15n - 2 \cos \theta \sin^3 \theta) \sin^3 \theta} \] (3.6)
(ii) for \( (28 - 11\sqrt{5})/6 \leq \cos 2\theta \leq 13/14 \); in addition to (3.5)
\[ 16m \sin^5 \theta \cos \theta \geq 3n^2 \sqrt{1521n^2 - 1152n \sin^2 \theta \cos \theta + 256 \sin^4 \theta \cos^2 \theta} \]
\[ -(117n^3 - 72n^2 \sin^2 \theta \cos \theta - 16n \cos^2 \theta \sin^4 \theta) \] (3.7)
(iii) for \( 13/14 \leq \cos 2\theta \); in addition to (3.5)
\[ m \geq \frac{-n \{81n^3 - 81n^2 \cos \theta \sin^2 \theta - 3n(8 \cos^2 \theta - 5) \sin^4 \theta + 10 \cos \theta \sin^6 \theta\}}{2(12n \cos \theta - 5 \sin^2 \theta) \sin^5 \theta} \] (3.8)
The above result gives the spiral region for the fourth control points \( P_3 \) with \( P_0 \) fixed.

Figure 2. Spiral region for the fourth control point \( P_3 \) with \( P_0 \) fixed
A shift of $P_3$ to the origin and $P_2$ on the positive part of the $x$-axis gives

$$P_0 = (x(1) \cos \theta + y(1) \sin \theta, x(1) \sin \theta - y(1) \cos \theta)$$

(3.9)

A combination of (3.2)-(3.9) yields the spiral region for the first control point $P_0$ with $P_3$ fixed.

Figure 3. Spiral region for the fourth control point $P_0$ with $P_3$ fixed

References


