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<td>Author(s)</td>
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<tr>
<td>Citation</td>
<td>数理解析研究所講究録 (2000), 1145: 194-203</td>
</tr>
<tr>
<td>Issue Date</td>
<td>2000-04</td>
</tr>
<tr>
<td>URL</td>
<td><a href="http://hdl.handle.net/2433/63933">http://hdl.handle.net/2433/63933</a></td>
</tr>
<tr>
<td>Type</td>
<td>Departmental Bulletin Paper</td>
</tr>
<tr>
<td>Textversion</td>
<td>publisher</td>
</tr>
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<td>Institution</td>
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Order of Accuracy of Functional Fitting
Runge-Kutta-Nyström Formula

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1 Introduction

In this paper we introduce functional fitting Runge-Kutta-Nyström method which integrates some set of functions exactly. The method proposed here is a generalization of exponentially or trigonometrically fitting Runge-Kutta-Nyström methods.

2 Functional fitting Runge-Kutta-Nyström method

Consider the variable coefficient Runge-Kutta-Nyström method

\[
\begin{align*}
\begin{cases}
y_{n+1} &= y_n + h y_n' + h^2 \sum_{i=1}^{s} \tilde{b}_i(t_n, h) f(t_n + c_i h, Y_i), \\
y_n' &= y_n' + h \sum_{i=1}^{s} b_i(t_n, h) f(t_n + c_i h, Y_i), \\
Y_i &= y_n + c_i h y_n' + h^2 \sum_{j=1}^{s} \bar{a}_{i,j}(t_n, h) f(t_n + c_j h, Y_j),
\end{cases}
\end{align*}
\]

(1)

for solving the second order ODE of the form

\[
y''(t) = f(t, y), \quad y(0) = y_0, \quad y'(0) = y'_0, \quad t \in [t_0, T].
\]

(2)

We will call the method functional fitting Runge-Kutta-Nyström (FRKN) method, when the method is designed to integrate some functions exactly. The coefficients \( \bar{a}_{i,j}, b_i \) and \( \tilde{b}_i \) of the FRKN to be considered here are determined by the simultaneous equation

\[
\begin{align*}
\begin{cases}
\bar{u}_m(t + h) &= \bar{u}_m(t) + h \bar{u}_m'(t) + h^2 \sum_{i=1}^{s} \tilde{b}_i(t, h) \bar{u}_m''(t + c_i h), \\
\bar{u}_m'(t + h) &= \bar{u}_m'(t) + h \sum_{i=1}^{s} b_i(t, h) \bar{u}_m''(t + c_i h), \\
\bar{u}_m(t + c_i h) &= \bar{u}_m(t) + c_i h \bar{u}_m'(t) + h^2 \sum_{j=1}^{s} \bar{a}_{i,j}(t, h) \bar{u}_m''(t + c_j h), \quad i = 1, 2, \ldots, s,
\end{cases}
\end{align*}
\]

(3)

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where the functions $u_m''(t) = \varphi_m(t)$ are linearly independent on $[t_0, T]$, that is the Wronskian matrix $W$ given by

$$W(\varphi_1, \varphi_2, \ldots, \varphi_s) \equiv \begin{pmatrix} \varphi_1(t) & \varphi_1'(t) & \cdots & \varphi_1^{(s-1)}(t) \\ \varphi_2(t) & \varphi_2'(t) & \cdots & \varphi_2^{(s-1)}(t) \\ \vdots & \vdots & \ddots & \vdots \\ \varphi_s(t) & \varphi_s'(t) & \cdots & \varphi_s^{(s-1)}(t) \end{pmatrix}$$

is nonsingular for all $t \in [t_0, T]$. For the uniqueness of the coefficients we have:

**Theorem 1** The coefficients $\overline{a}_{i,j}(t, h)$, $b_i(t, h)$ and $\overline{b}_i(t, h)$ determined by (3) are unique for small $h > 0$, if the functions $\varphi_m(t)$ are sufficiently smooth and linearly independent.

**Proof.** It is clear from (3) that these coefficients are unique, if the matrix given by

$$\Phi(t, h) = \begin{pmatrix} \varphi_1(t + c_1 h) & \varphi_1(t + c_2 h) & \cdots & \varphi_1(t + c_s h) \\ \varphi_2(t + c_1 h) & \varphi_2(t + c_2 h) & \cdots & \varphi_2(t + c_s h) \\ \vdots & \vdots & \ddots & \vdots \\ \varphi_s(t + c_1 h) & \varphi_s(t + c_2 h) & \cdots & \varphi_s(t + c_s h) \end{pmatrix}$$

is nonsingular. The matrix $\Phi$ can be expressed as

$$\Phi(t, h) = W(\varphi_1, \varphi_2, \ldots, \varphi_s) \begin{pmatrix} 1 & 1 & \cdots & 1 \\ c_1 h & c_2 h & \cdots & c_s h \\ \vdots & \vdots & \ddots & \vdots \\ (c_1 h)^{s-1} & (c_2 h)^{s-1} & \cdots & (c_s h)^{s-1} \end{pmatrix} + O(h^s).$$

Since $W$ is assumed to be nonsingular, and the second matrix in the right-hand side is also nonsingular from the assumption that $c_i$ are different from each other, then we have the nonsingularity of $\Phi(t, h)$ for sufficiently small $h > 0$.

Hereafter, we simply denote the coefficients by $\tilde{a}_{i,j}$, $b_i$ and $\tilde{b}_i$, and denote their power series expansions in $h$ by

$$\tilde{a}_{i,j} = a_{i,j}^{(0)} + a_{i,j}^{(1)} h + a_{i,j}^{(2)} h^2 + \cdots, \quad b_i = b_i^{(0)} + b_i^{(1)} h + b_i^{(2)} h^2 + \cdots,$$

$$\tilde{b}_i = \tilde{b}_i^{(0)} + \tilde{b}_i^{(1)} h + \tilde{b}_i^{(2)} h^2 + \cdots.$$  

If we take $u_m''(t) = t^{m-1}$ ($m = 1, 2, \ldots, s$) in (3), then $\tilde{a}_{i,j} = a_{i,j}^{(0)}$, $b_i = b_i^{(0)}$, and $\tilde{b}_i = \tilde{b}_i^{(0)}$, and the method reduces to the direct collocation Runge-Kutta-Nyström method proposed by Van der Houwen et al [5].

Here we consider the order of accuracy of the FRKN method. The order of accuracy of the FRKN is defined to be $p = \min\{p_1, p_2\}$, where $p_1$ and $p_2$ are the integers satisfying

$$E := y_{n+1} - y(t_{n+1}) = O(h^{p_1+1}), \quad E' := y'_{n+1} - y'(t_{n+1}) = O(h^{p_2+1}), \quad h \to 0,$$

and the stage order is defined to be the minimum of the $r_i (i = 1, 2, \ldots, s)$ satisfying

$$e_i \equiv Y_i - y(t_n + c_i h) = O(h^{r_i+1}), \quad h \to 0, \quad i = 1, 2, \ldots, s.$$
In these definitions, like in the case of constant coefficient methods, the localizing assumption $y_n = y(t_n) = y'(t_n)$ is of course made, and unlike in that case, the errors are considered in the situation that the coefficients are being changed as the functions of $h$, when $h \to 0$.

In order to analyze the order of accuracy of the FRKN, let us define the quantities:

\[ B(q) = \sum_{i=1}^{s} b_i c_i^{q-1} - \frac{1}{q}, \quad \bar{B}(q) = \sum_{i=1}^{s} \bar{b}_i c_i^{q-1} - \frac{1}{q(q+1)}, \]

\[ \bar{C}_i(q) = \sum_{j=1}^{\bar{a}_{i,j}} \bar{c}_i^{q-1} - \frac{c_i^{q+1}}{q(q+1)}, \quad i = 1, 2, \ldots, s. \]

In (3) expanding $u''_m(t) = \varphi_m(t)$ into their power series, we find

\[ \sum_{q=1}^{\infty} \frac{B(q)}{(q-1)!} h^q \varphi_m^{(q-1)}(t) = 0, \quad m = 1, 2, \ldots, s. \]

\[ \sum_{q=1}^{\infty} \frac{\bar{C}_i(q)}{(q-1)!} h^q \varphi_m^{(q-1)}(t) = 0, \quad i = 1, 2, \ldots, s + 1, \]

where we set $c_{s+1} = 1, \bar{a}_{s+1,j} = \bar{b}_j$ and $\bar{c}_{s+1}(q) = \bar{B}(q)$. For the orders of $B(q)$ and $\bar{C}_i(q)$ we have the following lemma:

**Lemma 1** Let the orders of $B(q), \bar{B}(q)$ and $\bar{C}_i(q)$ ($i = 1, 2, \ldots, s$) be

\[ B(q) = O(h^{\mu_q}), \quad \bar{B}(q) = O(h^{\bar{\mu}_q}), \quad \bar{C}_i(q) = O(h^{\nu_{i,q}}), \]

then for $q = 1, 2, \ldots, s$

\[ \mu_q \geq s + 1 - q, \quad \bar{\mu}_q \geq s + 1 - q, \quad \nu_{i,q} \geq s + 1 - q, \quad i = 1, 2, \ldots, s. \]

**Proof.** Let us define the power series expansion of $B(q)$ as

\[ B(q) = B^{(0)}(q) + B^{(1)}(q) + B^{(2)}(q)h^2 + \cdots, \]

then (9) means

\[ \sum_{l=1}^{\infty} \left( \sum_{q=1}^{l} \varphi_m^{(q-1)}(t) \frac{B^{(l-q)}(q)}{(q-1)!} \right) h^l = 0, \quad m = 1, 2, \ldots, s. \]

Since the coefficients of $h^l$ are 0 for all $l$, we have for $l = 1, 2, \ldots, s$

\[
W(\varphi_1, \varphi_2, \ldots, \varphi_s) \begin{pmatrix}
\beta_{1,1} & \beta_{1,2} & \cdots & \cdots & \beta_{1,s} \\
0 & \beta_{2,2} & \cdots & \cdots & \beta_{2,s} \\
\vdots & \vdots & \ddots & \ddots & \vdots \\
0 & 0 & \cdots & \cdots & 0 \end{pmatrix} = 0,
\]

where we set $\beta_{q,l} = B^{(l-q)}(q)/(q-1)!$. From the nonsingularity of $W$, we have

\[ \beta_{q,q} = \beta_{q,q+1} = \cdots = \beta_{q,s} = 0, \quad q = 1, 2, \ldots, s, \]

which proves the first inequality in (11). The second and third ones are proved in the same way.
Corollary 1. The constant terms of the expansions of $a_{i,j}$, $b_i$ and $\bar{b}_i$ satisfy the so-called simplifying assumption:

\[
\sum_{j=1}^{s} b_{i}^{(0)} c_{i}^{q-1} = \frac{1}{q} \quad q = 1, 2, \ldots, s, \quad (12)
\]

\[
\sum_{j=1}^{s} \bar{b}_{i}^{(0)} c_{i}^{q-1} = \frac{1}{q (q + 1)}, \quad q = 1, 2, \ldots, s, \quad (13)
\]

\[
\sum_{j=1}^{s} \bar{a}_{i,j} c_{j}^{q-1} = \frac{c_{i}^{q+1}}{q (q + 1)}, \quad i = 1, 2, \ldots, s, \quad (14)
\]

These equations determine $\bar{a}_{i,j}^{(0)}$, $b_{i}^{(0)}$, and $\bar{b}_i^{(0)}$ uniquely, since $c_i$ are assumed to be different from each other.

Lemma 2. If $\nu_{i,q}$, $\mu_q$, and $\bar{\mu}_q$ are equal to their lower bounds in (11), i.e.

\[
\nu_{i,q} = \mu_q = \bar{\mu}_q = s + 1 - q, \quad q = 1, 2, \ldots, s, \quad (15)
\]

then for any sufficiently smooth function $g(t)$, we have

\[
g(c_i h) = g(0) + c_i h g'(0) + h^2 \sum_{j=1}^{s} \bar{a}_{i,j} g''(c_j h) + O(h^{s+2})
\]

\[
g(c_i h) = g(0) + c_i h g'(0) + h^2 \sum_{j=1}^{s} \bar{a}_{i,j}^{(0)} g''(c_j h) + O(h^{s+2}), \quad i = 1, 2, \ldots, s + 1, \quad (16)
\]

\[
g'(h) = g'(0) + \sum_{i=1}^{s} b_i^{(0)} g''(c_i h) + O(h^{s+1}) = g'(0) + h \sum_{i=1}^{s} b_i^{(0)} g''(c_i h) + O(h^{s+1}). \quad (17)
\]

Proof. Let $g(t)$ be a sufficiently smooth function, then

\[
g(c_i h) = g(0) + c_i h g'(0) + h^2 \sum_{q=1}^{\infty} \frac{1}{(q - 1)!} (g''(0))^{(q-1)}(0) h^{q-1} \left( \sum_{j=1}^{s} \bar{a}_{i,j} c_{j}^{q-1} - \bar{C}_i(q) \right)
\]

\[
= g(0) + c_i h g'(0) + h^2 \sum_{j=1}^{s} \bar{a}_{i,j} g''(c_j h) - \sum_{q=1}^{s} \frac{\bar{C}_i(q)}{(q - 1)!} h^{q+1} g^{(q+1)}(0) + O(h^{s+2}). \quad (18)
\]

In this expression we have from the assumption of this lemma $\bar{C}_i(q) h^{q+1} = O(h^{s+2})$, which leads to the first relations in (16). The second relation is also proved by noting that (18) is valid even for the case that $\varphi_m(t) = t^{m-1}$ ($m = 1, 2, \ldots, s$), in which case $\bar{a}_{i,j} = \bar{a}_{i,j}^{(0)}$. The proof of (17) is done by the straightforward manner.

Next we consider the stage order of the FRKN for the case that (15) holds. If the solution $y(t)$ of (2) is sufficiently smooth, then we have from the result of Lemma 2

\[
e_i = (1 - h^2 f_y \bar{a}_{i,i})^{-1} h^2 \sum_{j=1}^{s} (\bar{a}_{i,j} e_j f_y + O(e_j^2)) + O(h^{s+2}), \quad i = 1, 2, \ldots, s, \quad (19)
\]
where \( f_y \) is the partial derivative of \( f \) with respect to \( y \), and is assumed to be bounded. Since \( e_i = O(h^{r_i+1}) \) we have from (19) \( r = \min_i \{ r_i \} = \min \{ r + 2, s + 1 \} \), which means \( r = s + 1 \), i.e. the stage order of the method is \( s + 1 \).

Next we consider the order of accuracy of the FRKN. We have also from (15)

\[
E = y_1 - y(h) = h^2 f_y \sum_{i=1}^{s} \tilde{b}_i e_i + O(h^{s+2}) = O(h^{s+2}),
\]

(20)

\[
E' = y'_1 - y'(h) = h f_y \sum_{i=1}^{s} b_i e_i + O(h^{s+1}) = O(h^{s+1}),
\]

which means that the order of accuracy of the method is \( s \). Thus we have proved:

**Theorem 2** The stage order of the functional fitting Runge-Kutta-Nyström method is \( s + 1 \), and the order of accuracy of the method is \( s \), when (15) holds.

### 3 Higher order formula

Here we consider the order of accuracy of the FRKN for the cases that the relations

\[
\mu_q \geq s + 1 - q, \quad \overline{\mu}_q \geq s + 1 - q
\]

hold for \( q = 1, 2, \ldots, s \). We note again that the constant terms \( b_i^{(0)}, \overline{b}_i^{(0)} \) and \( \overline{a}_{ij}^{(0)} \), which are determined uniquely by (12), (13) and (14), respectively, are the coefficients of the direct collocation Runge-Kutta-Nyström method proposed by Van der Houwen et. al [5].

According to [1] and [5], if we take the abscissae \( c_i \) such that

\[
\int_0^1 t^{\nu-1} \prod_{i=1}^{s} (t - c_i) \, dt = 0, \quad q = 1, 2, \ldots, \nu, \quad 1 \leq \nu \leq s,
\]

(21)

then for the \( b_i^{(0)} \) determined by (12), the stronger relation is in fact valid:

\[
\sum_{i=1}^{s} b_i^{(0)} c_i^{q-1} = \frac{1}{q}, \quad q = 1, 2, \ldots, s + \nu.
\]

(22)

Moreover for the \( \overline{b}_i^{(0)} \) determined (13) are related to the \( b_i^{(0)} \) by

\[
\overline{b}_i^{(0)} = b_i^{(0)} (1 - c_i), \quad i = 1, 2, \ldots, s.
\]

(23)

As a result we have instead of (13)

\[
\sum_{i=1}^{s} \overline{b}_i^{(0)} c_i^{q-1} = \frac{1}{q (q + 1)}, \quad q = 1, 2, \ldots, s + \nu - 1.
\]

(24)

Thus from (22) and (24) we have

\[
B^{(0)}(q) = 0, \quad q = 1, 2, \ldots, s + \nu, \quad \overline{B}^{(0)}(q) = 0, \quad q = 1, 2, \ldots, s + \nu - 1.
\]

(25)

When \( \nu > 1 \) in (21) we have the following lemma:
Lemma 3 If $\nu > 1$ then for $1 \leq \xi \leq \nu - 1$,
\[
\sum_{i=1}^{s}b_{i}^{(0)}c_{i}^{\xi-1}a_{i,j}^{(0)} = b_{j}^{(0)} \left( \frac{c_{j}^{\xi+1}}{\xi (\xi + 1)} - \frac{c_{j}}{\xi} + \frac{1}{\xi + 1} \right) \tag{26}.
\]

Proof. Let $\alpha_{j}$ be
\[
\alpha_{j} = \sum_{i=1}^{s}b_{i}^{(0)}c_{i}^{\xi-1}a_{i,j}^{(0)} - b_{j}^{(0)} \left( \frac{c_{j}^{\xi+1}}{\xi (\xi + 1)} - \frac{c_{j}}{\xi} + \frac{1}{\xi + 1} \right), \quad j = 1, 2, \ldots, s,
\]
then from (22) we have $\sum_{j=1}^{s}\alpha_{j}^{(0)}c_{j}^{\xi} = 0 \left( g = 1, 2, \ldots, s \right)$, which means $\alpha_{j} = 0 \left( j = 1, 2, \ldots, s \right)$, since $c_{i}$ are different from each other.

Next we define the quantity $D(q, \xi)$ by
\[
D(q, \xi) = \sum_{i=1}^{s}b_{i}^{(0)}c_{i}^{\xi-1}C_{i}^{(0)}(q) - \frac{1}{q(q+1)} \sum_{i=1}^{s}b_{i}^{(0)}c_{i}^{q+\xi}, \tag{27}
\]
where we set $d_{j} = \sum_{i=1}^{s}b_{i}^{(0)}c_{i}^{\xi-1}a_{i,j}$ and $d_{j}^{(0)} = d_{j}^{(0)}c_{j}^{\xi-1}a_{j}^{(0)} + \cdots$. For $D(q, \xi)$ we can find the two relations which are similar to (10) and (25). The first one is
\[
\sum_{q=1}^{\infty} \frac{D(q, \xi)}{(q-1)!} h^{q} \varphi_{m}(q-1)(t) = 0, \tag{28}
\]
which is easily derived by multiplying both sides of (10) by $b_{i}^{(0)}c_{i}^{\xi-1}$ and summing over $i$. The second one is
\[
D^{(0)}(q, \xi) = 0, \quad q = 1, 2, \ldots, s + \nu - \xi - 1, \quad \xi = 1, 2, \ldots, \nu - 1, \quad \nu > 1, \tag{29}
\]
which can be shown by Lemma 3.

Lemma 4 Consider the function $F(q)$ defined by
\[
F(q) = \sum_{i=1}^{s}f_{i}(h) c_{i}^{\xi-1} - \eta_{q}, \quad q = 1, 2, \ldots, s,
\]
where $f_{i}(h)$ is analytic at $h = 0$, and $\eta_{q}$ depends only on $q$. If the function $F(q)$ satisfies
\[
\sum_{q=1}^{\infty} \frac{F(q)}{(q-1)!} h^{q} \varphi_{m}^{(q-1)}(t) = 0, \quad m = 1, 2, \ldots, s, \tag{30}
\]
and for some $\kappa > 0$
\[
F^{(0)}(q) = 0, \quad q = 1, 2, \ldots, s + \kappa,
\]
then we have
\[
F(q) = O(h^{\tau_{q}}), \quad \tau_{q} = \max\{s + \kappa + 1 - q, \kappa + 1\}, \quad q = 1, 2, \ldots, s + \kappa.
\]
\[
\sum_{i=1}^{s} \left( f_{i}^{(0)} - f_{i}(h) \right) g(c_{i}h) = O(h^{s+\nu}),
\]
(31)

where \( f_{i}^{(0)} \) are the constant terms of the power series expansions of \( f_{i}(h) \), and \( g(t) \) is a sufficiently smooth function.

**Proof.** The proof of this theorem is done in the same manner as in the proof of Lemma 4 in [3].

**Corollary 2** For the orders of \( B(q) \), \( \overline{B}(q) \), and \( D(q, \xi) \), we have for \( \nu > 1 \)

\[
\begin{align*}
B(q) &= O(h^{\mu_q}), \quad \mu_q = \max\{s + \nu + 1 - q, \nu + 1\}, \quad q = 1, 2, \ldots, s + \nu, \\
\overline{B}(q) &= O(h^{\overline{\mu}_q}), \quad \overline{\mu}_q = \max\{s + \nu - q, \nu\}, \quad q = 1, 2, \ldots, s + \nu - 1, \\
D(q, \xi) &= O(h^{\lambda_{q,\xi}}), \quad \lambda_{q,\xi} = \max\{s + \nu - \xi - q, \nu - \xi\}, \quad q = 1, 2, \ldots, s + \nu - \xi - 1,
\end{align*}
\]

and from (31) we have

\[
\begin{align*}
\sum_{i=1}^{s} (b_{i}^{(0)} - b_{i}) g(c_{i}h) &= O(h^{s+\nu}), \\
\sum_{i=1}^{s} (\overline{b}_{i}^{(0)} - \overline{b}_{i}) g(c_{i}h) &= O(h^{s+\nu-1}), \\
\sum_{i=1}^{s} (d_{i}^{(0)} - d_{i}) g(c_{i}h) &= O(h^{s+\nu-\xi-1}), \quad \nu > 1.
\end{align*}
\]

**Lemma 5** If relation (21) holds, then for any sufficiently smooth function \( g(t) \)

\[
\begin{align*}
g(h) &= g(0) + h g'(0) + h^{2} \sum_{j=1}^{s} \tilde{b}_{i} g''(c_{i}h) + O(h^{s+\nu+1}) \\
&= g(0) + h g'(0) + h^{2} \sum_{j=1}^{s} b_{i}^{(0)} g''(c_{i}h) + O(h^{s+\nu+1}), \\
g'(h) &= g'(0) + h \sum_{i=1}^{s} b_{i} g''(c_{i}h) + O(h^{s+\nu+1}) = g'(0) + h \sum_{i=1}^{s} b_{i}^{(0)} g''(c_{i}h) + O(h^{s+\nu+1}).
\end{align*}
\]

**Proof.** This lemma is proved in the same way as Lemma 2.

**Lemma 6** If \( \nu > 1 \), then for \( \xi = 1, 2, \ldots, \nu - 1 \),

\[
\sum_{i=1}^{s} b_{i}^{(0)} c_{i}^{\xi-1} e_{i} = (h^{2} f_{q}) \sum_{i=1}^{s} b_{i}^{(0)} \left( \frac{c_{i}^{\xi+1}}{\xi (\xi + 1)} - \frac{c_{i}}{\xi} + \frac{1}{\xi + 1} \right) e_{i} + O(h^{s+\nu-\xi+1}).
\]

**Proof.** Multiplying both sides of

\[
y(c_{i}h) = y_{0} + c_{i}h y_{0} + h^{2} \sum_{j=1}^{s} \tilde{a}_{i,j} g''(c_{j}h) - \sum_{q=1}^{\frac{h^{q+1}}{(q-1)! \tilde{C}_{i}(q)y^{(q+1)}(0)},}
\]

(33)
by $b_i^{(0)} c_i^{\xi-1}$, and summing for $i$, we have

$$
\sum_{i=1}^{s} b_i^{(0)} c_i^{\xi-1} y(c_i h) = \frac{1}{\xi} y_0 + \frac{1}{\xi + 1} h y_0 + h^2 \sum_{i,j=1}^{s} b_i^{(0)} c_i^{\xi-1} a_{i,j} y''(c_j h) - \sum_{q=1}^{s+\nu-\xi-1} \frac{1}{(q - 1)!} D(q, \xi) y^{(q+1)}(0) + O(h^{s+\nu-\xi+1}).
$$

(34)

Taking into account the relation

$$
q + 1 + \lambda_{q,\xi} \geq s + \nu - \xi + 1, \quad \text{for } q = 1, 2, \ldots, s + \nu - \xi - 1,
$$

and using (32), we have

$$
\sum_{i=1}^{s} b_i^{(0)} c_i^{\xi-1} y(c_i h) = \frac{1}{\xi} y_0 + \frac{1}{\xi + 1} h y_0 + h^2 \sum_{i,j=1}^{s} b_i^{(0)} c_i^{\xi-1} a_{i,j} y''(c_j h) + O(h^{s+\nu-\xi+1}).
$$

(35)

Therefore we have from Lemma 3

$$
\sum_{i=1}^{s} b_i^{(0)} c_i^{\xi-1} e_i = h^2 \sum_{i,j=1}^{s} b_i^{(0)} c_i^{\xi-1} a_{i,j}^{-1}(f_y e_j + O(e_j^2)) + O(h^{s+\nu-\xi+1})
$$

(36)

$$
= (h^2 f_y) \sum_{j=1}^{s} b_j^{(0)} \left( \frac{c_j^{\xi+1}}{\xi (\xi + 1)} - \frac{c_j}{\xi} + \frac{1}{\xi + 1} \right) e_j + O(h^{s+\nu-\xi+1}),
$$

where $e_i = O(h^{s+2})$ is used.

Next we consider the order of accuracy of the method, for the two cases, $\nu = 1$ and $\nu > 1$.

If $\nu = 1$ then we have from (23) and the result of Lemma 5

$$
E' = y_1' - y'(h) = h f_y \sum_{i=1}^{s} b_i^{(0)} e_i + O(h^{s+2}) = O(h^{s+2}),
$$

$$
E = y_1 - y(h) = h^2 f_y \sum_{i=1}^{s} b_i^{(0)} e_i + O(h^{s+2}) = O(h^{s+2}),
$$

so that the method is of order $s + 1$. For $\nu > 1$, we have from Lemma 6

$$
E' = h f_y \sum_{i=1}^{s} b_i^{(0)} e_i + O(h^{s+\nu+1})
$$

$$
= h^3 (f_y)^2 \sum_{i=1}^{s} b_i^{(0)} \left( \frac{c_i^2}{2} - c_i + \frac{1}{2} \right) e_i + O(h^{s+\nu+1})
$$

(37)

$$
= \ldots
$$

$$
= \begin{cases}
    h^{\nu+1} (f_y)^{\frac{\nu+1}{2}} \sum_{i=1}^{s} b_i^{(0)} Q_{\nu}(c_i) e_i + O(h^{s+\nu+1}), & \nu = \text{even} \\
    h^{\nu} (f_y)^{\frac{\nu-1}{2}+1} \sum_{i=1}^{s} b_i^{(0)} Q_{\nu-1}(c_i) e_i + O(h^{s+\nu+1}), & \nu = \text{odd}
\end{cases}
$$

$$
= O(h^{s+\nu+1}).
$$
where $Q_{\nu}(c_{i})$ is a polynomial in $c_{i}$ of degree $\nu$. On the other hand, $E$ is given by

$$E = hE' - h^2 \sum_{i=1}^{s} b_{i}^{(0)} c_{i} e_{i} + O(h^{s+\nu+1}).$$

Evaluating the sum in this expression in the same way, we have $E = O(h^{s+\nu+1})$. Thus we have:

**Theorem 3** If the abscissae $c_{i}$ are taken to satisfy (21), then the order of accuracy of the FRKN is $s + \nu$, for any $\nu$ ($1 \leq \nu \leq s$).

Note that this theorem is a generalization of the theorem (Theorem 3.4 of [5]) which has proved that the order of accuracy of the direct collocation Runge-Kutta-Nyström method with the same abscissae is being $s + \nu$.

**Corollary 3** The attainable order of the FRKN method is $2s$.

### 4 Numerical examples

Consider the 3-stage FRKN method with the abscissae $c_{1} = 0$, $c_{2} = 0.5$, $c_{3} = 1$, and with $\varphi_{1}(t) = \cos \omega t$, $\varphi_{2}(t) = \sin \omega t$, $\varphi_{3}(t) = 1$, which are linearly independent functions when $\omega > 0$. This method is expected to be of order 4, since orthogonal condition (21) holds with $s = 3$ and $\nu = 1$. The equation to be solved is

$$y'' = -y + \varepsilon \cos t, \quad y(0) = 1, \quad y'(0) = 1,$$

which has the exact solution $y(t) = \cos t + \frac{\varepsilon}{2} t \sin t$. We solve the equation by the method with $\omega = 1$ and obtain the global errors at $t = 20$ (see Table 1). We can easily see from Table 1 that the order of accuracy of the method is being 4 for $\varepsilon = 0.05$, and that for $\varepsilon = 0.0$ the method is exact; the values in the column headed with $\varepsilon = 0.0$ must be the accumulations of the round-off errors, since the rounding unit of our computer is $2^{-52} \approx 2.22 \times 10^{-16}$.

Next we consider the well-known two-body problem [2]:

$$y_1'' = -y_1/r^3, \quad y_2'' = -y_2/r^3, \quad r = \sqrt{y_1^2 + y_2^2}$$

$$y_1(0) = 1 - e, \quad y_2(0) = 0, \quad y'_1(0) = 0, \quad y'_2(0) = \sqrt{1 + e/1 - e},$$

where $e (0 \leq e < 1)$ is an eccentricity. The exact solution of this system is given by

$$y_1(t) = \cos u - e, \quad y_2(t) = \sqrt{1 - e^2} \sin u,$$

where $u$ is the solution of Kepler's equation $u = t + e \sin u$. Here we calculate the global errors at $t = 20$ of the two methods, 3-stage FRKN method with $\omega = 1$ and 2-stage Gauss Runge-Kutta method, for various $h$ (see Table 2). From the table we can see that the FRKN method is accurate compared with the 2-stage Gauss Runge-Kutta method.
Table 1. Global errors at $t = 20$ of problem (38).

<table>
<thead>
<tr>
<th>$i$</th>
<th>$\log_{2} R_{i}$</th>
<th>$\log_{2}(R_{i}/R_{i-1})$</th>
<th>$\log_{2} R_{i}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>-15.1</td>
<td></td>
<td>-50.4</td>
</tr>
<tr>
<td>2</td>
<td>-19.0</td>
<td>-3.99</td>
<td>-49.5</td>
</tr>
<tr>
<td>3</td>
<td>-23.0</td>
<td>-4.00</td>
<td>-50.1</td>
</tr>
<tr>
<td>4</td>
<td>-27.0</td>
<td>-4.00</td>
<td>-51.4</td>
</tr>
<tr>
<td>5</td>
<td>-31.0</td>
<td>-4.00</td>
<td>-48.5</td>
</tr>
<tr>
<td>6</td>
<td>-35.0</td>
<td>-4.00</td>
<td>-49.4</td>
</tr>
<tr>
<td>7</td>
<td>-39.0</td>
<td>-4.00</td>
<td>-51.2</td>
</tr>
<tr>
<td>8</td>
<td>-43.0</td>
<td>-3.99</td>
<td>-48.8</td>
</tr>
<tr>
<td>9</td>
<td>-46.6</td>
<td>-3.62</td>
<td>-48.5</td>
</tr>
<tr>
<td>10</td>
<td>-46.1</td>
<td>.500</td>
<td>-47.2</td>
</tr>
</tbody>
</table>

$h = 2^{-i}$, $R_{i} = |y_{n} - y(nh)|$, where $nh = 20$.

Table 2. Global errors at $t = 20$ of the two-body problem.

<table>
<thead>
<tr>
<th>FRKN method</th>
<th>Gauss RK method</th>
</tr>
</thead>
<tbody>
<tr>
<td>$h = 0.200$</td>
<td>$h = 0.100$</td>
</tr>
<tr>
<td>$h = 0.050$</td>
<td>$h = 0.200$</td>
</tr>
<tr>
<td>$h = 0.100$</td>
<td>$h = 0.050$</td>
</tr>
</tbody>
</table>

$e = 0.00$  | 1.119e-13  | 4.186e-14  | 2.242e-13  | 5.839e-04  | 3.658e-05  | 2.290e-06  |
$e = 0.01$  | 1.886e-05  | 1.182e-06  | 7.402e-08  | 5.939e-04  | 3.623e-05  | 2.266e-06  |
$e = 0.10$  | 2.280e-04  | 1.429e-05  | 8.938e-07  | 8.345e-04  | 5.238e-05  | 3.278e-06  |
$e = 0.50$  | 5.665e-03  | 7.101e-04  | 4.897e-05  | 2.121e-02  | 1.493e-03  | 9.551e-05  |

References


