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Order of Accuracy of Functional Fitting Runge-Kutta-Nyström Formula

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1 Introduction

In this paper we introduce functional fitting Runge-Kutta-Nyström method which integrates some set of functions exactly. The method proposed here is a generalization of exponentially or trigonometrically fitting Runge-Kutta-Nyström methods.

2 Functional fitting Runge-Kutta-Nyström method

Consider the variable coefficient Runge-Kutta-Nyström method

$$\begin{cases} y_{n+1} = y_n + h\,y_n' + h^2 \sum_{i=1}^{s} \bar{b}_i(t_n, h)\, f(t_n + c_i h,\, Y_i), \\ y_{n+1}' = y_n' + h \sum_{i=1}^{s} b_i(t_n, h)\, f(t_n + c_i h, Y_i), \\ Y_i = y_n + c_i\, h y_n' + h^2 \sum_{j=1}^{s} \bar{a}_{i,j}(t_n,\, h)\, f(t_n + c_j h, Y_j), \end{cases} \tag{1}$$

for solving the second order ODE of the form

$$y''(t) = f(t, y), \quad y(0) = y_0, \quad y'(0) = y_0', \quad t \in [t_0, T]. \tag{2}$$

We will call the method *functional fitting Runge-Kutta-Nyström* (FRKN) method, when the method is designed to integrate some functions exactly. The coefficients $\bar{a}_{i,j}$, b_i and $\bar{b}_i$ of the FRKN to be considered here are determined by the simultaneous equation

$$\begin{cases} u_m(t+h) = u_m(t) + h\,u_m'(t) + h^2 \sum_{i=1}^{s} \bar{b}_i(t, h)\, u_m''(t + c_i h), \\ u_m'(t+h) = u_m'(t) + h \sum_{i=1}^{s} b_i(t, h)\, u_m''(t + c_i h), \\ u_m(t + c_i h) = u_m(t) + c_i\, h\, u_m'(t) + h^2 \sum_{j=1}^{s} \bar{a}_{i,j}(t, h)\, u_m''(t + c_j h), \quad i = 1, 2, \dots, s, \end{cases} \tag{3}$$

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where the functions $u''_m(t) = \varphi_m(t)$ are linearly independent on $[t_0, T]$, that is the Wronskian matrix W given by

$$W(\varphi_1, \varphi_2, \cdots, \varphi_s) \equiv \begin{pmatrix} \varphi_1(t) & \varphi'_1(t) & \cdots & \varphi_1^{(s-1)}(t) \\ \varphi_2(t) & \varphi'_2(t) & \cdots & \varphi_2^{(s-1)}(t) \\ \cdots & \cdots & \cdots & \cdots \\ \varphi_s(t) & \varphi'_s(t) & \cdots & \varphi_s^{(s-1)}(t) \end{pmatrix} \tag{4}$$

is nonsingular for all $t \in [t_0, T]$. For the uniqueness of the coefficients we have:

Theorem 1 *The coefficients $\bar{a}_{i,j}(t,h)$, $b_i(t,h)$ and $\bar{b}_i(t,h)$ determined by (3) are unique for small $h > 0$, if the functions $\varphi_m(t)$ are sufficiently smooth and linearly independent.*

Proof. It is clear from (3) that these coefficients are unique, if the matrix given by

$$\Phi(t, h) = \begin{pmatrix} \varphi_1(t + c_1 h) & \varphi_1(t + c_2 h) & \dots & \varphi_1(t + c_s h) \\ \varphi_2(t + c_1 h) & \varphi_2(t + c_2 h) & \dots & \varphi_2(t + c_s h) \\ \vdots & \vdots & \dots & \vdots \\ \varphi_s(t + c_1 h) & \varphi_s(t + c_2 h) & \dots & \varphi_s(t + c_s h) \end{pmatrix}$$

is nonsingular. The matrix Φ can be expressed as

$$\Phi(t, h) = W(\varphi_1, \varphi_2, \dots, \varphi_s) \begin{pmatrix} 1 & 1 & \cdots & 1 \\ c_1 h & c_2 h & \cdots & c_s h \\ \vdots & \vdots & \cdots & \vdots \\ \frac{(c_1 h)^{s-1}}{(s-1)!} & \frac{(c_2 h)^{s-1}}{(s-1)!} & \cdots & \frac{(c_s h)^{s-1}}{(s-1)!} \end{pmatrix} + O(h^s).$$

Since W is assumed to be nonsingular, and the second matrix in the right-hand side is also nonsingular from the assumption that c_i are different from each other, then we have the nonsingularity of $\Phi(t, h)$ for sufficiently small $h > 0$. ∎

Hereafter, we simply denote the coefficients by $\bar{a}_{i,j}$, b_i and $\bar{b}_i$, and denote their power series expansions in h by

$$\begin{aligned} \bar{a}_{i,j} &= \bar{a}_{i,j}^{(0)} + \bar{a}_{i,j}^{(1)} h + \bar{a}_{i,j}^{(2)} h^2 + \cdots, \qquad b_i = b_i^{(0)} + b_i^{(1)} h + b_i^{(2)} h^2 + \cdots, \\ \bar{b}_i &= \bar{b}_i^{(0)} + \bar{b}_i^{(1)} h + \bar{b}_i^{(2)} h^2 + \cdots \end{aligned} \tag{5}$$

If we take $u''_m(t) = t^{m-1}\,(m = 1, 2, \dots, s)$ in (3), then $\bar{a}_{i,j} = \bar{a}_{i,j}^{(0)}, b_i = b_i^{(0)}$, and $\bar{b}_i = \bar{b}_i^{(0)}$, and the method reduces to the direct collocation Runge-Kutta-Nyström method proposed by Van der Houwen et al [5].

Here we consider the order of accuracy of the FRKN method. The order of accuracy of the FRKN is defined to be $p = \min\{p_1, p_2\}$, where p_1 and p_2 are the integers satisfying

$$E := y_{n+1} - y(t_{n+1}) = O(h^{p_1+1}), \quad E' := y'_{n+1} - y'(t_{n+1}) = O(h^{p_2+1}), \quad h \to 0, \tag{6}$$

and the stage order is defined to be the minimum of the $r_i\,(i = 1, 2, \dots, s)$ satisfying

$$e_i \equiv Y_i - y(t_n + c_i h) = O(h^{r_i+1}), \quad h \to 0, \quad i = 1, 2, \dots, s. \tag{7}$$

In these definitions, like in the case of constant coefficient methods, the localizing assumption $y_n = y(t_n)$, $y'_n = y'(t_n)$ is of course made, and unlike in that case, the errors are considered in the situation that the coefficients are being changed as the functions of h, when $h \to 0$.

In order to analyze the order of accuracy of the FRKN, let us define the quantities:

$$\begin{aligned} B(q) &= \sum_{i=1}^{s} b_i\, c_i^{q-1} - \frac{1}{q}, \qquad \bar{B}(q) = \sum_{i=1}^{s} \bar{b}_i\, c_i^{q-1} - \frac{1}{q\,(q+1)}, \\ \bar{C}_i(q) &= \sum_{j=1}^{s} \bar{a}_{i,j}\, c_j^{q-1} - \frac{c_i^{q+1}}{q\,(q+1)}, \quad i = 1, 2, \dots, s. \end{aligned} \tag{8}$$

In (3) expanding $u''_m(t) = \varphi_m(t)$ into their power series, we find

$$\sum_{q=1}^{\infty} \frac{B(q)}{(q-1)!} h^q \varphi_m^{(q-1)}(t) = 0, \tag{9}$$

$$\sum_{q=1}^{\infty} \frac{\bar{C}_i(q)}{(q-1)!} h^q \varphi_m^{(q-1)}(t) = 0, \quad i = 1, 2, \dots, s+1, \tag{10}$$

where we set $c_{s+1} = 1$, $\bar{a}_{s+1,j} = \bar{b}_j$ and $\bar{C}_{s+1}(q) = \bar{B}(q)$. For the orders of $B(q)$ and $\bar{C}_i(q)$ we have the following lemma:

Lemma 1 *Let the orders of* $B(q)$, $\bar{B}(q)$ *and* $\bar{C}_i(q)$ $(i = 1, 2, \dots, s)$ *be*

$$B(q) = O(h^{\mu_q}), \quad \bar{B}(q) = O(h^{\bar{\mu}_q}), \quad \bar{C}_i(q) = O(h^{\nu_{i,q}}),$$

then for $q = 1, 2, \dots, s$

$$\mu_q \geq s + 1 - q, \quad \bar{\mu}_q \geq s + 1 - q, \quad \nu_{i,q} \geq s + 1 - q, \quad i = 1, 2, \dots, s. \tag{11}$$

Proof. Let us define the power series expansion of $B(q)$ as

$$B(q) = B^{(0)}(q) + B^{(1)}(q) + B^{(2)}(q)h^2 + \cdots,$$

then (9) means

$$\sum_{l=1} \left(\sum_{q=1}^{l} \varphi_m^{(q-1)}(t)\, \frac{B^{(l-q)}(q)}{(q-1)!} \right) h^l = 0, \quad m = 1, 2, \dots, s.$$

Since the coefficients of h^l are 0 for all l, we have for $l = 1, 2, \dots, s$

$$W(\varphi_1, \varphi_2, \dots, \varphi_s) \begin{pmatrix} \beta_{1,1} & \beta_{1,2} & \cdots & \cdots & \beta_{1,s} \\ 0 & \beta_{2,2} & \cdots & \cdots & \beta_{2,s} \\ 0 & 0 & \ddots & \cdots & \cdots \\ \vdots & \vdots & \ddots & \ddots & \cdots \\ 0 & 0 & \cdots & 0 & \beta_{s,s} \end{pmatrix} = 0,$$

where we set $\beta_{q,l} = B^{(l-q)}(q)/(q-1)!$ From the nonsingularity of W, we have

$$\beta_{q,q} = \beta_{q,q+1} = \cdots = \beta_{q,s} = 0, \quad q = 1, 2, \dots, s,$$

which proves the first inequality in (11). The second and third ones are proved in the same way. ∎

Corollary 1 *The constant terms of the expansions of $a_{i,j}$, b_i and $\bar{b}_i$ satisfy the so-called simplifying assumption:*

$$\sum_{j=1}^{s} b_i^{(0)} c_i^{q-1} = \frac{1}{q}, \quad q = 1, 2, \dots, s, \tag{12}$$

$$\sum_{j=1}^{s} \bar{b}_i^{(0)} c_i^{q-1} = \frac{1}{q(q+1)}, \quad q = 1, 2, \dots, s, \tag{13}$$

$$\sum_{j=1}^{s} \bar{a}_{i,j}^{(0)} c_j^{q-1} = \frac{c_i^{q+1}}{q(q+1)}, \quad i = 1, 2, \dots, s, \quad q = 1, 2, \dots, s. \tag{14}$$

These equations determine $\bar{a}_{i,j}^{(0)}$, $b_i^{(0)}$ and $\bar{b}_i^{(0)}$ uniquely, since c_i are assumed to be different from each other.

Lemma 2 *If $\nu_{i,q}, \mu_q$ and $\bar{\mu}_q$ are equal to their lower bounds in (11), i.e.*

$$\nu_{i,q} = \mu_q = \bar{\mu}_q = s + 1 - q, \quad q = 1, 2, \dots, s, \tag{15}$$

then for any sufficiently smooth function $g(t)$, we have

$$\begin{aligned} g(c_i h) &= g(0) + c_i h g'(0) + h^2 \sum_{j=1}^{s} \bar{a}_{i,j}\, g''(c_j h) + O(h^{s+2}) \\ &= g(0) + c_i h g'(0) + h^2 \sum_{j=1}^{s} \bar{a}_{i,j}^{(0)} g''(c_j h) + O(h^{s+2}), \quad i = 1, 2, \dots, s+1, \end{aligned} \tag{16}$$

$$g'(h) = g'(0) + h \sum_{i=1}^{s} b_i\, g''(c_i h) + O(h^{s+1}) = g'(0) + h \sum_{i=1}^{s} b_i^{(0)} g''(c_i h) + O(h^{s+1}). \tag{17}$$

Proof. Let $g(t)$ be a sufficiently smooth function, then

$$\begin{aligned} g(c_i h) &= g(0) + c_i h g'(0) + h^2 \sum_{q=1}^{\infty} \frac{1}{(q-1)!} (g'')^{(q-1)}(0)\, h^{q-1} \left(\sum_{j=1}^{s} \bar{a}_{i,j} c_j^{q-1} - \bar{C}_i(q) \right) \\ &= g(0) + c_i h g'(0) + h^2 \sum_{j=1}^{s} \bar{a}_{i,j}\, g''(c_j h) - \sum_{q=1}^{s} \frac{\bar{C}_i(q)}{(q-1)!} h^{q+1} g^{(q+1)}(0) + O(h^{s+2}). \end{aligned} \tag{18}$$

In this expression we have from the assumption of this lemma $\bar{C}_i(q) h^{q+1} = O(h^{s+2})$, which leads to the first relations in (16). The second relation is also proved by noting that (18) is valid even for the case that $\varphi_m(t) = t^{m-1}$ ($m = 1, 2, \dots, s$), in which case $\bar{a}_{i,j} = \bar{a}_{i,j}^{(0)}$. The proof of (17) is done by the straightforward manner. ∎

Next we consider the stage order of the FRKN for the case that (15) holds. If the solution $y(t)$ of (2) is sufficiently smooth, then we have from the result of Lemma 2

$$e_i = (1 - h^2 f_y \bar{a}_{i,i})^{-1} h^2 \sum_{\substack{j \neq i \\ j=1}}^{s} (\bar{a}_{i,j} e_j f_y + O(e_j^2)) + O(h^{s+2}), \quad i = 1, 2, \dots, s, \tag{19}$$

where f_y is the partial derivative of f with respect to y, and is assumed to be bounded. Since $e_i = O(h^{r_i+1})$ we have from (19) $r = \min_i\{r_i\} = \min\{r+2,\, s+1\}$, which means $r = s+1$, i.e. the stage order of the method is $s+1$.

Next we consider the order of accuracy of the FRKN. We have also from (15)

$$\begin{aligned} E &= y_1 - y(h) = h^2 f_y \sum_{i=1}^{s} \bar{b}_i\, e_i + O(h^{s+2}) = O(h^{s+2}), \\ E' &= y_1' - y'(h) = h f_y \sum_{i=1}^{s} b_i\, e_i + O(h^{s+1}) = O(h^{s+1}), \end{aligned} \tag{20}$$

which means that the order of accuracy of the method is s. Thus we have proved:

Theorem 2 *The stage order of the functional fitting Runge-Kutta-Nyström method is $s+1$, and the order of accuracy of the method is s, when (15) holds.*

3 Higher order formula

Here we consider the order of accuracy of the FRKN for the cases that the relations

$$\mu_q \geq s+1-q, \quad \bar{\mu}_q \geq s+1-q$$

hold for $q = 1, 2, \ldots, s$. We note again that the constant terms $b_i^{(0)}$, $\bar{b}_i^{(0)}$ and $\bar{a}_{ij}^{(0)}$, which are determined uniquely by (12), (13) and (14), respectively, are the coefficients of the direct collocation Runge-Kutta-Nyström method proposed by Van der Houwen et. al [5]. According to [1] and [5], if we take the abscissae c_i such that

$$\int_0^1 t^{q-1} \prod_{i=1}^{s} (t - c_i)\, \mathrm{d}t = 0, \quad q = 1, 2, \ldots, \nu, \quad 1 \leq \nu \leq s, \tag{21}$$

then for the $b_i^{(0)}$ determined by (12), the stronger relation is in fact valid:

$$\sum_{i=1}^{s} b_i^{(0)} c_i^{q-1} = \frac{1}{q}, \quad q = 1, 2, \ldots, s+\nu. \tag{22}$$

Moreover for the $\bar{b}_i^{(0)}$ determined (13) are related to the $b_i^{(0)}$ by

$$\bar{b}_i^{(0)} = b_i^{(0)}\,(1 - c_i), \quad i = 1, 2, \ldots, s. \tag{23}$$

As a result we have instead of (13)

$$\sum_{i=1}^{s} \bar{b}_i^{(0)} c_i^{q-1} = \frac{1}{q\,(q+1)}, \quad q = 1, 2, \ldots, s+\nu-1. \tag{24}$$

Thus from (22) and (24) we have

$$B^{(0)}(q) = 0, \quad q = 1, 2, \ldots, s+\nu, \quad \bar{B}^{(0)}(q) = 0, \quad q = 1, 2, \ldots, s+\nu-1. \tag{25}$$

When $\nu > 1$ in (21) we have the following lemma:

Lemma 3 *If $\nu > 1$ then for $1 \le \xi \le \nu - 1$,*

$$\sum_{i=1}^{s} b_i^{(0)} c_i^{\xi-1} \bar{a}_{i,j}^{(0)} = b_j^{(0)} \left(\frac{c_j^{\xi+1}}{\xi(\xi+1)} - \frac{c_j}{\xi} + \frac{1}{\xi+1} \right). \tag{26}$$

Proof. Let α_j be

$$\alpha_j = \sum_{i=1}^{s} b_i^{(0)} c_i^{\xi-1} \bar{a}_{i,j}^{(0)} - b_j^{(0)} \left(\frac{c_j^{\xi+1}}{\xi(\xi+1)} - \frac{c_j}{\xi} + \frac{1}{\xi+1} \right), \quad j = 1, 2, \ldots, s,$$

then from (22) we have $\sum_{j=1}^{s} \alpha_j c_j^{q-1} = 0\,(q = 1, 2, \ldots, s)$, which means $\alpha_j = 0$ $(j = 1, 2, \ldots, s)$, since c_i are different from each other. ∎

Next we define the quantity $D(q, \xi)$ by

$$D(q, \xi) = \sum_{i=1}^{s} b_i^{(0)} c_i^{\xi-1} \bar{C}_i(q) = \sum_{j=1}^{s} d_j c_j^{q-1} - \frac{1}{q(q+1)} \sum_{i=1}^{s} b_i^{(0)} c_i^{q+\xi}, \tag{27}$$

where we set $d_j = \sum_{i=1}^{s} b_i^{(0)} c_i^{\xi-1} \bar{a}_{i,j}$ and $d_j = d_j^{(0)} + d_j^{(1)} h + \cdots$. For $D(q, \xi)$ we can find the two relations which are similar to (10) and (25). The first one is

$$\sum_{q=1}^{\infty} \frac{D(q, \xi)}{(q-1)!} h^q \varphi_m^{(q-1)}(t) = 0, \tag{28}$$

which is easily derived by multiplying both sides of (10) by $b_i^{(0)} c_i^{\xi-1}$ and summing over i. The second one is

$$D^{(0)}(q, \xi) = 0, \quad q = 1, 2, \ldots, s + \nu - \xi - 1, \quad \xi = 1, 2, \ldots, \nu - 1, \quad \nu > 1, \tag{29}$$

which can be shown by Lemma 3.

Lemma 4 *Consider the function $F(q)$ defined by*

$$F(q) = \sum_{i=1}^{s} f_i(h) c_i^{q-1} - \eta_q, \quad q = 1, 2, \ldots, s,$$

where $f_i(h)$ is analytic at $h = 0$, and η_q depends only on q. If the function $F(q)$ satisfies

$$\sum_{q=1} \frac{F(q)}{(q-1)!} h^q \varphi_m^{(q-1)}(t) = 0, \quad m = 1, 2, \ldots, s, \tag{30}$$

and for some $\kappa > 0$

$$F^{(0)}(q) = 0, \quad q = 1, 2, \ldots, s + \kappa,$$

then we have

$$F(q) = O(h^{\tau_q}), \quad \tau_q = \max\{s + \kappa + 1 - q,\, \kappa + 1\}, \quad q = 1, 2, \ldots, s + \kappa,$$

$$\sum_{i=1}^{s}(f_i^{(0)} - f_i(h))\, g(c_i h) = O(h^{s+\kappa}), \tag{31}$$

where $f_i^{(0)}$ *are the constant terms of the power series expansions of* $f_i(h)$*, and* $g(t)$ *is a sufficiently smooth function.*

Proof. The proof of this theorem is done in the same manner as in the proof of Lemma 4 in [3].

Corollary 2 *For the orders of* $B(q)$, $\bar{B}(q)$ *and* $D(q,\, \xi)$*, we have for* $\nu > 1$

$$\begin{aligned} &B(q) = O(h^{\mu_q}), \quad \mu_q = \max\{s+\nu+1-q,\, \nu+1\}, \quad q = 1,2,\ldots,s+\nu, \\ &\bar{B}(q) = O(h^{\bar{\mu}_q}), \quad \bar{\mu}_q = \max\{s+\nu-q,\, \nu\}, \quad q = 1,2,\ldots,s+\nu-1, \\ &D(q,\, \xi) = O(h^{\lambda_{q,\xi}}), \quad \lambda_{q,\xi} = \max\{s+\nu-\xi-q,\, \nu-\xi\}, \quad q = 1,2,\ldots,s+\nu-\xi-1, \end{aligned}$$

and from (31) we have

$$\begin{aligned} &\sum_{i=1}^{s}(b_i^{(0)} - b_i)\, g(c_i h) = O(h^{s+\nu}), \qquad \sum_{i=1}^{s}(\bar{b}_i^{(0)} - \bar{b}_i)\, g(c_i h) = O(h^{s+\nu-1}), \\ &\sum_{i=1}^{s}(d_i^{(0)} - d_i)\, g(c_i h) = O(h^{s+\nu-\xi-1}), \quad \nu > 1. \end{aligned} \tag{32}$$

Lemma 5 *If relation (21) holds, then for any sufficiently smooth function* $g(t)$

$$\begin{aligned} g(h) &= g(0) + h\, g'(0) + h^2 \sum_{j=1}^{s} \bar{b}_i\, g''(c_i h) + O(h^{s+\nu+1}) \\ &= g(0) + h\, g'(0) + h^2 \sum_{j=1}^{s} \bar{b}_i^{(0)}\, g''(c_i h) + O(h^{s+\nu+1}), \end{aligned}$$

$$g'(h) = g'(0) + h \sum_{i=1}^{s} b_i\, g''(c_i h) + O(h^{s+\nu+1}) = g'(0) + h \sum_{i=1}^{s} b_i^{(0)}\, g''(c_i h) + O(h^{s+\nu+1}).$$

Proof. This lemma is proved in the same way as Lemma 2.

Lemma 6 *If* $\nu > 1$*, then for* $\xi = 1, 2, \ldots, \nu - 1$,

$$\sum_{i=1}^{s} b_i^{(0)}\, c_i^{\xi-1}\, e_i = (h^2 f_y) \sum_{i=1}^{s} b_i^{(0)} \left(\frac{c_i^{\xi+1}}{\xi\,(\xi+1)} - \frac{c_i}{\xi} + \frac{1}{\xi+1} \right) e_i + O(h^{s+\nu-\xi+1}).$$

Proof. Multiplying both sides of

$$y(c_i h) = y_0 + c_i h y_0' + h^2 \sum_{j=1}^{s} \bar{a}_{i,j} y''(c_j h) - \sum_{q=1} \frac{h^{q+1}}{(q-1)!} \bar{C}_i(q) y^{(q+1)}(0), \tag{33}$$

by $b_i^{(0)} c_i^{\xi-1}$, and summing for i, we have

$$\begin{aligned}\sum_{i=1}^{s} b_i^{(0)}\, c_i^{\xi-1}\, y(c_i h) = \frac{1}{\xi}\, y_0 + \frac{1}{\xi+1}\, h y_0' + h^2 \sum_{i,j=1}^{s} b_i^{(0)}\, c_i^{\xi-1} \bar{a}_{i,j}\, y''(c_j h) \\ - \sum_{q=1}^{s+\nu-\xi-1} \frac{h^{q+1}}{(q-1)!}\, D(q,\, \xi)\, y^{(q+1)}(0) + O(h^{s+\nu-\xi+1}).\end{aligned} \tag{34}$$

Taking into account the relation

$$q + 1 + \lambda_{q,\xi} \geq s + \nu - \xi + 1, \quad \text{for } \ q = 1, 2, \ldots, s + \nu - \xi - 1,$$

and using (32), we have

$$\sum_{i=1}^{s} b_i^{(0)}\, c_i^{\xi-1}\, y(c_i h) = \frac{1}{\xi}\, y_0 + \frac{1}{\xi+1}\, h y_0' + h^2 \sum_{i,j=1}^{s} b_i^{(0)}\, c_i^{\xi-1} \bar{a}_{i,j}^{(0)}\, y''(c_j h) + O(h^{s+\nu-\xi+1}). \tag{35}$$

Therefore we have from Lemma 3

$$\begin{aligned}\sum_{i=1}^{s} b_i^{(0)}\, c_i^{\xi-1}\, e_i &= h^2 \sum_{i,j=1}^{s} b_i^{(0)} c_i^{\xi-1}\, \bar{a}_{i,j}^{(0)}\, (f_y\, e_j + O(e_j^2)) + O(h^{s+\nu-\xi+1}) \\ &= (h^2 f_y) \sum_{j=1}^{s} b_j^{(0)} \left(\frac{c_j^{\xi+1}}{\xi\,(\xi+1)} - \frac{c_j}{\xi} + \frac{1}{\xi+1} \right) e_j + O(h^{s+\nu-\xi+1}),\end{aligned} \tag{36}$$

where $e_i = O(h^{s+2})$ is used. ∎

Next we consider the order of accuracy of the method, for the two cases, $\nu = 1$ and $\nu > 1$. If $\nu = 1$ then we have from (23) and the result of Lemma 5

$$E' = y_1' - y'(h) = h\, f_y \sum_{i=1}^{s} b_i^{(0)}\, e_i + O(h^{s+2}) = O(h^{s+2}),$$

$$E = y_1 - y(h) = h^2\, f_y \sum_{i=1}^{s} \bar{b}_i^{(0)}\, e_i + O(h^{s+2}) = O(h^{s+2}),$$

so that the method is of order $s + 1$. For $\nu > 1$, we have from Lemma 6

$$\begin{aligned}E' &= h\, f_y \sum_{i=1}^{s} b_i^{(0)}\, e_i + O(h^{s+\nu+1}) \\ &= h^3\, (f_y)^2 \sum_{i=1}^{s} b_i^{(0)} \left(\frac{c_i^2}{2} - c_i + \frac{1}{2} \right) e_i + O(h^{s+\nu+1}) \\ &= \cdots \\ &= \begin{cases} h^{\nu+1}\, (f_y)^{\frac{\nu}{2}+1} \displaystyle\sum_{i=1}^{s} b_i^{(0)}\, Q_\nu(c_i)\, e_i + O(h^{s+\nu+1}), & \nu = \text{even} \\ h^{\nu}\, (f_y)^{\frac{\nu-1}{2}+1} \displaystyle\sum_{i=1}^{s} b_i^{(0)}\, Q_{\nu-1}(c_i)\, e_i + O(h^{s+\nu+1}), & \nu = \text{odd} \end{cases} \\ &= O(h^{s+\nu+1}),\end{aligned} \tag{37}$$

where $Q_\nu(c_i)$ is a polynomial in c_i of degree ν. On the other hand, E is given by

$$E = hE' - h^2 f_y \sum_{i=1}^{s} b_i^{(0)} c_i e_i + O(h^{s+\nu+1}).$$

Evaluating the sum in this expression in the same way, we have $E = O(h^{s+\nu+1})$. Thus we have:

Theorem 3 *If the abscissae c_i are taken to satisfy (21), then the order of accuracy of the FRKN is $s + \nu$, for any ν $(1 \le \nu \le s)$.*

Note that this theorem is a generalization of the theorem (Theorem 3.4 of [5]) which has proved that the order of accuracy of the direct collocation Runge-Kutta-Nyström method with the same abscissae is being $s + \nu$.

Corollary 3 *The attainable order of the FRKN method is $2s$.*

4 Numerical examples

Consider the 3-stage FRKN method with the abscissae $c_1 = 0$, $c_2 = 0.5$, $c_3 = 1$, and with $\varphi_1(t) = \cos\omega t$, $\varphi_2(t) = \sin\omega t$, $\varphi_3(t) = 1$, which are linearly independent functions when $\omega > 0$. This method is expected to be of order 4, since orthogonal condition (21) holds with $s = 3$ and $\nu = 1$. The equation to be solved is

$$y'' = -y + \varepsilon \cos t, \quad y(0) = 1, \quad y'(0) = 1, \tag{38}$$

which has the exact solution $y(t) = \cos t + \frac{1}{2}\varepsilon t \sin t$. We solve the equation by the method with $\omega = 1$ and obtain the global errors at $t = 20$ (see Table 1). We can easily see from Table 1 that the order of accuracy of the method is being 4 for $\varepsilon = 0.05$, and that for $\varepsilon = 0.0$ the method is exact; the values in the column headed with $\varepsilon = 0.0$ must be the accumulations of the round-off errors, since the rounding unit of our computer is $2^{-52} \simeq 2.22 \times 10^{-16}$.

Next we consider the well-known two-body problem [2]:

$$\begin{aligned} &y_1'' = -y_1/r^3, \quad y_2'' = -y_2/r^3, \quad r = \sqrt{y_1^2 + y_2^2} \\ &y_1(0) = 1 - e, \quad y_2(0) = 0, \quad y_1'(0) = 0, \quad y_2'(0) = \sqrt{\frac{1+e}{1-e}}, \end{aligned} \tag{39}$$

where $e\,(0 \le e < 1)$ is an eccentricity. The exact solution of this system is given by

$$y_1(t) = \cos u - e, \quad y_2(t) = \sqrt{1 - e^2} \sin u, \tag{40}$$

where u is the solution of Kepler's equation $u = t + e \sin u$. Here we calculate the global errors at $t = 20$ of the two methods, 3-stage FRKN method with $\omega = 1$ and 2-stage Gauss Runge-Kutta method, for various h (see Table 2). From the table we can see that the FRKN method is accurate compared with the 2-stage Gauss Runge-Kutta method.

Table 1. Global errors at $t = 20$ of problem (38).

	$\varepsilon = 0.05$		$\varepsilon = 0.0$
i	$\log_2 R_i$	$\log_2(R_i/R_{i-1})$	$\log_2 R_i$
1	-15.1		-50.4
2	-19.0	-3.99	-49.5
3	-23.0	-4.00	-50.1
4	-27.0	-4.00	-51.4
5	-31.0	-4.00	-48.5
6	-35.0	-4.00	-49.4
7	-39.0	-4.00	-51.2
8	-43.0	-3.99	-48.8
9	-46.6	-3.62	-48.5
10	-46.1	.500	-47.2

$h = 2^{-i}$, $R_i = |y_n - y(nh)|$, where $nh = 20$.

Table 2. Global errors at $t = 20$ of the two-body problem.

	FRKN method			Gauss RK method		
	h=0.200	h=0.100	h=0.050	h=0.200	h=0.100	h=0.050
e=0.00	1.119e-13	4.186e-14	2.242e-13	5.839e-04	3.658e-05	2.290e-06
e=0.01	1.886e-05	1.182e-06	7.402e-08	5.939e-04	3.623e-05	2.266e-06
e=0.10	2.280e-04	1.429e-05	8.938e-07	8.345e-04	5.238e-05	3.278e-06
e=0.50	5.665e-03	7.101e-04	4.897e-05	2.121e-02	1.493e-03	9.551e-05

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