数理解析研究所講究録
1145 巻 2000 年 188-193

Numerical Inversion of the Laplace Transform Using a Continuous Euler Transformation

京都大学数理解析研究所 大浦拓哉 (Takuya Ooura)

Research Institute for Mathematical Sciences, Kyoto University

1 Introduction

The Laplace transform of function $g(t)$ is defined by

$$G(s)=\int_{0}^{\infty}e^{-st}g(t)\,dt \tag{1.1}$$

and its inversion is given by

$$g(t)=\frac{1}{2\pi i}\int_{\gamma-i\infty}^{\gamma+i\infty}e^{st}G(s)\,ds\ ,\quad \gamma>\sigma\ , \tag{1.2}$$

where σ is the abscissa of convergence for (1.1). It is known that numerical evaluation of the integral (1.2) is difficult [2]. The integral (1.2) is a Fourier type integral:

$$g(t)=\frac{e^{\gamma t}}{2\pi}\int_{-\infty}^{\infty}e^{itx}G(\gamma+ix)\,dx \tag{1.3}$$

with a slowly convergent integrand. Accordingly its computation accompanies a serious difficulty [1]. Recently, the author proposed a continuous Euler transformation that accelerates the convergence of the integral of such type and we showed that the continuous Euler transformation is efficient in computing the Fourier transforms of slowly decaying function by using FFT [5].

In this paper, we apply the continuous Euler transformation to the numerical evaluation of the integral (1.3) and propose a new method to compute numerical inversion of the Laplace transform using the continuous Euler transformation and FFT.

2 Continuous Euler transformation

An alternating series

$$S=\sum_{n=0}^{\infty}(-1)^{n}f(n)=\sum_{n=0}^{\infty}e^{i\pi n}f(n) \tag{2.1}$$

is a discrete approximation to a Fourier type integral

$$I=\int_{0}^{\infty}e^{i\pi x}f(x)\,dx \tag{2.2}$$

with a mesh size 1. So we may expect that a continuous version of the Euler transformation, if any, will accelerate the convergence of integrals of Fourier type.

2.1 Definition of the Conventional Euler Transformation

The Euler transformation truncated at the N-th term of an alternating series

$$S = \sum_{n=0}^{\infty} (-1)^n f(n) \tag{2.3}$$

is defined by

$$S_{\text{Euler}}^{(N)} = \sum_{m=0}^{N-1} w_m^{(N)} (-1)^m f(m) \,, \quad w_m^{(N)} = \sum_{n=m+1}^{N} \frac{1}{2^N} \binom{N}{n} \,, \tag{2.4}$$

where $w_m^{(N)}$ are the weights of the linear sequence transformation [7]. The weights $w_m^{(N)}$ happen to be the probability of the binomial distribution.

2.2 Definition of the Continuous Euler Transformation

We define the continuous Euler transformation for the integral

$$I = \int_0^{\infty} e^{i\omega x} f(x)\, dx \,, \quad \omega > 0 \tag{2.5}$$

by

$$I_w^{(L)} = \int_0^L w(x;p,q) f(x) e^{i\omega x}\, dx \,, \quad w(x;p,q) = \int_{x/p-q}^{\infty} \phi(t)\, dt. \tag{2.6}$$

Where $\phi(t)$ is a prescribed function satisfying

$$\int_{-\infty}^{\infty} \phi(t)\, dt = 1 \,, \quad \lim_{t \to \pm\infty} \phi(t) = 0 \tag{2.7}$$

and p and q are constants depending on L and ω.

2.3 Convergence of the Continuous Euler Transformation

In this section, We take the following w

$$w(x;p,q) = \int_{x/p-q}^{\infty} \frac{1}{\sqrt{\pi}} e^{-t^2}\, dt = \frac{1}{2} \operatorname{erfc}(x/p - q) \tag{2.8}$$

and choose $L = 2pq$. We then have the following.

Theorem 1 We assume that $f(z)$ is regular in a domain $|\arg(z + 1/2)| \le \delta$ with $0 < \delta < \pi/2$, that $|f(z)| \le M$ in the domain, and that

$$\lim_{R \to \infty} \max_{|\theta| \le \delta} |f(R - 1/2 + iR \tan\theta)| = 0.$$

Then, for arbitrary α such that $\alpha \le \tan\delta$, $0 < \alpha < 1$ we have

$$|I - I_w^{(L)}| < M \left(\frac{\sqrt{\pi} p}{\sqrt{2}\sqrt{1-\alpha^2}} e^{(q-\omega\alpha p/2)^2/(1-\alpha^2)} + \frac{\sqrt{\pi} p}{4} + \frac{\sqrt{2}}{\omega\alpha} e^{(q-\omega\alpha p)q} \right) e^{-q^2}. \tag{2.9}$$

The proof of Theorem 1 is in [5, 3]. Now, we choose p and q such that

$$p = \frac{\sqrt{L}}{\sqrt{\omega\alpha}} \,, \quad q = \frac{\sqrt{\omega\alpha L}}{2} \,, \tag{2.10}$$

then the error of the approximation $I_w^{(L)}$ is bounded as

$$|I - I_w^{(L)}| < M\left(\frac{\sqrt{\pi L}}{\sqrt{2\omega\alpha}\sqrt{1-\alpha^2}} + \frac{\sqrt{\pi L}}{4\sqrt{\omega\alpha}} + \frac{\sqrt{2}}{\omega\alpha}\right)e^{-\omega\alpha L/4}. \tag{2.11}$$

Hence, the error of $I_w^{(L)}$ always decays exponentially as $L \to \infty$ and the continuous Euler transformation accelerates the convergence of Fourier type integrals for those f described in Theorem 1.

2.4 Efficient Weight Function

To improve the convergence of the continuous Euler transformation, the weight function

$$w(x;p,q) = \int_{x/p-q}^{\infty} \phi(t)\,dt \tag{2.12}$$

should be taken so that

1. $|\Phi(\omega)|$ decays rapidly for large $|\omega|$
2. $|\phi(t)|$ decays rapidly for large $|t|$,

where $\Phi(\omega)$ is the Fourier transform of $\phi(t)$ [4]. An example of an efficient weight function is

$$w(x;p,q) = \frac{1}{\pi I_0(\beta)}\int_{x/p-q}^{\infty} \frac{\sinh\sqrt{\beta^2-t^2}}{\sqrt{\beta^2-t^2}}dt \tag{2.13}$$

with parameters $L = 2pq$, $q = \beta$, $p \geq \omega^{-1}$ [3]. The error of $I_w^{(L)}$ using this weight function decays $O(e^{-L/(2p)})$ as $L \to \infty$. If we set $p = \omega^{-1}$, then the error decays $O(e^{-\omega L/2})$ and the coefficient of the exponent is twice as great as the coefficient of (2.11).

3 Numerical Inversion of the Laplace Transform

3.1 Method using the Continuous Euler Transformation and FFT

We first apply the continuous Euler transformation to (1.3), and obtain

$$g_w^{(L)}(t) = \frac{e^{\gamma t}}{2\pi}\int_{-L_-}^{L_+} w(|x|;p,q)e^{itx}G(\gamma+ix)\,dx. \tag{3.1}$$

We next approximate it by the discrete summation with mesh size h to obtain

$$g_w^{(N,h)}(t) = \frac{he^{\gamma t}}{2\pi}\sum_{n=-N_-}^{N_+} w(|nh|;p,q)e^{itnh}G(\gamma+inh)\,, \tag{3.2}$$

where $N_\pm = \lfloor L_\pm/h \rfloor$ ($\lfloor x \rfloor$ denotes the largest integer $\leq x$).

To keep the precision of the continuous Euler transformation, we should make p, q depend on $L_\pm$ and ω. Here we fix the parameters p, q with a view to using FFT. For $L_+ = Nh/2 - h$, $L_- = Nh/2$ and $t = 2\pi k/(Nh)$, (3.2) becomes

$$g_w^{(N,h)}(\frac{2\pi k}{Nh}) = \frac{he^{\gamma t}}{2\pi}\sum_{n=-N/2}^{N/2-1} w(|nh|;p,q)G(\gamma+inh)e^{2\pi ink/N}\,, \tag{3.3}$$

which can be computed by FFT.

3.2 Error Estimation

The approximation $g_w^{(N,h)}$ includes the following errors

A. Error of the continuous Euler transformation: $g(t) - g_w^{(L)}(t)$

B. Error of the discrete approximation: $g_w^{(L)}(t) - g_w^{(N,h)}(t)$.

By Theorem 1, the error committed by by A is

$$|g(t) - g_w^{(L)}(t)| < \frac{M}{\pi}\left(\frac{\sqrt{\pi}pe^{(q-t\alpha p/2)^2/(1-\alpha^2)}}{\sqrt{2}\sqrt{1-\alpha^2}} + \frac{\sqrt{\pi}p}{4} + \frac{\sqrt{2}e^{(q-t\alpha p)q}}{t\alpha}\right)e^{\gamma t - q^2}, \tag{3.4}$$

where $w(x;p,q) = \frac{1}{2}\mathrm{erfc}(x/p - q)$ and $L = Nh/2 = 2pq$.

The error due to B is an error of the discrete Fourier transform. We can estimate this error by an extension of the theorem on the discrete Fourier transform [6, chapter 3.3] and obtain

$$|g_w^{(L)}(t) - g_w^{(N,h)}(t)| < \frac{\hat{M}L}{2\pi}\mu + \frac{d_\pm\hat{\varepsilon}}{2\pi}e^{\gamma t} + \frac{ML}{2\pi}e^{\gamma t - q^2} \tag{3.5}$$

$$\mu = \frac{\exp(\gamma t - d_+t - 2\pi d_+/h)}{1 - \exp(-2\pi d_+/h)} + \frac{\exp(\gamma t + d_-t - 2\pi d_-/h)}{1 - \exp(-2\pi d_-/h)}, \tag{3.6}$$

where $G(\gamma + iz)$ is regular in the domain $-d_- \leq \mathrm{Im}\ z \leq d_+$, $\hat{M}$ is an upper bound of $|G(\gamma + iz)|$ in the domain, and $\hat{\varepsilon}$ is an upper bound of $|G(\gamma + iz)|$ on the contour of integration C_+ and C_-.

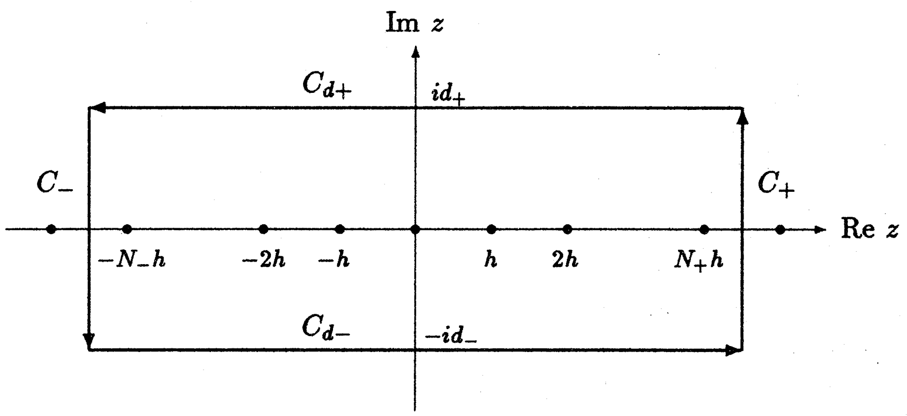


Fig. 1: Contour of Integration

3.3 Numerical Example

The following functions are used for test:

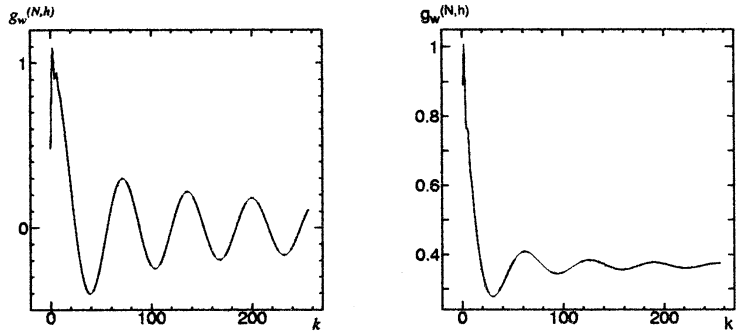
$$G_1(s) = \frac{1}{\sqrt{1+s^2}}, \quad g(t) = J_0(t) \tag{3.7}$$

$$G_2(s) = \frac{1}{s\exp(1/(s+\sqrt{1+s^2}))}. \tag{3.8}$$

The result using the weight function

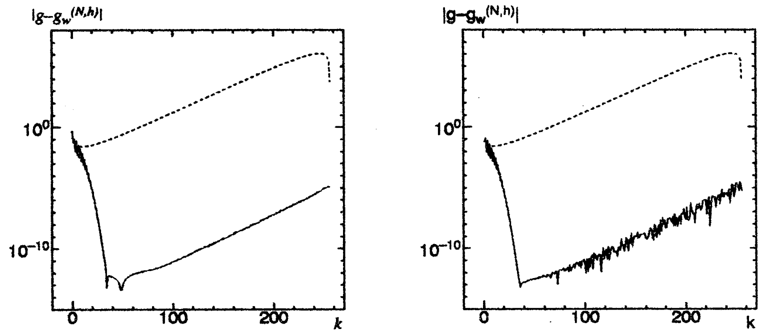
$$w_1(x;p,q) = \frac{1}{2}\mathrm{erfc}(x/p - q) \tag{3.9}$$

is shown in Fig. 2 and the error is shown in Fig. 3.



(a) $G(s) = \frac{1}{\sqrt{1+s^2}}$ (b) $G(s) = \frac{1}{s\exp(1/(s+\sqrt{1+s^2}))}$

Fig. 2: Approximations to the Inversion of the Laplace Transform: $g_w^{(N,h)}(\frac{2\pi k}{Nh})$, $N = 512$, $h = 0.125$



(a) $G(s) = \frac{1}{\sqrt{1+s^2}}$ (b) $G(s) = \frac{1}{s\exp(1/(s+\sqrt{1+s^2}))}$

Fig. 3: Absolute error of the Inversion of the Laplace Transform: $g_w^{(N,h)}(\frac{2\pi k}{Nh})$, $N = 512$, $h = 0.125$

We chose $N = 512$, $h = 0.125$, $\gamma = 1$, $L = Nh/2 = 2pq$ and computed in double precision that has 53 bit accuracy. Truncation error of the continuous Euler transformation was set to 10^{-12} and the parameter q is chosen so that $\exp(-q^2) = 10^{-12}$. The broken line in Fig. 3 denotes the error of the direct transform that does not use $w(x;p,q)$.

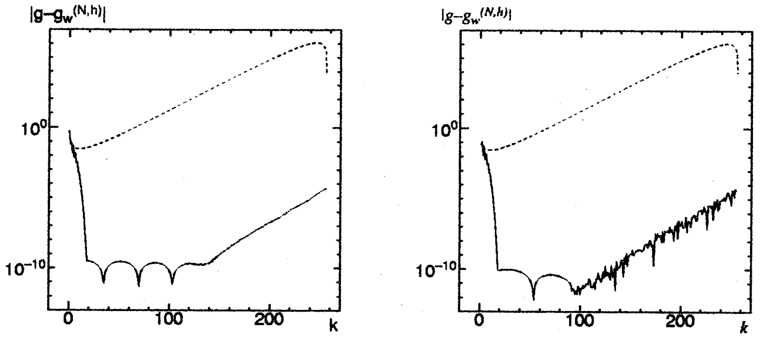
By (3.5), the error bound increases exponentially for large t. By (3.4), the error bound becomes very large if $q - tp/2 > 0$, i.e. $t < 2q/p = 3.45$. This error estimation is consistent with the numerical result.

Next, we computed the same transform using the efficient weight function

$$w_2(x;p,q) = \frac{1}{\pi I_0(\beta)} \int_{x/p-q}^{\infty} \frac{\sinh\sqrt{\beta^2 - t^2}}{\sqrt{\beta^2 - t^2}} dt \tag{3.10}$$

with the parameters $N = 512$, $h = 0.125$, $\gamma = 1$, $L = Nh/2 = 2pq$ and $e^{-\beta} = e^{-q} = 10^{-13}$. The error is shown in Fig. 4.

The error of the approximation using $w_2(x;p,q)$ is large in $t < 1.7$ that is only half the interval using $w_1(x;p,q)$.



(a) $G(s)=\frac{1}{\sqrt{1+s^2}}$ (b) $G(s)=\frac{1}{s\exp(1/(s+\sqrt{1+s^2}))}$

Fig. 4: Absolute error of the Inversion of the Laplace Transform: $g_w^{(N,h)}(\frac{2\pi k}{Nh})$ using w_2, $N=512$, $h=0.125$

3.4 Execution Time

We measured the execution time on SUN Ultra SPARC-I 200MHz machine with SUN Workshop cc 4.2.1 compiler. Actual execution time is shown in Table 1.

Table 1: Execution Time to Transform

Function to Transform	$G_1(s)$	$G_2(s)$
Our Method	3.3 μ sec./N	5.2 μ sec./N
Hosono's Method	33.2 μ sec./N	93.7 μ sec./N

The weight function we used is $w_2(x;p,q)$ with the same parameters as in section 3.3. We also compared with Hosono's method [2] with 30-point Euler transformation.

References

[1] P. J. Davis and P. Rabinowitz, Methods of Numerical Integration (Second Edition), Academic Press, 1984.

[2] T. Hosono, Numerical Inversion of the Laplace Transform in High Precision (in Japanese), RIMS Kokyuroku, No.422, Kyoto University, 1981.

[3] T. Ooura, Study on Numerical Integration of Fourier Type Integrals (in Japanese), Doctoral Thesis, University of Tokyo, 1997.

[4] T. Ooura, A Continuous Euler Transformation and Acceleration of the Convergence of Fourier Type Integrals (in Japanese), RIMS Kokyuroku, No.1084, Kyoto University, 1999.

[5] T. Ooura, A Continuous Euler Transformation and its Application to the Fourier Transform of a Slowly Decaying Function, J. Comput. Appl. Math., Accepted for publication, (The Same Title, RIMS Preprint, No.1233, Kyoto University, 1999).

[6] F. Stenger, Numerical Methods Based on Sinc and Analytic Functions, Springer Verlag, 1993.

[7] J. Wimp, Sequence Transformations and Their Applications, Academic Press, 1981.