数理解析研究所講究録
1145巻 2000年 70-77

A discrete compact injection and its application to the convection problems

Satoshi KAIZU (海津　聰) *
Faculty of Education
Ibaraki University, Mito 310-8512 Japan

1 Motivation and results

Let $W_0^{1,p}(\Omega), 1 \leq p \leq \infty$, be the Sobolev space over a bounded polygon or polyhedron Ω with boundary in $\partial\Omega$ in $\mathbf{R}^d, d \geq 2$. Then we have a compact injection $W_0^{1,p}(\Omega) \overset{c}{\hookrightarrow} L^q(\Omega)$ for

$$\begin{cases} \frac{1}{p} - \frac{1}{d} < \frac{1}{q} \leq 1 & \text{for } \frac{1}{d} < \frac{1}{p}, \\ 0 < \frac{1}{q} \leq 1 & \text{for } \frac{1}{d} = \frac{1}{p}, \end{cases} \tag{1}$$

(the Kondrasov theorem in the case $d \geq 2$). Let $\mathcal{T}_h$ be a triangulation of Ω and $W_{0,h}$ be the space of non conforming finite element space of degree one with seminorm $|v_h|_{1,p,h}$ defined by

$$|v_h|_{1,p,h}^p = \sum_{K \in \mathcal{T}_h} \|\tilde{\nabla} v_h\|_{L^p(K)}^p,$$

where the precise definition of $\tilde{\nabla}$ is given in Crouzeix and Raviart [3]. Note that the seminorm $|v_h|_{1,p,h}$ is considered as the norm for the space $W_{0,h}$ denoted by $W_{0,h}^{1,p}$. Then we see that there exists a discrete compact injenction $W_{0,h}^{1,p} \overset{c}{\hookrightarrow} L^q(\Omega)$, where the relation between p and q is described as in (1) (cf. R. Temam [8], for example). We can regard h as the value of the maximum of elements $K \in \mathcal{T}_h$. Now we recall an external approximation of normed spaces mentioned in section 3.1 of Chapter 1 of the book R. Temam [8] and shall modify it slightly.

*email:kaizu@mito.ipc.ibaraki.ac.jp

Let $F = \{L^p(\Omega)\}^{d+1}$ and $\omega_0 : W^{1,p}(\Omega) \ni v \to \omega_0(v_h) = (v, \nabla v) \in F$ be an isomorphism from $W_0^{1,p}(\Omega)$ into the space F. Since $W_{0,h}^{1,p}$ is not included in the space $W_0^{1,p}(\Omega)$ and $\partial v_h / \partial x_i$ is the sum of Dirac distribution on the faces of elements $K \in \mathcal{T}_h$ and of a step function $D_{ih} v_h$ defined almost everywhere by

$$D_{ih} v_h = \frac{\partial v_h}{\partial x_i} \quad \forall x \in K, \forall K \in \mathcal{T}_h.$$

We set $\tilde{\nabla} v_h = \left(D_{ih} v_h \right)_{1 \le i \le d}$ and $\omega_h : W_{0,h}^{1,p} \ni v_h \to \omega_h v_h = (v_h, \tilde{\nabla} v_h) \in F$.

As mentioned we have a discrete compact injection from $W_{0,h}^{1,p}$ into $L^q(\Omega)$ for each positive number h. Let $\mathcal{H} = \{h_n\}_n$ be a sequence of positive numbers decreasing to zero, and let $W_{0,\mathcal{H}}^{1,p} = \cup_{n=1}^{\infty} W_{0,h_n}^{1,p}$. For each h_n, there exists an injection : $W_{0,h_n}^{1,p} \hookrightarrow L^q(\Omega)$ under the same relation (1) between p and q by the discrete Sobolev imbedding theorem (cf. [8]). Then our problem is described as follows.

Is it true that the injenction $W_{0,\mathcal{H}}^{1,p} \hookrightarrow L^q(\Omega)$ is compact ?

Let $\mathcal{T}_n = \mathcal{T}_{h_n}$. We show that the question above is solved affirmatively under two conditions on the sequence $\mathcal{T}_{\mathcal{H}} = \{\mathcal{T}_n\}_{n=1}^{\infty}$, that is, (1) $\mathcal{T}_{\mathcal{H}}$ is *quasi uniform*, and (2) $\mathcal{T}_{\mathcal{H}}$ is *quasi uniform in any direction*. We shall fixed these two ideas below.

Let h_K and $h_{0,K}$ be the maximum of $K \in \mathcal{T}_n$ and also the maximum of the spheres included in the same K, respectively, let $\sigma_K = h_K / h_{0,K}$ and $\sigma_n = \max_{K \in \mathcal{T}_n} \sigma_n$. Before describing the quasi uniformity and the quasi uniformity in any direction on $\mathcal{T}_{\mathcal{H}}$, we recall that $\mathcal{T}_{\mathcal{H}}$ is regular if $\mathcal{T}_{\mathcal{H}}$ satisfies

$$\limsup_{n \to \infty} \sigma_n = \sigma_0 < \infty. \tag{2}$$

Let $h_{max,n} = \max_K h_K$, $h_{min,n} = \min_K h_K$ and $\theta_n = h_{max,n} / h_{min,n}$.

Definition 1 *If*

$$\limsup_{n \to \infty} \theta_n = \theta_0 < \infty, \tag{3}$$

then we say that $\mathcal{T}_{\mathcal{H}}$ is quasi uniform.

Let us introduce another notion. For arbitrary $n \in \mathbf{N}$, $K_0 \in \mathcal{T}_n$, $z_0 \in \mathbf{R}^d$ we set

$$\mathcal{K}(K_0, z_0) = \{K \in \mathcal{T}_n \;\big|\; \exists w \in K_0,\ \exists t \in [0,1],\ w + t z_0 \in K\ \}$$

and let $\#(K_0, z_0)$ be the number of elements $K \in \mathcal{K}(K_0, z_0)$.

Definition 2 *If there exist constants* $c_{\mathcal{H},1}, c_{\mathcal{H},2}$*, independent of* $n \in \mathbf{N}$*,* $K_0 \in \mathcal{T}_n$*,* $z_0 \in \mathbf{R}^d$*, such that*

$$\#(K_0, z_0) \leq c_{\mathcal{H},1} \frac{|z_0|}{h_n} + c_{\mathcal{H},2},$$

then we say that $\mathcal{T}_{\mathcal{H}}$ *is quasi uniform in any direction.*

The lemma below is essential in this paper.

Lemma 1 *If* $\mathcal{T}_{\mathcal{H}}$ *is regular and quasi uniform, then it is quasi uniform in any direction.*

My main theorem is described as follows.

Theorem 1 *If* $\mathcal{T}_{\mathcal{H}}$ *is regular and quasi uniform, then the injection :*

$$W^{1,p}_{0,\mathcal{H}} \hookrightarrow L^q(\Omega)$$

is compact where p *and* q *satisfies* (1).

Let $t \in (0, T)$ be a time variable and $\dot{v}(t)$ be the time derivative of a function $v(t)$. For $1 < r < \infty$, we introduce a space by

$$\mathcal{Z}^{1,r}(0,T;\ W^{1,p}_{0,\mathcal{H}}) = \left\{ v \in L^r\left(0,T; W^{1,p}_{0,\mathcal{H}}\right) \ \middle|\ \dot{v} \in L^r\left(0,T; L^p(\Omega)\right) \right\}$$

Then applying Theorem 2.1 in Chapter III, section 2 in [8] to the above Theorem 1 we directly get

Corollary 1 *For a regular and quasi uniform family* $\mathcal{T}_{\mathcal{H}}$ *we have a compact injection*

$$\mathcal{Z}^{1,r}\left(0,T;\ W^{1,p}_{0,\mathcal{H}}\right) \hookrightarrow L^r\left(0,T; L^p(\Omega)\right).$$

Further we show an application of Corollary 1 for a finite element scheme approximating the convection problem $(\mathrm{P})_0$: find ρ such that

$$\frac{\partial \rho}{\partial t} + \vec{u} \cdot \nabla \rho = 0 \quad \text{in } Q = \Omega \times (0,T), \tag{4}$$

$$\rho(x,0) = \rho^0(x) \quad \Omega, \tag{5}$$

where ρ^0 is an initial data satisfying $0 < M_1 \leq \rho^0 \leq M_2 < \infty$, with some positive constants M_1 and M_2, and u is a known velocity field satisfying

$$\vec{u} \in L^\infty\left(0,T; \{L^2(\Omega)\}^d\right) \cap L^2\left(0,T; \{H^1_0(\Omega)\}^d\right), \tag{6}$$

$$\nabla \cdot \vec{u} = 0 \quad \text{in } Q \tag{7}$$

in the sense of distribution.

Since we only assume (6) and (7), the velocity $\vec{u}$ is not smooth in the classical sense, we can not know generally how regular a solution ρ of the problem $(\mathrm{P})_0$ is. Besides we should choose an approximation of the known velocity $\vec{u}$ to construct approximations of ρ. Actually, by our finite element scheme (8), under the conditions (6) and (7), we have a weak solution $\rho \in L^\infty\big(0,T;L^\infty\big(\Omega\big)\big)$ of the problem $(\mathrm{P})_0$ such that $\dot{\rho} \in L^2\big(0,T;M^*\big)$ (cf. Theorem 2). Thus our weak solution ρ is not smooth enough, however applying Corollary 1 obtains a more regularity than as in Theorem 2. Under smooth velocities we can see methods to construct solutions in C. Bardos [1].

More minutely, we shall consider our scheme (8) and the solutions of $(\mathrm{P})_0$ as follows. As mentioned previously, we do not know how to construct the classcial solution ρ by a finite element scheme under the conditions (6) and (7), although it is possible to get a weak solution of $(\mathrm{P})_0$ in the sense described below, as a limit of discrete solutions r^n of the problem $(\mathrm{P})_h$: find $r^n \in G_h$ (cf. Theorem 2) such that

$$\frac{(\delta r^n,\alpha)}{\tau}+\sum_{F\subset\Omega}\int_F U^n\cdot\nu_F\big[r^n\big]_U^D\alpha d\sigma=0 \quad \forall\alpha\in G_h \tag{8}$$

where G_h is a function space of the totality of functions constant on each trianlge $K \in \mathcal{T}_h$. Further $F(\subset \partial K)$ is a $d-1$ dimensional simplex, $\delta r^n = r^n - r^{n-1}$. $U^n = (U_j^n)_{1\le j\le d}$, $U_j^n \in W_{0,h}$ and $W_{0,h}$ is the space of the totality of non-conforming finite elements of degree one (cf. [4]). Besides see [6], [5] and [3]. Let ν_F be the normal unit vector to F such that $U^n\cdot\nu_F \ge 0$ and for $K_i \in \mathcal{T}_h, i=1,2$. Then either of K_1 and K_2 is called to be the upwind element and the other is said to be the downwind element associated with F, where $F = \partial K_1 \cap \partial K_2$. Besides we set $\big[r^n\big]_U^D = r^n\big|_{K_D} - r^n\big|_{K_U}$, where K_D and K_U are the downwind and the upwind elements assoociated with F, respectively. We assume

$$\sup_{m=1,2,\cdot,N}\left\{\left\|U^m\right\|^2+\sum_{n=1}^m\left(\left\|\delta U^n\right\|^2+\tau\left\|U^n\right\|_h^2\right)\right\}\le c_0<\infty, \tag{9}$$

where $\|U^n\|_h = \sqrt{\sum_{K\in\mathcal{T}_h}\int_K\left|\nabla U^n\right|^2dx}$. Let $U^A(t) = U^n(t)$ and

$$r^L(t)=\Big(\big(t-t_{n-1}\big)r^n+\big(t_n-t\big)r^{n-1}\Big)/\tau$$

for $t_{n-1} < t \le t_n$. We further assume

$$\left\|U^A-u\right\|_{L^2(0,T;\{L^2(\Omega)\}^d)}\to 0 \quad \text{as } \tau\to 0, h\to 0. \tag{10}$$

Then we can show the follwoing theorem (see [4]).

Theorem 2 *We assume* (6), (7), (9) *and* (10). *Then the sequences* $\{r^A\}$ *and* $\{\dot{r}^L\}$ *are bounded in the space* $L^\infty\Big(0,T;L^\infty\big(\Omega\big)\Big)$ *and* $L^2\Big(0,T;M^*\Big)$ *with* $M = H^1(\Omega)$, *respectively. Further, there exist subsequences, still denoted by* $\{r^A\}$ *and* $\{\dot{r}^L\}$ *convergent to* ρ *and* $\dot{\rho}$ *in* $L^\infty\Big(0,T;L^\infty\big(\Omega\big)\Big)$ $*-$*weakly and* $L^2\Big(0,T;M^*\Big)$ *weakly, respectively. Here* ρ *is a weak solution of the problem* $(\mathrm{P})_0$ *associated with the space* M *in the sense below.*

In the above theorem we have adopted the notion of weak solutions of $(\mathrm{P})_0$ described below.

Let $M = W^{1,p}(\Omega)$ with some $1 \le p \le \infty$ and M^* be the dual space of the space M. Then ρ is called a weak solution of $(\mathrm{P})_0$ associated with M, provided that ρ is a solution of the problem $(\mathrm{P})_1$: find $\rho \in L^\infty\Big(0,T;L^\infty\big(\Omega\big)\Big)$ and $\mu \in L^2(0,T;M^*)$ such that

$$\begin{cases} \displaystyle\int_0^T <\mu,\beta> \zeta(t)dt - \int_0^T (\rho, \vec{u}\cdot\nabla\beta)\zeta(t)dt = 0 \qquad \forall\beta \in M, \zeta \in C_0^\infty(0,T), \\ \displaystyle\int_0^T <\mu,\beta> \zeta(t)dt + (\rho^0,\beta) = -\int_0^T (\rho,\beta)\dot{\zeta}(t)dt \quad \forall\beta \in M, \zeta \in C^\infty[0,T] \end{cases}$$

with $\zeta(T) = 0$. Here we write $\mu = \dot{\rho}$.

Recall that $M_1^* \neq M_2^*$ and $M_1^* \subset M_2^*$ for $M_1 = W^{1,q'}(\Omega), M_2 = W^{1,q''}(\Omega)$, $\infty \ge q'' > q' \ge 1$. It would be said that $g \in M_2$ is smoother than $f \in (M_1 \setminus M_2)$ and $G \in M_1^*$ is smoother than $F \in (M_2^* \setminus M_1^*)$. Thus, for $\rho_1 \in M_1^*$ and $M_1 = W^{1,p_1}(\Omega)$, $\rho_1 \in M_1^*$ is smoother than $\rho_2 \in M_2^*$. This means that a weak solution associated with M_1 is smoother than a weak solution associated with M_2.

Under these preparation combinig with Corollary 1 and Theorem 2 we get as an application of Theoem 1 that the weak solution satisfies more regularity than as described in Theorem 2.

Theorem 3 *The weak solution* ρ *of the problem* $(\mathrm{P})_0$ *gotten in* Theorem 2 *is a weak solution associated with* $W^{1,q}(\Omega)$ *provided* $2 \ge q > q_0 = 2d/(d+2)$ *for* $d \ge 3$ *and* $2 \ge q > 1$ *for* $d = 2$.

2 Proof

Once we have obtained Theorem 2, then Theorem 3 is implied by Theorem 2 and Corollary 1 together with the Hölder inequality. Therefore, it is essential to prove Lemma 1 and Theorem 1.

Proof of Lemma 1 For a set $G \subset \mathbf{R}^d$ and $y \in \mathbf{R}^d$ we denote $d(y,G)$ the metric between y and G. Let $U(2h,K_0) = \{y \in \mathbf{R}^d \mid d(y,K_0) \le 2h\}$. We can assume that $0 \in K_0$ and $z_0 = (O', \delta_0), O' \in \mathbf{R}^{d-1}$. Further we can take a positive constant $c'_{\mathcal{H}}$, independent of $h \in \mathcal{H}$, such that

$$K_0 \subset U(2h, K_0) \subset [-c'_{\mathcal{H}}h, c'_{\mathcal{H}}h]^d.$$

Let $Q(K_0; z_0) = \cup\{ K \mid K \in \mathcal{K}_{K_0, z_0} \}$. Then we have $Q(K_0; z_0) \subset U\big(h, Q(K_0; z_0)\big)$ and

$$U\left(h, \left(K_0; z_0\right)\right) \subset [-c'_{\mathcal{H}}h, c'_{\mathcal{H}}h]^{d-1} \times [-c'_{\mathcal{H}}h, c'_{\mathcal{H}}h + \delta_0] = S(K_0; z_0).$$

Notice that any element $K \in \mathcal{K}(K_0; z_0)$ belongs to $S(K_0; z_0)$.

On the other hand the Lebesgue measure $|K|$ of any K is estimated by $c''_{\mathcal{H}} h^d_{min,n} \le |K|$, because $\mathcal{T}_{\mathcal{H}}$ is regular. Thus, for small $h_n = h_{max,n}$, we get

$$\begin{aligned}\#(K_0; z_0) &\le \frac{|S(K_0; z_0)|}{c''_{\mathcal{H}} h^d_{min,n}} \le \frac{c'_{\mathcal{H}} h^{d-1}_{max,n}(\delta_0 + 2c'_{\mathcal{H}} h_{max,n})}{c''_{\mathcal{H}} h^d_{min,n}} \\ &\le 2\left(\frac{c'_{\mathcal{H}}\theta_0}{c''_{\mathcal{H}}}\right)^{d-1} \theta_0 \left(\frac{\delta_0}{h_{\min,n}} + 2\frac{c'_{\mathcal{H}}}{c''_{\mathcal{H}}}\right) \\ &\le 2\left(\frac{c'_{\mathcal{H}}\theta_0}{c''_{\mathcal{H}}}\right)^{d-1} \theta_0 \left(\frac{2\theta_0\delta_0}{h_{\max,n}} + 2\frac{c'_{\mathcal{H}}}{c''_{\mathcal{H}}}\right).\end{aligned}$$

■

Proof of Theorem 1 First notice that we have a discrete version for the Sobolev imbedding theorem as follows: there exists a positive constant $C_{\mathcal{H}}$, independent of $v_h \in W^{1,p}_{0,\mathcal{H}}$ and of $h_n \in \mathcal{H}$, such that

$$\|v_h\|_{\bar{q},\Omega} \le C_{\mathcal{H}} \left(|v_h|_{1,p} + \|v_h\|_p\right) \tag{11}$$

provided that $\mathcal{H}$ is regular, where $\bar{q}$ is given by $1/\bar{q} = 1/p - 1/d$ for $1/p > 1/d$, or otherwise $\bar{q}$ is an arbitrary number such that $1 \le \bar{q} < \infty$.

This is proved on each estimate on each elements $K \in \mathcal{T}_{h_n}, h_n \in \mathcal{H}$ by the standard method in the interpolation theory described in [2], then these estimates are summarized to the domain Ω by using the Hölder inequality. Thus, we get (11).

To prove Theorem 1 in a short form, we show only the major part of the proof and the remaining part is refered to the proof of the Kondrasov theorem.

First, for an arbitrary small positive number ϵ there exists a subset Ω^* such that $\Omega^* \subset \overline{\Omega^*} \subset \Omega$ and $|\Omega \setminus \Omega^*| \le (\epsilon/(3C'_{\mathcal{H}}))^{q^*/(q^*-1)}$, where q^* is the dual exponent of q.

We assume that $|v_n|_{1,p,\mathcal{H}} \le 1, n = 1, 2, 3, \cdots$, for $\{v_n\}_n \subset W^{1,p}_{0,\mathcal{H}}$. To conclude the proof it is sufficient to show for $p = 1$ (cf. [7]) that there exist positive numbers δ_0 and h_0 such that, for any $z \in \mathbf{R}^d$, $|z| \le \delta_0$ and

$$\int_{\Omega^*} \Big|v_n(x+z) - v_n(x)\Big| dx \le \sum_{K_0} \int_{K_0} \Big|v_n(x+z) - v_n(x)\Big| dx$$

$$= I(z) \leq \frac{\epsilon}{3}\left|v_n\right|_{1,1,\mathcal{H}}$$

for $v_n \in W_{0,\mathcal{H}_0}$, where $\mathcal{H}_0 = \{h_n \in \mathcal{H} \mid h_n \leq h_0\ \}$. For $x \in K_0 (\in \mathcal{T}_{h_n})$ we show that there exists a constant such that c_{K_0}, independent of $x \in K_0$, but dependent on v and $|z|$, such that $|v_n(x+z) - v_n(x)| \leq c_{K_0}$ for all $x \in K_0$. Then

$$I(z) \leq \sum_{K_0 \in \mathcal{T}_{h_n}} c_{K_0}|K_0|.$$

For simplicity $w = v_n$ and for an arbitrary $x \in K_0$ let

$$\mathcal{K}(x;z) = \Big\{ K \in \mathcal{T}_{h_n} \mid \exists t \in [0,1] \text{ such that } x + tz \in K \ \Big\}.$$

Precisely the family $\mathcal{K}(x;z)$ is different from other points $x' \in K_0$ and K_0 are decomposed into several equivalents family by the class of $\mathcal{M} = \{\mathcal{K}(x;z) \mid \forall x \in K_0\ \}$. However, for the sake of simplicity, we assume that $\mathcal{M}$ contains a sigle element family $\mathcal{K}(x;z)$ for $x \in K_0$.

Let n_0 be the number of elements belonging to $\mathcal{K}(x;z)$. We can choose numbers $0 = t_0 \leq t_1 \leq t_2 \leq \cdots \leq t_{n_0} = 1$ such that $x_i = x + t_i z$ for $i = 0,1,2,\cdots,n_0$, $x_i \in \partial K_{i-1} \cap \partial K_i, i = 1,2,3,\cdots,n_0 - 1$ and $\mathcal{K}(x;z) = \{K_i \mid i = 0,1,2,\cdots,n_0-1\}$. Let $w_i = w\Big|_{K_i}, i = 0,1,2,\cdots,n_0-1$. Generally we have $w_{i-1}(x_i) \neq w_i(x_i)$ for $i = 1,2,3,\cdots,n_0-1$. Since $w \in W_{0,h}$ is nonconforming element of degree one, there exists a point $y_i \in \partial K_{i-1} \cap \partial K_i, i = 1,2,3,\cdots,n_0-1$ such that $w_{i-1}(y_i) = w_i(y_i)$ for $i = 1,2,3,\cdots,n_0-1$, because $w \in W_{0,h}$ made of nonconforming elements of degree one. Thus

$$|w(x+z) - w(x)| \leq \Big|w_0(x_0) - w_0(x_1)\Big| + \Big|w_0(x_1) - w_0(y_1)\Big|$$

$$+ \sum_{i=1}^{n_0-1} \left(\Big|w_i(y_i) - w_i(x_i)\Big| + \Big|w_i(x_i) - w_i(x_{i+1})\Big| + \Big|w_i(x_{i+1}) - w_i(y_{i+1})\Big| \right)$$

$$+ \Big|w_{n_0}(y_{n_0-1}) - w_{n_0}(x_{n_0-1})\Big| + \Big|w_{n_0}(x_{n_0-1}) - w_{n_0}(x_{n_0})\Big|$$

$$\leq 3\,h \sum_{i=0}^{n_0} \Big|\text{grad } w_i\Big|_{0,\infty,K_i} = c_{K_0}.$$

Let $w_k = w\Big|_K$. Then, therefore

$$I(z) \leq 3h \sum_{K_0 \in \mathcal{T}_h} |K_0| \sum_{K \in \mathcal{K}(K_0;z)} \Big|\text{grad } w_K\Big|_{0,\infty,K}.$$

In the last summation, each summand $\Big|\text{grad } w_K\Big|_{0,\infty,K}$ is added many times at most the number n_1 of elements belonging to $\mathcal{K}(K;-z)$. Further we can replace $|K_0|$ with $c'''_{\mathcal{H}}|K|$, where $c'''_{\mathcal{H}}$ is independent of $h \in \mathcal{H}$ and $K \in \mathcal{T}_h$. Recall that $n_1 \leq c_{1,\mathcal{H}}|z|/h + c_{2,\mathcal{H}}$, because $\{\mathcal{T}_h\}_{h\in\mathcal{H}}$ is normal. Thus,

$$3h \sum_{K_0\in\mathcal{T}_h} |K_0| \sum_{K\in\mathcal{K}(K_0;z)} \left|\text{grad } w_K\right|_{0,\infty,K}$$

$$\leq 3h\ c'''_{\mathcal{H}}|K| \left(c_{1,\mathcal{H}}\frac{|z|}{h} + c_{2,\mathcal{H}}\right) \sum_{K\in\mathcal{T}_h} |\text{grad } w|_{0,\infty,K}|K|$$

$$\leq 3c'''_{\mathcal{H}}|K|\ \left(c_{1,\mathcal{H}}\left|z\right| + c_{2,\mathcal{H}}\ h_0\right) \left|w\right|_{1,1,h}.$$

So we can choose δ_0 and h_0 to get $I(z) \leq \epsilon/3$ and the proof is concluded. ∎

References

[1] Bardos, C., Problèmes aux limites pour les équations aux dérivées partielles du premier ordre à coefficients reels, Ann. Scient. Ec. Norm. Sup., 4^e serie, t. 3, 185-233, 1970.

[2] Ciarlet, Philippe G. and P. A. Raviart, General Lagrange and Hermite interpolation in R^n with applications to finite element methods, Arch. Rational Mech. Anal., Vol. 46, 1991.

[3] Crouzeix, M. and P.-A. Raviart, Conforming and nonconforming finite element methods for solving the stationary Stokes equations I, R. A. I. R. O., 7, Sér. Anal. Numer., pp. 33-76, 1973.

[4] Kaizu, S., A convergent finite element scheme of convection problems and its application to the density dependent Stokes problems.

[5] Johnson, C. and J. Pitkäranta, An analysis of the discontinuous Galerkin method for a scalar hyperbolic equaion, Math. Compu. 46, pp. 1-26, 1986.

[6] Lesaint, L. and P.-A. Raviart, On a finite element method for solving the neutron transport eqution, in Math. Aspects of finite elements in partial differentaial equations, ed. C. de Boor, Academic Press, New York, 1974.

[7] Jindrich Necas, Méthodes Directes en théirue des éqyatuibs elliptiques, Masson, Paris, 1967.

[8] Roger Temam, Navier-Stokes equations, North-Holland, Amsterdam, 1979.