数理解析研究所講究録
1146 巻 2000 年 73-91

ON SOME STABILITY THEOREM OF THE NAVIER-STOKES EQUATION IN THE THREE DIMENSIONAL EXTERIOR DOMAIN

YOSHIHIRO SHIBATA (WASEDA UNIV.)
MASAO YAMAZAKI (HITOTSUBASHI UNIV.)

Problem, History and our Motivation of study. The motion of nonstationary flow of an incompressible viscous fluid past an isolated rigid body is formulated by the following initial boundary value problem of the Navier–Stokes equation:

$$
(1) \qquad \begin{cases} \mathbf{u}_t - \Delta\,\mathbf{u} + (\mathbf{u}\cdot\nabla)\,\mathbf{u} + \nabla\,\mathfrak{p} = \mathbf{f}, \ \ \nabla\cdot\mathbf{u} = 0 \quad \text{in } (0,\infty)\times\Omega, \\ \mathbf{u}|_{\partial\Omega} = \mathbf{0}, \quad \mathbf{u}|_{t=0} = \mathbf{a} \\ \lim_{|x|\to\infty} \mathbf{u}(t,x) = \mathbf{u}_\infty. \end{cases}
$$

Here, Ω is the exterior domain in $\mathbb{R}^3$ identified with the region filled by a viscous incompressible fluid; $\partial\Omega$ denotes the boundary of Ω which is assumed to be a smooth and compact hypersurface; $\mathbf{u} = {}^T(u_1, u_2, u_3)$ (${}^TM$ means the transposed M) and $\mathfrak{p}$ denote the unknown velocity vector and pressure, respectively, while $\mathbf{f} = {}^T(f_1, f_2, f_3)$ and $\mathbf{a} = {}^T(a_1, a_2, a_3)$ denote the given external force and initial velocity, respectively. $\mathbf{u}_\infty$ is the given speed of the motion of the fluid at infinity and $\mathbf{0} = {}^T(0,0,0)$.

Here and hereafter, we use the standard notation in the vector analysis. For example, we put

$$
\Delta u = {}^T(\Delta u_1, \Delta u_2, \Delta u_3), \ \ \Delta u_j = \sum_{\ell=1}^3 \frac{\partial^2 u_j}{\partial x_\ell^2}, \ \ \nabla = {}^T(\partial_1, \partial_2, \partial_3), \ \ \partial_\ell = \frac{\partial}{\partial x_\ell}
$$

$$
(\mathbf{u}\cdot\nabla)\mathbf{v} = {}^T((\mathbf{u}\cdot\nabla)v_1, (\mathbf{u}\cdot\nabla)v_2, (\mathbf{u}\cdot\nabla)v_3), \ \ (\mathbf{u}\cdot\nabla)v_j = \sum_{\ell=1}^3 u_\ell \frac{\partial v_j}{\partial x_\ell},
$$

$$
\nabla\cdot\mathbf{u} = \operatorname{div}\mathbf{u} = \sum_{\ell=1}^3 \frac{\partial u_\ell}{\partial x_\ell}, \quad \mathbf{u} = {}^T(u_1, u_2, u_3), \ \ \mathbf{v} = {}^T(v_1, v_2, v_3)
$$

$$
\mathbf{u}\otimes\mathbf{v} = \begin{pmatrix} u_1v_1, \ u_2v_1, \ u_3v_1 \\ u_1v_2, \ u_2v_2, \ u_3v_2 \\ u_1v_3, \ u_2v_3, \ u_3v_3 \end{pmatrix}, \ \ \nabla\cdot F = \begin{pmatrix} \sum_{\ell=1}^3 \partial_\ell f_{1\ell} \\ \sum_{\ell=1}^3 \partial_\ell f_{2\ell} \\ \sum_{\ell=1}^3 \partial_\ell f_{3\ell} \end{pmatrix}, \ \ F = \begin{pmatrix} f_{11}, \ f_{12}, \ f_{13} \\ f_{21}, \ f_{22}, \ f_{23} \\ f_{31}, \ f_{32}, \ f_{33} \end{pmatrix}.
$$

Putting $\mathbf{u} = \mathbf{u}_\infty + \mathbf{v}$, instead of (1), here we consider the following problem :

$$
(2) \qquad \begin{cases} \mathbf{v}_t - \Delta\,\mathbf{v} + (\mathbf{u}_\infty\cdot\nabla)\mathbf{v} + (\mathbf{v}\cdot\nabla)\,\mathbf{v} + \nabla\,\mathfrak{p} = \mathbf{f}, \ \ \nabla\cdot\mathbf{u} = 0 \quad \text{in } (0,\infty)\times\Omega, \\ \mathbf{v}|_{\partial\Omega} = -\mathbf{u}_\infty, \quad \mathbf{v}|_{t=0} = \mathbf{a} - \mathbf{u}_\infty \\ \lim_{|x|\to\infty} \mathbf{v}(t,x) = \mathbf{0}. \end{cases}
$$

In this note we consider the case where the external force $\mathbf{f}$ is independent of time t, namely $\mathbf{f} = \mathbf{f}(x)$. The results reported here can be extended to the time depending external force by using the method due to Yamazaki [33]. But, since we would like to show some basical idea, we consider only the case of time independent external forces. And moreover, we will discuss the problem from the point of the stability of stationary solutions. Because, when the external force is independent of time, we can expect that the flow becomes stable asymptotically in time because of the viscousity. Therefore, as the stationary problem of (2) we consider the following time independent problem :

$$
(3) \qquad \begin{cases} -\Delta \mathbf{w} + (\mathbf{u}_\infty \cdot \nabla)\mathbf{w} + (\mathbf{w} \cdot \nabla)\mathbf{w} + \nabla \pi = \mathbf{f}, \ \nabla \cdot \mathbf{w} = 0 \quad \text{in } \Omega, \\ \mathbf{w}\big|_{\partial\Omega} = -\mathbf{u}_\infty, \quad \lim\limits_{|x|\to\infty} \mathbf{w}(x) = \mathbf{0}. \end{cases}
$$

Concerning (2), Leray [26] and Hopf [19] proved the existence of square-integrable weak solutions for an arbitrary square-integrable initial velocity, whose uniqueness is a still unknown and challenging problem. Concerning the stationary flow to (1), namely $\mathbf{u} = \mathbf{u}(x)$ and therefore $\mathbf{u}_t = \mathbf{0}$, Leray [25] proved the existence of a smooth steady solution with a finite Dirichlet integral. But, the solutions obtained by Leray and Hopf did not provide much qualitative information. In particular, nothing was proven about the asymptotic structure of the wake behind the body $\mathcal{O} = \mathbb{R}^3 - \overline{\Omega}$. This is a topic of great interest in itself. In 1965, Finn [8] - [13] gave a new existence theorem of (3) for the case of small data, which provided a great deal of qualitative asymptotic information, especially exhibited a phenomenon of wake behind the body $\mathcal{O}$. The solution that he obtained was called physically reasonable. To investigate the relationship between Finn's physically reasonable solutions and Leray's solution is also very interesting problem, which was first studied by Babenko [1] (also Galdi [14], Farwig [7]). In his review paper [13], Finn proposed a further investigation of the relationship between the class of the physically reasonable solutions and corresponding nonstationary solutions solving (2), which is called the stability problem below.

If we put $\mathbf{v}(t,x) = \mathbf{w}(x) + \mathbf{z}(t,x)$ and $p(t,x) = \pi(x) + q(t,x)$ in (2), the stability problem is to solve the following problem :

$$
(4) \qquad \begin{cases} \mathbf{z}_t - \Delta\mathbf{z} + (\mathbf{u}_\infty \cdot \nabla)\mathbf{z} + (\mathbf{w} \cdot \nabla)\mathbf{z} + (\mathbf{z} \cdot \nabla)\mathbf{w} \\ \qquad\qquad + (\mathbf{z} \cdot \nabla)\mathbf{z} + \nabla q = \mathbf{0} \text{ and } \nabla \cdot \mathbf{z} = 0 \quad \text{in } (0,\infty) \times \Omega, \\ \mathbf{z}\big|_{\partial\Omega} = \mathbf{0}, \quad \mathbf{z}\big|_{t=0} = \mathbf{b} \overset{def}{=} \mathbf{a} - \mathbf{u}_\infty - \mathbf{w}, \quad \lim\limits_{|x|\to\infty} \mathbf{z}(t,x) = \mathbf{0}. \end{cases}
$$

This problem was first solved in the L_2–framework by Heywood [16]. In fact, he proved an unique existence theorem of solutions to (4) in the L_2 framework with $\mathbf{b} \in L_2(\Omega)$ with small norm. This was sharpened, particularly with respect to the time decay rate, by Masuda [28], Heywood [17, 18], Miyakawa [29] and Maremonti [27]. But, as Finn already showed in [10], if $\mathbf{w}(x)$ is a physically reasonable solution and if $\mathbf{u}_\infty \neq \mathbf{0}$ and $\mathbf{f} = \mathbf{0}$, then $\mathbf{w}(x)$ is not square-integrable over Ω. Therefore, it seems reasonable to seek a solution $\mathbf{z}(t,x)$ of the problem (4) such that $\mathbf{z}(t,x)$ belongs to the class to which $\mathbf{w}(x)$ belongs for each $t > 0$. Especially such a class is not the set of square-integrable functions over Ω.

In this direction, Kato [21] solved the problem (1) in the L_n–framework when $\Omega = \mathbb{R}^n$ $(n \geqq 2)$, $\mathbf{u}_\infty = \mathbf{0}$, $\mathbf{f} = \mathbf{0}$ and the L_n norm of $\mathbf{a}$ is very small. He employed various L_p norms and L_p–L_q estimates for the semigroup generated by the Stokes operator. Iwashita [18] (cf. also Borchers and Miyakawa [3], Giga and Sohr [15]) extended Kato's method to the case where $\Omega \neq \mathbb{R}^n$ $(n \geqq 3)$, $\mathbf{f} = \mathbf{0}$ and $\mathbf{u}_\infty = \mathbf{0}$ and the L_n norm of $\mathbf{a}$ is very small. Our argument about the stability theorem is also based on L_p - L_q type decay estimates of the Oseen semigroup. In connection with the stability problem, from the results due to Kato and Iwashita we have the stability of trivial solution $\mathbf{0}$ of the stationary problem of (3) with respect to small L_n perturbation when $\mathbf{u}_\infty = \mathbf{0}$.

Our interest here is to consider the stability problem when $\mathbf{f} = \mathbf{f}(x)$ is non-trivial. When $\mathbf{u}_\infty = \mathbf{0}$ and $\mathbf{f} = \nabla \cdot F(x)$, F having suitable decay property at infinity, Borchers and Miyakawa [6] and Kozono and Yamazaki [23, 24] proved the stability of physically reasonable solutions of (3) with $\mathbf{u}_\infty = \mathbf{0}$ with respect to small $L_{n,\infty}(\Omega)$ perturbation, that is the problem (4) admits a unique solution $\mathbf{z}(t,x) \in \mathrm{BC}((0,\infty)\,;\, L_{n,\infty}(\Omega))$ when $\|\mathbf{w}\|_{L_{n,\infty}(\Omega)}$ and $\|\mathbf{b}\|_{L_{n,\infty}(\Omega)}$ are small enough, $\nabla \cdot \mathbf{b} = 0$, $n \geqq 3$ and $\mathbf{u}_\infty = \mathbf{0}$

On the other hand, when $\mathbf{u}_\infty \neq \mathbf{0}$, Shibata [32] proved the stability of physically reasonable solutions of (3) with respect to small $L_3(\Omega)$ perturbation, that is the problem (4) admits a unique solution $\mathbf{z}(t,x) \in \mathrm{BC}([0,\infty)\,;\, L_3(\Omega))$ when some weighted norm of $\mathbf{w}(x)$, $\|\mathbf{b}\|_{L_3(\Omega)}$ and $|\mathbf{u}_\infty|$ are small enough and $\nabla \cdot \mathbf{b} = 0$. But, the smallness assumption of $\mathbf{w}$ depends on $|\mathbf{u}_\infty|$, and therefore from Shibata [32] we can not consider the limit process : $|\mathbf{u}_\infty| \to 0$. One of the reason is that the solution class for non-zero $\mathbf{u}_\infty$ is different from the $\mathbf{u}_\infty = \mathbf{0}$ case. Since the solution class is the same when $\mathbf{f} = 0$, from Shibata [32] we can see that the solution of (1) in the non-zero $\mathbf{u}_\infty$ case tends to the solution in the case when $\mathbf{u}_\infty = \mathbf{0}$ in $L_3(\Omega)$ norm for each $t > 0$ (moreover, in $L_\infty(\Omega)$ norm) when $|\mathbf{u}_\infty| \to 0$.

The motivation of our study here is to consider the limit process : $|\mathbf{u}_\infty| \to 0$ when $\mathbf{f}(x)$ is non-trivial. Since $L_{3,\infty}(\Omega)$ seems to be the optimal space when $\mathbf{u}_\infty = \mathbf{0}$, we have to consider (4) also in $L_{3,\infty}(\Omega)$ when $|\mathbf{u}_\infty| \neq \mathbf{0}$. Unfortunately, we have not yet obtained any answer about the limit process. Here, we can report only that when $|\mathbf{u}_\infty|$ is small enough, our solutions to (3) and (4) have uniform estimations with respect to $|\mathbf{u}_\infty|$. From this, we can obtain some weak star limit, but it is very weak conclusion concerning the limit process and therefore we omit the precise statement. We hope that such direction of study of the Navier-Stokes equation has own interest and that our study gives an interesting aspect in the study of the Navier-Stokes equation.

Statement of main results. In order to state our main results precisely, first of all we introduce the definition of the Lorenz spaces $L_{p,q}(\Omega)$ for $1 \leqq p < \infty$ as follows :

$$f \in L_{p,q}(G) \overset{def}{\Longleftrightarrow} \begin{cases} \|f\|_{L_{p,q}(G)} = \left\{ \displaystyle\int_0^\infty [t^{1/p} f^*(t)]^q \frac{dt}{t} \right\}^{1/q} & 1 \leqq q < \infty,; \\ \|f\|_{L_{p,\infty}(G)} = \displaystyle\sup_{\sigma > 0} \sigma m(\sigma, f)^{1/p} < \infty & q = \infty, \end{cases}$$

where

$$f^*(t) = \inf\{\sigma > 0 \mid m(\sigma, f) \leqq t\}\,; \;\; m(\sigma, f) = |\{x \in G \mid |f(x)| > \sigma\}|$$

and $|\cdot|$ denotes the Lebesgue measure.

Below, we consider only the case where the external force $\mathbf{f}$ is given by the following potential form :

$$\mathbf{f}(x) = \nabla \cdot F(x), \quad F = \begin{pmatrix} F_{11}, F_{12}, F_{13} \\ F_{21}, F_{22}, F_{23} \\ F_{31}, F_{32}, F_{33} \end{pmatrix}.$$

Note that under the assumption : $\nabla \cdot \mathbf{w} = 0$ we have

$$(\mathbf{w} \cdot \nabla)\mathbf{w} = \nabla \cdot (\mathbf{w} \otimes \mathbf{w}).$$

Below, we say that $(\mathbf{w}, \pi)$ is a solution to (3) if

$$< \nabla\mathbf{w}, \nabla\varphi > + < (\mathbf{u}_\infty \cdot \nabla)\mathbf{w}, \varphi > - < \mathbf{w} \otimes \mathbf{w}, \nabla\varphi > - < \pi, \nabla \cdot \varphi >= - < F, \nabla\varphi >$$

for any $\varphi \in C_0^\infty(\Omega)^3$, and

$$\nabla \cdot \mathbf{w} = 0 \text{ in } \Omega, \qquad \mathbf{w}|_{\partial\Omega} = -\mathbf{u}_\infty, \qquad \lim_{|x|\to\infty} \mathbf{w}(x) = \mathbf{0}.$$

Here and hereafter we put

$$\begin{aligned} \nabla\varphi &= \begin{pmatrix} \partial_1\varphi_1, \partial_2\varphi_1, \partial_3\varphi_1 \\ \partial_1\varphi_2, \partial_2\varphi_2, \partial_3\varphi_2 \\ \partial_1\varphi_3, \partial_2\varphi_3, \partial_3\varphi_3 \end{pmatrix} && \text{for } \varphi = {}^T(\varphi_1, \varphi_2, \varphi_3); \\ < p, q >&= \int_\Omega p(x)q(x)\,dx && \text{when } p \text{ and } q \text{ are scalor;} \\ < \Phi, \Psi >&= \sum_{j,k=1}^{3} < \Phi_{jk}, \Psi_{jk} > && \text{for } 3 \times 3 \text{ matrices } \Phi = (\Phi_{jk}),\ \Psi = (\Psi_{jk})\ ; \\ < \mathbf{u}, \mathbf{v} >&= \sum_{j=1}^{3} < u_j, v_j > && \text{for } \mathbf{u} = {}^T(u_1, u_2, u_3),\ \mathbf{v} = {}^T(v_1, v_2, v_3)\ ; \end{aligned}$$

Theorem 1. (1)(Existence) *There eixsts an $\epsilon > 0$ such that $F = (F_{jk})$, $F_{jk} \in L_{3/2,\infty}(\Omega)$ and*

$$\sum_{j,k=1}^{3} \|F_{jk}\|_{L_{3/2,\infty}(\Omega)} + |\mathbf{u}_\infty| \leqq \epsilon,$$

then the problem (3) *admits a solution $(\mathbf{w}, \pi) \in L_{3,\infty}(\Omega)^3 \times L_{3/2,\infty}(\Omega)$ such that $\nabla\mathbf{w} \in L_{3/2,\infty}(\Omega)^{3\times 3}$, and moreover*

$$\|\nabla\mathbf{w}\|_{L_{3/2,\infty}(\Omega)} + \|\mathbf{w}\|_{L_{3,\infty}(\Omega)} + \|\pi\|_{L_{3/2,\infty}(\Omega)} \leqq C\epsilon$$

where C is independent of F, $\mathbf{w}$, π, ϵ and $\mathbf{u}_\infty$.

(2) (Uniqueness) *There exists an $\epsilon' > 0$ such that if $(\mathbf{w}_j, \pi_j)$, $j = 1, 2$, are solutions of (3) with the same external force $\mathbf{f}$ such that $\mathbf{w}_j \in L_{3,\infty}(\Omega)$, $\nabla \mathbf{w}_j \in L_{3/2,\infty}(\Omega)$, $\pi_j \in L_{3/2,\infty}(\Omega)$ and moreover*

$$\|\mathbf{w}_j\|_{L_{3,\infty}(\Omega)} \leqq \epsilon'$$

then $\mathbf{w}_1 = \mathbf{w}_2$ and $\pi_1 = \pi_2$.

Now, we will disuss the stability. Namely, we will discuss an exsitence of solutions of (4) with some uniform estimates with respect to $\mathbf{u}_\infty$. Moreover, we will discuss some decay property of solutions to (4). The problem (4) will be considered as a perturbation of the following evolutional Oseen equation :

$$(7) \qquad \begin{aligned} &\mathbf{u}_t - \Delta \mathbf{u} + (\mathbf{u}_\infty \cdot \nabla)\mathbf{u} + \nabla p = 0, \quad \nabla \cdot \mathbf{u} = 0 \quad \text{in } (0,\infty) \times \Omega, \\ &\mathbf{u}|_{\partial\Omega} = 0, \qquad \mathbf{u}|_{t=0} = \mathbf{a}. \end{aligned}$$

Let $\mathbb{P}$ denote the regular projection from $L_{p,q}(\Omega)^3$ into $L_{\sigma,p,q}(\Omega)^3 = \{\mathbf{v} \in L_{p,q}(\Omega) \mid \nabla \cdot \mathbf{v} = 0\}$ (cf. Kozono and Yamazaki [23, 24], Borchers and Miyakawa [6]). If we operate $\mathbb{P}$ to (7), we have

$$(8) \qquad \begin{aligned} &\mathbf{u}_t + \mathbb{P}(-\Delta + (\mathbf{u}_\infty \cdot \nabla))\mathbf{u} = 0 \quad \text{in } (0,\infty) \times \Omega, \\ &\mathbf{u}|_{\partial\Omega} = \mathbf{0}, \qquad \mathbf{u}|_{t=0} = \mathbf{a}. \end{aligned}$$

The Oseen operator $\mathbb{P}(-\Delta + (\mathbf{u}_\infty \cdot \nabla))$ generates an analytic semigroup $\{T_{\mathbf{u}_\infty}(t)\}_{t\geqq 0}$, which was proved by Miyakawa [29]. The following theorem concerning the decay property of $\{T_{\mathbf{u}_\infty}(t)\}_{t\geqq 0}$ is a key of our stability theorem.

Theorem 2. ($L_{p,r}$ - $L_{q,r}$ estimate) (i) *When $t \geqq 2$ and $|\mathbf{u}_\infty| \leqq \sigma$, we have the following estimates :*

$$(1) \qquad \|T_{\mathbf{u}_\infty}(t)\mathbf{a}\|_{L_{q,r}(\Omega)} \leqq Ct^{-\nu}\|\mathbf{a}\|_{L_{p,r}(\Omega)},$$
$$1 < p < \infty, \ \nu = \frac{3}{2}\left(\frac{1}{p} - \frac{1}{q}\right), \ 1 < p \leqq q < \infty, \ 1 \leqq r \leqq \infty.$$

$$(2) \qquad \|T_{\mathbf{u}_\infty}(t)\mathbf{a}\|_{L_\infty(\Omega)} \leqq C t^{-3/2p}\|\mathbf{a}\|_{L_{p,r}(\Omega)}, \qquad 1 < p < \infty, \ 1 \leqq r \leqq \infty.$$

$$(3) \qquad \|\nabla T_{\mathbf{u}_\infty}(t)\mathbf{a}\|_{L_{q,r}(\Omega)} \leqq Ct^{-(\nu+1/2)}\|\mathbf{a}\|_{L_{p,r}(\Omega)}, \ 1 < p \leqq q \leqq 3, \ 1 \leqq r \leqq \infty.$$

$$(4) \qquad \|\nabla T_{\mathbf{u}_\infty}(t)\mathbf{a}\|_{L_{q,r}(\Omega)} \leqq Ct^{-3/2p}\|\mathbf{a}\|_{L_{p,r}(\Omega)},$$
$$1 < p \leqq q, \ 3 < q < \infty, \ 1 \leqq r \leqq \infty.$$

$$(5) \qquad \|\nabla T_{\mathbf{u}_\infty}(t)\mathbf{a}\|_{L_\infty(\Omega)} \leqq Ct^{-3/2p}\|\mathbf{a}\|_{L_{p,r}(\Omega)}, \qquad 1 < p < \infty, \ 1 \leqq r \leqq \infty.$$

Here, C is independent of $\mathbf{u}_\infty$ while C depends on p, q, r and σ.
(ii) *When $0 < t \leqq 2$ and $|\mathbf{u}_\infty| \leqq \sigma$, we have the following estimates :*

$$(1) \qquad \|T_{\mathbf{u}_\infty}(t)\mathbf{a}\|_{L_{q,r}(\Omega)} \leqq Ct^{-\nu}\|\mathbf{a}\|_{L_{p,r}(\Omega)}, \qquad 1 < p \leqq q < \infty, \ 1 \leqq r \leqq \infty.$$

$$(2) \qquad \|\nabla T_{\mathbf{u}_\infty}(t)\mathbf{a}\|_{L_{q,r}(\Omega)} \leqq Ct^{-(\nu+1/2)}\|\mathbf{a}\|_{L_{q,r}(\Omega)}, \ 1 < p \leqq q < \infty, \ 1 \leqq r \leqq \infty.$$

Here, C is also independent of $\mathbf{u}_\infty$ while C depends on p, q, r and σ.

Applying $\mathbb{P}$ to (4) formally we have

$$\mathbf{z}_t + \mathbb{P}(-\Delta + \mathbf{u}_\infty \cdot \nabla)\mathbf{z} + \nabla\cdot[\mathbf{w}\otimes\mathbf{z} + \mathbf{z}\otimes\mathbf{w} + \mathbf{z}\otimes\mathbf{z}] = \mathbf{0},$$
$$\mathbf{z}\big|_{\partial\Omega} = \mathbf{0}, \quad \mathbf{z}\big|_{t=0} = \mathbf{b}.$$

Applying the Duhamel's principle, we have

$$\mathbf{z}(t) = T_{\mathbf{u}_\infty}(t)\mathbf{b} - \int_0^t T_{\mathbf{u}_\infty}(t-s)\mathbb{P}\nabla\cdot[\mathbf{w}\otimes\mathbf{z}(s) + \mathbf{z}(s)\otimes\mathbf{w} + \mathbf{z}(s)\otimes\mathbf{z}(s)]\,ds.$$

Testing the equation by $\varphi \in C_{0,\sigma}^\infty(\Omega) = \{\varphi \in C_0^\infty(\Omega) \mid \nabla\cdot\varphi = 0\}$, we have

$$\begin{aligned} <\mathbf{z}(t),\varphi> =& <T_{\mathbf{u}_\infty}(t)\mathbf{b},\varphi> \\ & - \int_0^t <T_{\mathbf{u}_\infty}(t-s)\mathbb{P}\nabla\cdot[\mathbf{w}\otimes\mathbf{z}(s) + \mathbf{z}(s)\otimes\mathbf{w} + \mathbf{z}(s)\otimes\mathbf{z}(s)],\varphi>\,ds \\ =& <T_{\mathbf{u}_\infty}(t)\mathbf{b},\varphi> \\ & + \int_0^t <\mathbf{w}\otimes\mathbf{z}(s) + \mathbf{z}(s)\otimes\mathbf{w} + \mathbf{z}(s)\otimes\mathbf{z}(s), \nabla[T_{-\mathbf{u}_\infty}(t-s)\varphi]>\,ds. \end{aligned}$$

Therefore, we introduce the following definition.

Definition 3. Let $3 < p < \infty$. We call $\mathbf{z}$ a mild solution of (4) in the class S_p if $\mathbf{z}$ satisfies the following conditions :

(i) $\quad \mathbf{z} \in BC((0,\infty); L_{3,\infty}(\Omega)),\ \nabla\cdot\mathbf{z} = 0,\ t^{(1/2-3/2p)}\mathbf{z}(t,\cdot) \in BC((0,\infty); L_{p,\infty}(\Omega))$;

(ii)
$$\begin{aligned} <\mathbf{z}(t),\varphi> =& <T_{\mathbf{u}_\infty}(t)\mathbf{b},\varphi> \\ & + \int_0^t <\mathbf{w}\otimes\mathbf{z}(s) + \mathbf{z}(s)\otimes\mathbf{w} + \mathbf{z}(s)\otimes\mathbf{z}(s), \nabla[T_{-\mathbf{u}_\infty}(t-s)\varphi>\,ds; \end{aligned}$$

(iii)
$$\lim_{t\to 0+} <\mathbf{z}(t),\varphi> = <\mathbf{b},\varphi> \qquad \forall\varphi \in C_{0,\sigma}^\infty(\Omega).$$

If a mild solution is regular in the usual sense, then it satisfies (4). To prove the regularity is now rather standard (cf. Kozono and Yamazaki [24], also Yamazaki [33]), and therefore we only give a sketch of our proof about the following existence theorem of mild solutions below.

Theorem 4. *Let $3 < p < \infty$. Then, there exists a $\sigma > 0$ such that if $\|\mathbf{b}\|_{L_{3,\infty}(\Omega)} + |\mathbf{u}_\infty| \leqq \sigma$ and $\nabla \cdot \mathbf{b} = 0$, then (4) admits a mild solution $\mathbf{z}$ in class S_p. Moreover, $\mathbf{z}$ satisfies the following estimate :*

$$[\mathbf{z}]_{3,\infty,t} + [\mathbf{z}]_{p,\infty,t} \leqq C\sigma \qquad \forall t \in (0,\infty), \tag{9}$$

where $C > 0$ is a constant independent of $\mathbf{u}_\infty$ and $\mathbf{b}$,

$$\begin{aligned} [\mathbf{z}]_{3,\infty,t} &= \sup_{0<s<t} \|\mathbf{z}(s,\cdot)\|_{L_{3,\infty}(\Omega)}, \\ [\mathbf{z}]_{p,\infty,t} &= \sup_{0<s<t} s^{(1/2-3/2p)} \|\mathbf{z}(s,\cdot)\|_{L_{p,\infty}(\Omega)}. \end{aligned} \tag{10}$$

Remark. By Marcinkiewitz interpolation theorem, for any $r \in (3,p)$ we have

$$\|\mathbf{z}(t,\cdot)\|_{L_r(\Omega)} \leqq C_r\, t^{-(1/2-3/2r)}\sigma \qquad \forall t \in (0,\infty).$$

Open Problem. Show the following decay property of our mild solution $\mathbf{z}$:

$$\sup_{0<s<t} s^{1/2} \|\mathbf{z}(s,\cdot)\|_{L_\infty(\Omega)} \leqq C\sigma,$$
$$\sup_{0<s<t} s^{1/2} \|\nabla\mathbf{z}(s,\cdot)\|_{L_{3,\infty}(\Omega)} \leqq C\sigma.$$

Sketch of Our Proof.

A Sketch of Our Proof of Theorem 1.

The linearized equation of (3) is the following Oseen equation in Ω :

$$\begin{cases} -\Delta\mathbf{u} + (\mathbf{u}_\infty \cdot \nabla)\mathbf{u} + \nabla\pi = \nabla \cdot F, \quad \nabla \cdot \mathbf{u} = 0 \qquad \text{in } \Omega, \\ \mathbf{u}|_{\partial\Omega} = \mathbf{0}. \end{cases} \tag{11}$$

In oreder to show the unique existence and estimates of solutions to (11), when $\mathbf{u}_\infty = \mathbf{0}$, Kozono and Yamazaki [23] used the duality argument. But, when $\mathbf{u}_\infty \neq \mathbf{0}$, this method does not seem to match with the Oseen equation, because of the first order term $\mathbf{u}_\infty \cdot \nabla$. We used a compact perturbation method, the idea of which going back to Shibata [31]. Namely, combining the unique existence and estimates of solutions in the whole space case and in the bounded domain case by using the cut-off technique, we reduce the problem to the Fredholm type equation on the right hand side. And then, the sharp uniqueness theorem for the Oseen equation in Ω implies the invertibility of this Fredmolm equation. Since we have to keep the divergence free condition, we use Bogovski lemma ([3, 4] and also [14, 20]). Essentially the same argument is found also in Shibata [32], Iwashita [20] and Kobayashi and Shibata [22]. While we have proved a linear theorem with very general exponents p and q, here we only state the following theorem in order to explain our basical idea.

Linear Theorem. *Let* $3/2 \leqq p < 3$ *and* $F = (F_{ij})$ (3×3 *matrix*) *with* $F_{ij} \in L_{p,\infty}(\Omega)$. *Then, there exists an* $\epsilon > 0$ *independent of* F *such that if* $|\mathbf{u}_\infty| \leqq \epsilon$, *then* (11) *admits a unique solution* $(\mathbf{u}, \pi) \in L_{3p/(3-p),\infty}(\Omega)^3 \times L_{p,\infty}(\Omega)$ *with* $\nabla \mathbf{u} \in L_{p,\infty}(\Omega)^{3\times 3}$.

Moreover, there exists a constant C *independent of* $\mathbf{u}_\infty$, F, $\mathbf{u}$ *and* π *such that*

$$\|\mathbf{u}\|_{L_{3p/(3-p),\infty}(\Omega)} + \|\nabla \mathbf{u}\|_{L_{p,\infty}(\Omega)} + \|\pi\|_{L_{p,\infty}(\Omega)} \leqq C\|F\|_{L_{p,\infty}(\Omega)}. \tag{12}$$

In oeder to solve (3) by using Linear Theorem, we construct a vector of $C_0^\infty(\mathbb{R}^3)$ functions $\mathbf{b}_{\mathbf{u}_\infty}(x)$ such that

$$\begin{aligned} &\nabla \cdot \mathbf{b}_{\mathbf{u}_\infty}(x) = 0, \quad \mathbf{b}_{\mathbf{u}_\infty}|_{\partial\Omega} = -\mathbf{u}_\infty, \quad \mathbf{b}_{\mathbf{u}_\infty}(x) = \mathbf{0} \quad (\,|x| \geqq {}^\exists R), \\ &|\partial_x^\alpha \mathbf{b}_{\mathbf{u}_\infty}(x)| \leqq C_\alpha |\mathbf{u}_\infty| \quad \forall \alpha. \end{aligned}$$

Such a vector-valued function is easily constructed by using the Bogovskii theorem ([3, 4] and also [14, 20]). Put $\mathbf{u} = \mathbf{b}_{\mathbf{u}_\infty} + \mathbf{v}$ and then (11) is reduced to the following equation :

$$\left\{ \begin{array}{l} -\Delta \mathbf{v} + (\mathbf{u}_\infty \cdot \nabla)\mathbf{v} + \nabla \cdot [(\mathbf{b}_{\mathbf{u}_\infty} + \mathbf{v}) \otimes (\mathbf{b}_{\mathbf{u}_\infty} + \mathbf{v})] + \nabla \pi = \nabla \cdot F \quad \text{in } \Omega, \\ \nabla \cdot \mathbf{u} = 0 \qquad \text{in } \Omega, \qquad \mathbf{u}|_{\partial\Omega} = \mathbf{0}, \qquad \lim_{|x| \to \infty} \mathbf{v}(x) = \mathbf{0}. \end{array} \right. \tag{12}$$

As the underliying space, we put

$$\begin{aligned} \mathcal{I}_\sigma = \{(\mathbf{u}, \pi) \in L_{3,\infty}(\Omega)^3 \times L_{3/2,\infty}(\Omega) \mid \nabla \mathbf{u} \in L_{3/2,\infty}(\Omega)^{3\times 3},\ \mathbf{u}|_{\partial\Omega} = \mathbf{0},\ \nabla \cdot \mathbf{u} = 0 \\ \|\mathbf{u}\|_{L_{3,\infty}(\Omega)} + \|\nabla \mathbf{u}\|_{L_{3/2,\infty}(\Omega)} + \|\nabla \pi\|_{L_{3/2,\infty}(\Omega)} \leqq \sigma\}, \end{aligned}$$

because the exponent p for which the assertions that $\mathbf{w} \in L_{3p/(3-p)}(\Omega)$ implies $\mathbf{w} \otimes \mathbf{w} \in L_p(\Omega)$ and that $\nabla \mathbf{w} \in L_p(\Omega)$ imples $\mathbf{w} \in L_{3p/(3-p)}(\Omega)$ is equal to $3/2$ only. By using Linear Theorem and the contraction mapping principle, we can prove the existence of solutions to (12) in $\mathcal{I}_\sigma$ immediately under suitable choice of a small positive number σ.

From now on, we give

A Sketch of Our Proof of Linear Theorem. *1st step : Analysis of solutions in* $\mathbb{R}^3$. By Fourier transform we can write a solution $(\mathbf{u}, \pi)$ to the equation in the whole space :

$$(-\Delta \mathbf{u} + (\mathbf{u}_\infty \cdot \nabla))\mathbf{u} + \nabla \pi = \nabla \cdot F, \quad \nabla \cdot \mathbf{u} = 0 \quad \text{in } \mathbb{R}^3$$

by the following form :

$$\mathbf{u}(x) = E_{\mathbf{u}_\infty}[F](x) = \mathcal{F}^{-1}\left[\sum_{j=1}^{3} \frac{i\xi_j}{|\xi|^2 + i\mathbf{u}_\infty \cdot \xi}\left(\hat{F}_j(\xi) - \frac{\xi(\xi \cdot \hat{F}_j(\xi))}{|\xi|^2}\right)\right](x),$$

$$\pi(x) = \Pi[F](x) = \mathcal{F}^{-1}\left[\sum_{j=1}^{3} \frac{\xi_j(\xi \cdot \hat{F}_j(\xi))}{|\xi|^2}\right](x).$$

Since

$$\left|\xi^\alpha \left(\frac{\partial}{\partial \xi}\right)^\alpha (|\xi|^2 + i|\mathbf{u}_\infty|\xi_1)^{-1}\right| \leqq C_\alpha ||\xi|^2 + i|\mathbf{u}_\infty|\xi_1|^{-1} \quad \forall\, \alpha,$$

where C_α is independent of $\mathbf{u}_\infty$, by the orthogonal transformation in ξ and the Lizorkin theorem about the Fourier multiplier oprator we can see easily that

$$\|\mathbf{u}\|_{L_{3p/(3-p)}(\mathbb{R}^3)} + \|\nabla \mathbf{u}\|_{L_p(\mathbb{R}^3)} + \|\pi\|_{L_p(\mathbb{R}^3)} \leqq C_p \|F\|_{L_p(\mathbb{R}^3)}.$$

Since $L_{p,\infty}(\mathbb{R}^3) = (L_{p_1}, L_{p_2})_{\theta,\infty}$, $1/p = (1-\theta)/p_1 + \theta/p_2$ in the real interpolation sense, we have

$$\|\mathbf{u}\|_{L_{3p/(3-p),\infty}(\mathbb{R}^3)} + \|\nabla \mathbf{u}\|_{L_{p,\infty}(\mathbb{R}^3)} + \|\pi\|_{L_{p,\infty}(\mathbb{R}^3)} \leqq C_p \|F\|_{L_{p,\infty}(\mathbb{R}^3)}. \tag{13}$$

After cutting off the solutions, we have to handle with the following equation :

$$-\Delta \mathbf{u} + (\mathbf{u}_\infty \cdot \nabla)\mathbf{u} + \nabla \pi = \mathbf{f}, \quad \nabla \cdot \mathbf{u} = 0 \quad \text{in } \mathbb{R}^3, \tag{14}$$

where $\mathbf{f} \in L_{p,\infty}(\mathbb{R}^3)$ with supp $\mathbf{f} \subset B_b = \{x \in \mathbb{R}^3 \mid |x| < b\}$. Let $(E(\mathbf{u}_\infty)(x), P(x))$ denote the Oseen fundamental solution whose exact formula was given by Oseen [30] (cf. also [14, 22, 32]), and then the solution of (14) is given by the convolution formula : $\mathbf{u} = E(\mathbf{u}_\infty) * \mathbf{f}$ and $\pi = P * \mathbf{f}$. Since

$$|E(\mathbf{u}_\infty)(x)| \leqq \frac{C}{|x|}, \quad |\nabla E(\mathbf{u}_\infty)| \leqq \begin{cases} \dfrac{C}{|x|^{3/2} s_{\mathbf{u}_\infty}(x)^{1/2}} & (\mathbf{u}_\infty \neq \mathbf{0}) \\ \dfrac{C}{|x|^2} & (\mathbf{u}_\infty = \mathbf{0}), \end{cases} \quad |p(x)| \leqq \frac{C}{|x|^2}$$

where $s_{\mathbf{u}_\infty}(x) = |x| - \mathbf{u}_\infty \cdot x/|\mathbf{u}_\infty|$ and C is independent of $\mathbf{u}_\infty$, we have

$$\|E(\mathbf{u}_\infty)\|_{L_{3,\infty}(\mathbb{R}^3)} \leqq C, \quad \|\nabla E(\mathbf{u}_\infty)\|_{L_{3/2,\infty}(\mathbb{R}^3)} \leqq C, \quad \|p\|_{L_{3/2,\infty}(\mathbb{R}^3)} \leqq C,$$

where C is independent of $\mathbf{u}_\infty$. Therefore, by the generalized Young inequality we see that

$$\begin{aligned} \|\mathbf{u}\|_{L_{3p/(3-p),\infty}(\mathbb{R}^3)} &\leqq \|E(\mathbf{u}_\infty)\|_{L_{3/2,\infty}(\mathbb{R}^3)} \|\mathbf{f}\|_{L_q(\mathbb{R}^3)} \leqq C_b \|\mathbf{f}\|_{L_{p,\infty}(\mathbb{R}^3)}, \\ \|\nabla \mathbf{u}\|_{L_{p,\infty}(\mathbb{R}^3)} &\leqq \|\nabla E(\mathbf{u}_\infty)\|_{L_{3/2,\infty}(\mathbb{R}^3)} \|\mathbf{f}\|_{L_q(\mathbb{R}^3)} \leqq C_b \|\mathbf{f}\|_{L_{p,\infty}(\mathbb{R}^3)}, \\ \|p\|_{L_{p,\infty}(\mathbb{R}^3)} &\leqq \|P\|_{L_{3/2,\infty}(\mathbb{R}^3)} \|\mathbf{f}\|_{L_q(\mathbb{R}^3)} \leqq C_b \|\mathbf{f}\|_{L_{p,\infty}(\mathbb{R}^3)}, \end{aligned}$$

where $1 + (3-p)/3p = 1/3 + 1/q$, $1 + 1/p = 2/3 + 1/q$ and $1 \leqq q < p$. To obtain that $q \geqq 1$, we need the assumption : $p \geqq 3/2$. The restriction : $p < 3$ comes from the Sobolev inequality :

$$\|\mathbf{u}\|_{L_{3p/(3-p),\infty}(\mathbb{R}^3)} \leqq C_p \|\nabla \mathbf{u}\|_{L_p(\mathbb{R}^3)}.$$

2nd step : Solutions in a bounded domain. Let D be a bounded domain in $\mathbb{R}^3$ with smooth boundary ∂D. By interpolating the well-known theorem concerning the Stokes equation and Oseen equation in a bounded domain, we have tha following theorem.

Theorem. *Given* $F = (F_{ij}) \in L_{p,\infty}(D)^{3\times 3}$, $F_0 \in L_{p,\infty}(D)$ *and* $c \in \mathbb{R}$, *there exists a unique solution* $(\mathbf{w}, \pi) \in W^1_{p,\infty}(D)^3 \times L_{p,\infty}(D)$ *to the equation :*

$$\begin{aligned} &< \nabla \mathbf{w}, \nabla\varphi > + < (\mathbf{u}_\infty \cdot \nabla)\mathbf{w}, \varphi > - < \pi, \nabla\cdot\varphi > \\ &\qquad =< F, \nabla\varphi > + < F_0, \varphi > \quad \forall\varphi \in C_0^\infty(D), \\ &\int_D \pi\, dx = c, \quad \nabla\cdot\mathbf{w} = 0 \quad in\ \Omega, \quad \mathbf{w}|_{\partial\Omega} = \mathbf{0}. \end{aligned}$$

Moreover, if $|\mathbf{u}_\infty| \leqq \sigma_0$ *and* $1 < p < 3$, *then there exists a constant* C *depending on* p, D *and* σ_0 *such that*

$$\|\mathbf{w}\|_{L_{3p/(3-p),\infty}(D)} + \|\nabla\mathbf{w}\|_{L_{p,\infty}(D)} + \|\pi\|_{L_{p,\infty}(D)} \leqq C\|(F, F_0)\|_{L_{p,\infty}(D)}$$

If $F = 0$, *then* $\mathbf{w} \in W^2_{p,\infty}(D)$, $\pi \in W^1_{p,\infty}(D)$ *and*

$$\|\mathbf{w}\|_{W^2_{p,\infty}(D)} + \|\pi\|_{W^1_{p,\infty}(D)} \leqq C\|F_0\|_{L_{p,\infty}(D)}.$$

Here and hereafter,

$$W^m_{p,\infty}(G) = \{w \in L_{p,\infty}(G) \mid \|w\|_{W^m_{p,\infty}(G)} = \sum_{|\alpha|\leqq m} \|\partial_x^\alpha w\|_{L_{p,\infty}(G)} < \infty\}.$$

For the latter purpose, we write the solution given in the above theorem as follows :

$$\mathbf{w} = \mathcal{L}(D, \mathbf{u}_\infty)[F, F_0, c], \quad \pi = \mathfrak{p}(D, \mathbf{u}_\infty)[F, F_0, c].$$

3rd step : Bogovskii Operator. Let $1 < p < \infty$ and let D be a bounded domain in $\mathbb{R}^3$ with smooth boundary ∂D.

$$\begin{aligned} W^m_{p,\infty,0}(D) &= \{u \in W^m_{p,\infty}(D) \mid \partial_x^\alpha u|_{\partial D} = 0 \quad (|\alpha| \leqq m-1)\}, \\ \overset{o}{W}{}^m_{p,\infty,0}(D) &= \{u \in W^m_{p,\infty,0}(D) \mid \int_D u\, dx = 0\}. \end{aligned}$$

Interpolating the well-known Bogovskii theorem ([3, 4] and also [14, 20]), we can construct a linear operator $\mathbb{B} : \overset{o}{W}{}^m_{p,\infty,0}(D) \longrightarrow W^{m+1}_{p,\infty,0}(D)^3$ such that for $f \in \overset{o}{W}{}^m_{p,\infty,0}(D)$ we have $\nabla\cdot\mathbb{B}[f] = f$ in D and

$$\|\mathbb{B}[f]\|_{W^{m+1}_{p,\infty}(D)} \leqq C\|f\|_{W^m_{p,\infty}(D)}$$

where the constant C depends on m, p and D. Since $\mathbb{B}[f] \in W^{m+1}_{p,\infty,0}(D)^3$, we can extend $\mathbb{B}[f]$ to the whole space by $\mathbf{0}$ outside D, and then $\mathbb{B}[f] \in W^{m+1}_{p,\infty}(\mathbb{R}^3)^3$, supp $\mathbb{B}[f] \subset D$, $\nabla\cdot\mathbb{B}[f] = f_0$ in $\mathbb{R}^3$ and

$$\|\mathbb{B}[f]\|_{W^{m+1}_{p,\infty}(\mathbb{R}^3)} \leqq C\|f\|_{W^m_{p,\infty}(D)}$$

where $f_0(x)$ also denotes the 0 extension of f to the whole space.

4th step : A Reduction to the Equation of the Fredholm type. Devide soluton to (11) into three parts :

$$\mathbf{u} = \mathbf{v}_\infty + \mathbf{v}_0 + \mathbf{v}_c, \quad \pi = \pi_\infty + \pi_0 + \pi_c.$$

$\mathbf{v}_\infty$ and π_∞ are defined in the following manner. Let φ_∞ and ψ_∞ be functions in $C^\infty(\mathbb{R}^3)$ such that

$$\varphi_\infty = \begin{cases} 1 & |x| \geqq R \\ 0 & |x| \leqq R-1 \end{cases}, \qquad \psi_\infty = \begin{cases} 1 & |x| \geqq R-1 \\ 0 & |x| \leqq R-2 \end{cases}.$$

Note that $\psi_\infty = 1$ on supp φ_∞. Put

$$\mathbf{v}_\infty = \psi_\infty E_{\mathbf{u}_\infty}[\varphi_\infty F] - \mathbb{B}[\nabla \psi_\infty \cdot E_{\mathbf{u}_\infty}[\varphi_\infty F]], \quad \pi_\infty = \psi_\infty \Pi[\varphi_\infty F].$$

Put $\varphi_0 = 1 - \psi_\infty$ and let $\psi_0 \in C_0^\infty(\mathbb{R}^3)$ such that

$$\psi_0(x) = \begin{cases} 1 & |x| \leqq R \\ 0 & |x| \geqq R+1 \end{cases}, \qquad \psi_0(x) = 1 \ \text{ on supp } \varphi_0.$$

Take R so large that $B_{R-4} \supset \partial\Omega$. Put $D = \Omega_{R+2} = \Omega \cap B_{R+2}$, and therefore

$$\begin{aligned} v_0 &= \psi_0 \mathcal{L}(D, \mathbf{u}_\infty)[\varphi_0 F, 0, 0] - \mathbb{B}[\nabla \psi_0 \cdot \mathcal{L}(D, \mathbf{u}_\infty)[\varphi_0 F, 0, 0]], \\ \pi_0 &= \psi_0 \mathfrak{p}(D, \mathbf{u}_\infty)[\varphi_0 F, 0, 0]. \end{aligned}$$

Then, we arrive at the following equation to $(\mathbf{v}_c, \pi_c)$:

$$\text{(15)} \qquad \begin{cases} -\Delta \mathbf{v}_c + (\mathbf{u}_\infty \cdot \nabla)\mathbf{v}_c + \nabla \pi_c = r(\mathbf{u}_\infty)[f], \quad \nabla \cdot \mathbf{v}_c = 0 \ \text{ in } \Omega, \\ \mathbf{v}_c|_{\partial\Omega} = \mathbf{0} \end{cases}$$

where $r(\mathbf{u}_\infty)[F] \in L_{p,\infty}(\Omega)$, supp $r(\mathbf{u}_\infty)[F] \subset D' = \{x \in \mathbb{R}^3 \mid R-2 \leqq |x| \leqq R+1\}$ and $\|r(\mathbf{u}_\infty)[F]\|_{L_{p,\infty}(\Omega)} \leqq C\|F\|_{L_{p,\infty}(\Omega)}$ with some constant $C > 0$ independent of $\mathbf{u}_\infty$ whenever $|\mathbf{u}_\infty| \leqq \sigma_0$. From this point of view, we are going to solve the following equation :

$$\text{(16)} \qquad \begin{cases} -\Delta \mathbf{u} + (\mathbf{u}_\infty \cdot \nabla)\mathbf{u} + \nabla \pi = \mathbf{f}, \quad \nabla \cdot \mathbf{u} = 0 \ \text{ in } \Omega, \\ \mathbf{u}|_{\partial\Omega} = \mathbf{0} \end{cases}$$

where $\mathbf{f} \in L_{p,\infty}(\Omega)$ and supp $\mathbf{f} \subset D'$. The equation (16) is solved by the compact perturbation method. In fact, put

$$\begin{aligned} P(\mathbf{u}_\infty)\mathbf{f} &= (1-\varphi)E(\mathbf{u}_\infty) * \mathbf{f}^0 + \varphi \mathcal{L}(\Omega_{R+2}, 0)[0, \mathbf{f}|_{\Omega_{R+2}}, c] \\ &\quad + \mathbb{B}[(\nabla\varphi) \cdot (E(\mathbf{u}_\infty) * \mathbf{f}^0)] - \mathbb{B}[(\nabla\varphi) \cdot \mathcal{L}(\Omega_{R+2}, 0)[0, \mathbf{f}|_{\Omega_{R+2}}, c]] \\ Q\mathbf{f} &= (1-\varphi)p * \mathbf{f}^0 + \varphi \mathfrak{p}(\Omega_{R+2}, 0)[0, \mathbf{f}|_{\Omega_{R+2}}, c] \end{aligned}$$

where

$$c = \int_{\Omega_{R+2}} \pi * \mathbf{f}^0 \, dx, \quad \varphi(x) = \begin{cases} 1 & |x| \leqq R-2 \\ 0 & |x| \geqq R+1 \end{cases}, \quad \mathbf{f}^0(x) = \begin{cases} \mathbf{f}(x) & x \in \Omega \\ \mathbf{0} & x \notin \Omega \end{cases}$$

and $\mathbf{f}|_{\Omega_{R+2}}$ is the restriction of $\mathbf{f}$ to Ω_{R+2}. $P(\mathbf{u}_\infty)\mathbf{f}$ and $Q\mathbf{f}$ satisfy the following equation :

$$(-\Delta + \mathbf{u}_\infty \cdot \nabla)P(\mathbf{u}_\infty)\mathbf{f} + \nabla(Q\mathbf{f}) = \mathbf{f} + S(\mathbf{u}_\infty)\mathbf{f}, \quad \nabla \cdot P(\mathbf{u}_\infty)\mathbf{f} = 0 \quad \text{in } \Omega.$$
$$P(\mathbf{u}_\infty)\mathbf{f}|_{\partial\Omega} = \mathbf{0}$$

where

$$\begin{aligned} S(\mathbf{u}_\infty)\mathbf{f} = & \, 2(\nabla\varphi) \cdot \nabla E(\mathbf{u}_\infty) * \mathbf{f}^0 + (\Delta\varphi)E(\mathbf{u}_\infty) * \mathbf{f}^0 + [(\mathbf{u}_\infty \nabla \cdot)\varphi]E(\mathbf{u}_\infty) * \mathbf{f}^0 \\ & + 2(\nabla\varphi) \cdot \mathcal{L}(\Omega_{R+2}, 0)[0, \mathbf{f}|_{\Omega_{R+2}}, c] - (\Delta\varphi)\mathcal{L}(\Omega_{R+2}, 0)[0, \mathbf{f}|_{\Omega_{R+2}}, c] - \\ & + (\mathbf{u}_\infty \cdot \nabla)(\varphi\mathcal{L}(\Omega_{R+2}, 0)[0, \mathbf{f}|_{\Omega_{R+2}}, c]) \\ & + (-\Delta + \mathbf{u}_\infty \cdot \nabla)(\mathbb{B}[(\nabla\varphi) \cdot E(\mathbf{u}_\infty) * \mathbf{f}^0] - \mathbb{B}[(\nabla\varphi) \cdot \mathcal{L}(\Omega_{R+2}, 0)[0, \mathbf{f}|_{\Omega_{R+2}}, c]]) \\ & - (\nabla\varphi)(p * \mathbf{f}^0 - \mathfrak{p}(\Omega_{R+2}, 0)[0, \mathbf{f}|_{\Omega_{R+2}}, c]). \end{aligned}$$

Since $S(\mathbf{u}_\infty)\mathbf{f} \in W^1_{p,\infty}(\Omega)$ and supp $S(\mathbf{u}_\infty)\mathbf{f} \subset D'$, $S(\mathbf{u}_\infty)$ is a compact operator from $L_{p,\infty,D'}(\Omega)$ into itself, where

$$L_{p,\infty,D'}(\Omega) = \{\mathbf{f} \in L_{p,\infty}(\Omega)^3 \mid \text{supp } \mathbf{f} \subset D'\}.$$

By using the representation formula of $E(\mathbf{u}_\infty) * \mathbf{f}^0$, we see easily that

$$\|S(\mathbf{u}_\infty) - S(\mathbf{0})\|_{\mathcal{L}(L_{p,\infty,D'}(\Omega))} \leqq C|\mathbf{u}_\infty|^{1/2} \tag{17}$$

when $|\mathbf{u}_\infty| \leqq 1$, where $\mathcal{L}(L_{p,\infty,D'}(\Omega))$ is the set of bounded linear operators from $L_{p,\infty,D'}(\Omega)$ into itself.

Our uniqueness theorem is the following one.

Uniqueness Theorem. *Let $1 < p < \infty$. If $(\mathbf{u}, \pi) \in \mathcal{S}'(\Omega)^4 \cap (W^2_{p,loc}(\Omega)^3 \times W^1_{p,loc}(\Omega))$ satisfies the homogeneous equation :*

$$-\Delta\mathbf{u} + (\mathbf{u}_\infty \cdot \nabla)\mathbf{u} + \nabla\pi = \mathbf{0}, \quad \nabla \cdot \mathbf{u} = 0 \quad \textit{in } \Omega, \quad \mathbf{u}|_{\partial\Omega} = \mathbf{0}$$

and the growth order condition :

$$\lim_{R\to\infty} R^{-3} \int_{R \leqq |x| \leqq 2R} |\mathbf{u}(x)| \, dx = 0, \quad \lim_{R\to\infty} R^{-3} \int_{R \leqq |x| \leqq 2R} |\pi(x)| \, dx = 0,$$

then $\mathbf{u} = \mathbf{0}$ and $\pi = 0$. Here, we put

$$\mathcal{S}'(\Omega) = \{u \mid \exists U \in \mathcal{S}' \textit{such that } u = U \textit{on } \Omega\}.$$

Remark. if $1 < p < 3$ and $\mathbf{u} \in L_{3p/(3-p),\infty}(\Omega)$, $\nabla\mathbf{u} \in L_{p,\infty}(\Omega)$ and $\pi \in L_{p,\infty}(\Omega)$, then $(\mathbf{u}, \pi)$ satisfies the growth order condition. But, in general the uniqueness does not hold for the exterior domain when $\mathbf{u} \in L_{p,loc}(\Omega)^3$ with $\nabla\mathbf{u} \in L_{p,\infty}(\Omega)^{3\times3}$ and $p \geqq 3$.

By using the Fredholm alternative theorem for the $I+$ compact operator, we have the following lemma.

Key Lemma. *There exists an* $\epsilon > 0$ *such that if* $|\mathbf{u}_\infty| \leqq \epsilon$*, then the inverse operator* $(I + S(\mathbf{u}_\infty))^{-1}$ *of* $I + S(\mathbf{u}_\infty)$ *exists in* $\mathcal{L}(L_{p,\infty,D'}(\Omega))$*. Moreover, we have*

$$\|(I + S(\mathbf{u}_\infty))^{-1}\|_{\mathcal{L}(L_{p,\infty,D'}(\Omega))} \leqq C$$

where C *is independent of* $\mathbf{u}_\infty$ *whenever* $|\mathbf{u}_\infty| \leqq \epsilon$*.*

Proof. By (17), it is sufficient to show the lemma in the case where $\mathbf{u}_\infty = \mathbf{0}$. In view of Fredholm alternative theorem, we have only to show the injectivity of $I + S(\mathbf{u}_\infty)$. Therefore, we take $\mathbf{f} \in L_{p,\infty,D'}(\Omega)$ such that $(I + S(\mathbf{u}_\infty))\mathbf{f} = \mathbf{0}$. And, we will show that $\mathbf{f} = \mathbf{0}$. By the definition of $S(\mathbf{u}_\infty)$ we have $-\Delta P(\mathbf{0})\mathbf{f} + \nabla Q\mathbf{f} = \mathbf{0}$ in Ω, $\nabla \cdot P(\mathbf{0})\mathbf{f} = 0$ in Ω and $P(\mathbf{0})\mathbf{f}|_{\partial\Omega} = \mathbf{0}$. By the uniqueness theorem, $P(\mathbf{0})\mathbf{f} = 0$ and $Q\mathbf{f} = 0$. And then, employing the argument due to Shibata [31] and also Iwashita [20], we see that $\mathbf{f} = \mathbf{0}$.

By Key lemma, the solution $(\mathbf{v}_c, \pi_c)$ of (15) can be written by the formula :

$$\mathbf{v}_c = P(\mathbf{u}_\infty)(I + S(\mathbf{u}_\infty))^{-1} r(\mathbf{u}_\infty)[\mathbf{f}], \quad \pi_c = Q(I + S(\mathbf{u}_\infty))^{-1} r(\mathbf{u}_\infty)[\mathbf{f}],$$

which completes our proof of Linear Theorem.

A Sketch of Our Proofs of Theorems 2 *and* 4

In order to show Theorem 2, we use the following estimate due to Kobayashi and Shibata [] :

$$\sum_{j=0}^{1} \|\partial_t^j T_{\mathbf{u}_\infty}(t)\mathbf{a}\|_{W_\infty^m(\Omega_R)} \leqq C_{p,m,R}(1+t)^{-3/2p}\|\mathbf{a}\|_{L_p(\Omega)} \tag{18}$$

for any $1 < p < \infty$, $m \geqq 0$ and $R >> 1$ with a suitable constant $C_{p,m,R}$ independent of $\mathbf{u}_\infty$. Interpolating this inequality, we have

$$\sum_{j=0}^{1} \|\partial_t^j T_{\mathbf{u}_\infty}(t)\mathbf{a}\|_{W_\infty^m(\Omega_R)} \leqq C_{p,m,R}(1+t)^{-3/2p}\|\mathbf{a}\|_{L_{p,q}(\Omega)} \tag{19}$$

for any $1 < p < \infty$ and $1 \leqq q \leqq \infty$. Let $S_{\mathbf{u}_\infty}(t)\mathbf{a}$ denote a solution of the evolutional Oseen equation in the whole space. By the usual L_p - L_q estimate and the interpolation theorem, we have

$$\|\partial_t^j \partial_x^\alpha S_{\mathbf{u}_\infty}(t)\mathbf{a}\|_{L_{q,r}(\mathbb{R}^3)} \leqq C_{p,q,r,j,\alpha} t^{-(\nu+j+|\alpha|/2)}\|\mathbf{a}\|_{L_{p,r}(\mathbb{R}^3)}, \quad \nu = \frac{3}{2}\left(\frac{1}{p} - \frac{1}{q}\right) \tag{20}$$

for $1 < p \leqq q < \infty$, $1 \leqq r \leqq \infty$, and

$$\|\partial_t^j \partial_x^\alpha S_{\mathbf{u}_\infty}(t)\mathbf{a}\|_{L_\infty(\mathbb{R}^3)} \leqq C_{p,q,r,j,\alpha} t^{-(3/2p+j+|\alpha|/2)}\|\mathbf{a}\|_{L_{p,r}(\mathbb{R}^3)} \tag{21}$$

for $1 < p < \infty$ and $1 \leqq r \leqq \infty$, when $t > 0$. By using the cut-off function and combining (18), (19) and (20) and employing the same argument due to Kobayashi and Shibata [22] and also Iwashita [20], we have Theorem 2.

Now, we will give a sketch of our proof of Theorem 4. We proved Theorem 4 by the contraction mapping principle. As the underlying space, we put

$$\mathcal{I}_\sigma = \{\mathbf{u}(t,\cdot) \in BC((0,\infty);\ L_{3,\infty}(\Omega)^3) \mid \nabla\cdot\mathbf{u} = 0 \text{ in } \Omega,$$
$$[\mathbf{u}]_{3,\infty,t} + [\mathbf{u}]_{p,\infty,t} \leqq \sigma \ \text{ for } \forall\, t > 0\}.$$

Given $\mathbf{u}(t) = \mathbf{u}(t,\cdot) \in \mathcal{I}_\sigma$, let us define $\mathbf{v}(t) = \mathbf{v}(t,\cdot)$ for each $t > 0$ by the formula :

$$< \mathbf{v}(t), \varphi > = < T_{\mathbf{u}_\infty}(t)\mathbf{b}, \varphi >$$
$$- \int_0^t < \mathbf{w}\otimes\mathbf{u}(s) + \mathbf{u}(s,\cdot)\otimes\mathbf{w} + \mathbf{u}(s)\otimes\mathbf{u}(s), \nabla[T_{-\mathbf{u}_\infty}(t-s)\varphi] > ds$$

for all $\varphi \in C^\infty_{0,\sigma}(\Omega)$. What we have to show is that

$$(22)\quad | < \mathbf{v}(t),\varphi > | \leqq C\{\|\mathbf{b}\|_{L_{3,\infty}(\Omega)} + \|\mathbf{w}\|_{L_{3,\infty}(\Omega)}[\mathbf{u}]_{3,\infty,t} + [\mathbf{u}]^2_{3,\infty,t}\}\|\varphi\|_{L_{3/2,1}(\Omega)},$$
$$(23)\quad | < \mathbf{v}(t),\varphi > | \leqq Ct^{-(1/2-3/2p)}\{\|\mathbf{b}\|_{L_{3,\infty}(\Omega)} + \|\mathbf{w}\|_{L_{3,\infty}(\Omega)}[\mathbf{u}]_{p,\infty,t}$$
$$+ [\mathbf{u}]_{3,\infty,t}[\mathbf{u}]_{p,\infty,t}\}\|\varphi\|_{L_{q,1}(\Omega)},\quad \frac{1}{p}+\frac{1}{q} = 1.$$

Since we can get the continuity of $\mathbf{v}(t,\cdot)$ with respect to $t > 0$ by considering the difference : $< \mathbf{v}(t_1) - \mathbf{v}(t_2), \varphi >$, we see that $\mathbf{v} \in \mathcal{I}_\sigma$. Taking σ smaller if necessary, we can also see easily that the map : $\mathbf{u} \mapsto \mathbf{v}$ is a contraction one from $\mathcal{I}_\sigma$ into iteself, which completes the proof of Theorem 4.

Therefore, we shall explain how to get (22) and (23) below. The key is the following lemma.

LEMMA. *If $1 < q < r \leqq 3$ and $1/q - 1/r = 1/3$, then we have*

$$\int_0^\infty \|\nabla[T_{\mathbf{u}_\infty}(t)\varphi]\|_{L_{r,1}(\Omega)}\, dt \leqq C_{r,q}\|\varphi\|_{L_{q,1}(\Omega)}.$$

Remark. From the usual L_p - L_q estimate, we have

$$\|\nabla[T_{\mathbf{u}_\infty}(t)\varphi]\|_{L_r(\Omega)} \leqq C_{r,q}t^{-1}\|\varphi\|_{L_q(\Omega)}$$

when $1/q - 1/r = 1/3$, which does not imply the integrability. In order to get the integrability, we used a little bit smaller spaces $L_{r,1}$ and $L_{q,1}$ than L_r and L_q, which is a crusial part of our argument.

Proof of LEMMA. Observe that

$$\int_0^\infty \|\nabla[T_{\mathbf{u}_\infty}(t)\varphi]\|_{L_{r,1}(\Omega)}\, dt = \sum_{j=-\infty}^{\infty}\int_{2^{j-1}}^{2^j} \|\nabla[T_{\mathbf{u}_\infty}(t)\varphi]\|_{L_{r,1}(\Omega)}\, dt \leqq \frac{1}{2}\sum_{j=-\infty}^{\infty} 2^j m_j$$

where

$$m_j = \sup_{2^{j-1} \leqq t \leqq 2^j} \|\nabla[T_{\mathbf{u}_\infty}(t)\varphi]\|_{L_{r,1}(\Omega)}.$$

By $L_{p,1}$ - $L_{q,1}$ estimate,

$$\|\nabla[T_{\mathbf{u}_\infty}(t)\varphi]\|_{L_{r,1}(\Omega)} \leqq d_{p_k} t^{-\frac{3}{2}\left(\left(\frac{1}{p_k}-\frac{1}{r}\right)+\frac{1}{2}\right)} \|\varphi\|_{L_{p_k,1}(\Omega)}$$

with suitable constant d_{p_k} independent of $\mathbf{u}_\infty$ for $k = 0$, 1, where $1 < p_0 < q < p_1 < r \leqq 3$. Since $2^{j-1} \leqq t \leqq 2^j$, we see that

$$m_j \leqq d_{p_k} 2^{\left(\frac{3}{2}\left(\frac{1}{p_k}-\frac{1}{r}\right)+\frac{1}{2}\right)} (2^j)^{-\left(\frac{3}{2}\left(\frac{1}{p_k}-\frac{1}{r}\right)+\frac{1}{2}\right)} \|\varphi\|_{L_{p_k,1}(\Omega)}.$$

Put

$$C_{p_k} = d_{p_k} 2^{\left(\frac{3}{2}\left(\frac{1}{p_k}-\frac{1}{r}\right)+\frac{1}{2}\right)} \quad \text{and} \quad s_k = \frac{3}{2}\left(\frac{1}{p_k} - \frac{1}{r}\right) + \frac{1}{2},$$

and then

$$\sup_{j \in \mathbb{Z}} (2^j)^{s_k} m_j \leqq C_{p_k} \|\varphi\|_{L_{p_k,1}(\Omega)}, \quad k = 0, 1.$$

By the real interpolation, we see that

$$(\ell_\infty^{s_0}, \ell_\infty^{s_1})_{\theta,1} = \ell_1^s, \quad s = (1-\theta)s_0 + \theta s_1, \quad 0 < \theta < 1$$

(cf. J. Bergh and J. Löfström [2 , Theorem 5.6.1]). Therefore, we have

$$\sum_{j=-\infty}^{\infty} 2^{js} m_j \leqq C_q \|\varphi\|_{L_{q,1}(\Omega)}, \quad \frac{1}{q} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1}.$$

In particular,

$$s = (1-\theta)s_0 + \theta s_1 = \frac{3}{2}\left(\frac{1}{q} - \frac{1}{r}\right) + \frac{1}{2} = 1$$

because $1/q - 1/r = 1/3$, and therefore we have

$$\sum_{j=-\infty}^{\infty} 2^j m_j \leqq C_q \|\varphi\|_{L_{q,1}(\Omega)},$$

which completes the proof of the lemma.

To show (22), observe that

$$\|T_{\mathbf{u}_\infty}(t)\mathbf{b}\|_{L_{3,\infty}(\Omega)} \leqq C\|\mathbf{b}\|_{L_{3,\infty}(\Omega)};$$

$$\left|\int_0^t < \mathbf{w} \otimes \mathbf{u}(s), \nabla[T_{-\mathbf{u}_\infty}(t-s)\varphi] > ds\right|$$

$$\leqq \|\mathbf{w}\|_{L_{3,\infty}(\Omega)} \int_0^t \|\mathbf{u}(s)\|_{L_{3,\infty}(\Omega)} \|\nabla[T_{-\mathbf{u}_\infty}(t-s)\varphi]\|_{L_{3,1}(\Omega)}\, ds$$
$$\leqq \|\mathbf{w}\|_{L_{3,\infty}(\Omega)} [\mathbf{u}]_{3,\infty,t} \int_0^\infty \|\nabla[T_{-\mathbf{u}_\infty}(t-s)\varphi]\|_{L_{3,1}(\Omega)}\, ds$$

using LEMMA and noting that $2/3 - 1/3 = 1/3$,

$$\leqq C\|\mathbf{w}\|_{L_{3,\infty}(\Omega)} [\mathbf{u}]_{3,\infty,t} \|\varphi\|_{L_{3/2,1}(\Omega)};$$

$$\left|\int_0^t < \mathbf{u}(s) \otimes \mathbf{u}(s), \nabla[T_{-\mathbf{u}_\infty}(t-s)\varphi] > ds\right|$$
$$\leqq \int_0^t \|\mathbf{u}(s)\|^2_{L_{3,\infty}(\Omega)} \|\nabla[T_{-\mathbf{u}_\infty}(t-s)\varphi]\|_{L_{3,1}(\Omega)}\, ds$$
$$\leqq C[\mathbf{u}]^2_{3,\infty,t} \int_0^\infty \|\nabla[T_{-\mathbf{u}_\infty}(t-s)\varphi]\|_{L_{3,1}(\Omega)}\, ds$$
$$\leqq C[\mathbf{u}]^2_{3,\infty,t} \|\varphi\|_{L_{3/2,1}(\Omega)}.$$

To show (23), observe that

$$\|T_{\mathbf{u}_\infty}(t)\mathbf{b}\|_{L_{p,\infty}(\Omega)} \leqq Ct^{-(\frac{1}{2}-\frac{3}{2p})}\|\mathbf{b}\|_{L_{3,\infty}(\Omega)}.$$

Choose r so that $1/3 + 1/p + 1/r = 1$, and then $1/q - 1/r = 1/3$. Therefore,

$$\left|\int_0^t < \mathbf{w} \otimes \mathbf{u}(s), \nabla[T_{-\mathbf{u}_\infty}(t-s)\varphi] > ds\right|$$
$$\leqq \|\mathbf{w}\|_{L_{3,\infty}(\Omega)} \int_0^t \|\mathbf{u}(s)\|_{L_{p,\infty}(\Omega)} \|\nabla[T_{-\mathbf{u}_\infty}(t-s)\varphi]\|_{L_{r,1}(\Omega)}\, ds$$
$$\leqq \|\mathbf{w}\|_{L_{3,\infty}(\Omega)} [\mathbf{u}]_{p,\infty,t} \int_0^t s^{-(\frac{1}{2}-\frac{3}{2p})} \|\nabla[T_{-\mathbf{u}_\infty}(t-s)\varphi\|_{L_{r,1}(\Omega)}\, ds$$
$$\leqq Ct^{-(\frac{1}{2}-\frac{3}{2p})}\|\mathbf{w}\|_{L_{3,\infty}(\Omega)} [\mathbf{u}]_{p,\infty,t} \|\varphi\|_{L_{q,1}(\Omega)}.$$

In fact, since

$$\|\nabla[T_{-\mathbf{u}_\infty}(t-s)\varphi\|_{L_{r,1}(\Omega)} \leqq C(t-s)^{-1}\|\varphi\|_{L_{q,1}(\Omega)}$$

as follows from that $(3/2)(1/q - 1/r) + 1/2 = 1$, we have

$$\int_0^{t/2} s^{-(\frac{1}{2}-\frac{3}{2p})} \|\nabla[T_{-\mathbf{u}_\infty}(t-s)\varphi]\|_{L_{r,1}(\Omega)}\, ds$$
$$\leqq C \int_0^{t/2} s^{-(\frac{1}{2}-\frac{3}{2p})} (t-s)^{-1}\, ds \|\varphi\|_{L_{q,1}(\Omega)}$$
$$\leqq C(t/2)^{-1} \int_0^{t/2} s^{-(\frac{1}{2}-\frac{3}{2p})}\, ds \|\varphi\|_{L_{q,1}(\Omega)}$$
$$\leqq C(t/2)^{-1}(t/2)^{(\frac{1}{2}+\frac{3}{2p})} \|\varphi\|_{L_{q,1}(\Omega)}$$
$$\leqq C^{-(\frac{1}{2}-\frac{3}{2p})} \|\varphi\|_{L_{q,1}(\Omega)}.$$

On the other hand,

$$\begin{aligned}
&\int_{t/2}^{t} s^{-\left(\frac{1}{2}-\frac{3}{2p}\right)} \|\nabla[T_{-\mathbf{u}_\infty}(t-s)\varphi]\|_{L_{r,1}(\Omega)}\,ds \\
&\quad \leqq (t/2)^{-\left(\frac{1}{2}-\frac{3}{2p}\right)} \int_{t/2}^{t} \|\nabla[T_{-\mathbf{u}_\infty}(t-s)\varphi]\|_{L_{r,1}(\Omega)}\,ds \\
&\quad \leqq Ct^{-\left(\frac{1}{2}-\frac{3}{2p}\right)} \int_{0}^{\infty} \|\nabla[T_{-\mathbf{u}_\infty}(s)\varphi]\|_{L_{r,1}(\Omega)}\,ds \\
&\quad \leqq Ct^{-\left(\frac{1}{2}-\frac{3}{2p}\right)} \|\varphi\|_{L_{q,1}(\Omega)},
\end{aligned}$$

and therefore we have

$$\int_{0}^{t} s^{-\left(\frac{1}{2}-\frac{3}{2p}\right)} \|\nabla[T_{-\mathbf{u}_\infty}(t-s)\varphi]\|_{L_{r,1}(\Omega)} \leqq Ct^{-\left(\frac{1}{2}-\frac{3}{2p}\right)} \|\varphi\|_{L_{q,1}(\Omega)}.$$

In the same manner, we have

$$\begin{aligned}
&\left|\int_{0}^{t} < \mathbf{u}(s)\otimes\mathbf{u}(s), \nabla[T_{-\mathbf{u}_\infty}(t-s)\varphi] > ds\right| \\
&\leqq \int_{0}^{t} \|\mathbf{u}(s)\|_{L_{3,\infty}(\Omega)} \|\mathbf{u}(s)\|_{L_{p,\infty}(\Omega)} \|\nabla[T_{-\mathbf{u}_\infty}(t-s)\varphi]\|_{L_{r,1}(\Omega)}\,ds \\
&\leqq C[\mathbf{u}]_{3,\infty,t}[\mathbf{u}]_{p,\infty,t} \int_{0}^{t} s^{-\left(\frac{1}{2}-\frac{3}{2p}\right)} \|\nabla[T_{-\mathbf{u}_\infty}(t-s)\varphi]\|_{L_{r,1}(\Omega)}\,ds \\
&\leqq Ct^{-\left(\frac{1}{2}-\frac{3}{2p}\right)} [\mathbf{u}]_{3,\infty,t}[\mathbf{u}]_{p,\infty,t} \|\varphi\|_{L_{q,1}(\Omega)}.
\end{aligned}$$

Combining these estimations implies (23), which completes the proof of Theorem 4.

References

[1] K. I. Babenko, *On stationary solutions of the problem of flow past a body of a viscous incompressible fluid*, Math. USSR Sb. **20** (1973), 1–25.

[2] J. Bergh, and J. Löfström, *Interpolation Spaces. An Introduction*, Springer, Berlin-Heidelberg-New York, 1976.

[3] M. E. Bogovskii, *Solution of the first boundary value problem for the equation of continuity of an incompressible medium*, Sov. Math. Dokl. **20** (1979), 1094–1098.

[4] ———, *Solution for some vector analysis problems connected with operators div and grad*, Theory of cubature formulas and application of functional anaylysis to problems of mathematical physics, Trudy Sem. S. L. Sobolev, #1, 80,, Novosibirsk: Acad. Nauk SSSR, Sibirsk. Otdel., Inst. Mat., 1980, pp. 5–40.

[5] W. Borchers and T. Miyakawa, *Algebraic L^2 decay for the Navier-Stokes flows in exterior domains*, Acta Math. **165** (1990), 189–227.

[6] ———, *On stability of exterior stationary Navier–Stokes flows*, Acta Math. **174** (1995), 311–382.

[7] R. Farwig, *The stationary Navier-Stokes equations in a 3d-exterior domain*, Lecture Notes in Num. Appl. Anal. **16**, Recent Topics on Mathematical Theory of Viscous Incompressible Fluid, (H. Kozono and Y. Shibata, eds.), Kinokuniya, Tokyo, 1998, pp. 53–115.

[8] R. Finn, *On steady-state solutions of the Navier-Stokes partial differential equations*, Arch. Rational Mech. Anal. **3** (1959), 139–151.

[9] ______, *Estimates at infinity for stationary solutions of the Navier-Stokes equations*, Bull. Math. dela Soc. Sci. Math. Phys. de la R. P. Roumaine **3 (51)** (1959), 387–418.

[10] ______, *An energy theorem for viscous fluid motions*, Arch. Rational Mech. Anal. **6** (1960), 371–381.

[11] ______, *On the steady-state solutions of the Navier–Stokes equations, III*, Acta Math. **105** (1961), 197–244.

[12] ______, *On the exterior stationary problem for the Navier–Stokes equations and associated perturbation problems*, Arch. Rational Mech. Anal. **19** (1965), 363–406.

[13] ______, *Stationary solutions of the Navier–Stokes equations*, Proc. Symp. Appl. Math. **19** (1965), 121–153, Amer. Math. Soc..

[14] G. P. Galdi, *An introduction to the mathematical theory of the Navier–Stokes equations*, Vol I, Linearized Steady Problems; Vol II nonlinear Steady Problems, Springer Tracts in Natural Phylosophy Vol. **38**, **39**, Springer–Verlag, New York at al, 1994.

[15] Y. Giga and H. Sohr, *On the Stokes operator in exterior domains*, J. Fac. Sci. Univ. Tokyo, Sect. IA **36** (1988), 103–130.

[16] J. G. Heywood, *On stationary solutions of the Navier–Stokes equations as limits of non–stationary solutions*, Arch. Rational Mech. Anal. **37** (1970), 48–60.

[17] ______, *The exterior nonstationary problem for the Navier–Stokes equations*, Acta Math. **129** (1972), 11–34.

[18] ______, *The Navier–Stokes equations : On the existence, regularity and decay of solutions,*, Indiana Univ. Math. J. **29** (1980), 639–681.

[19] E. Hopf, *Über die Anfganswertaufgabe für die hydrodynamischen Grundgleichungen*, Math. Nachr. **4** (1950-51), 213–231.

[20] H. Iwashita, *L_q–L_r estimates for solutions of the nonstationary Stokes equations in an exterior domain and the Navier–Stokes initial value problems in L_q spaces*, Math. Ann. **285** (1989), 265–288.

[21] T. Kato, *Strong L^p-solutions of the Navier–Stokes equation in $\mathbf{R}^m$ with applications to weak solutions*, Math. Z. **187** (1984), 471–480.

[22] T. Kobayashi and Y. Shibata, *On the Oseen equation in exterior domains*, Math. Ann. **310** (1998), 1–45.

[23] H. Kozono and M. Yamazaki,, *Exterior problem for the stationary Navier-Stokes equations in the Lorentz spaces*, Math. Ann. **310** (1998), 278–305.

[24] ______, *On a loarger class of stable solutions to the Navier-Stokes equations in exterior domains*, Math. Zeit. **228** (1998), 751–785.

[25] J. Leray, *Étude de diverses équations intétrales non linéaires et de quelques problèms que pose l'hydrodynamique*, J. Math. Pures Appl. IX. Sér. **12** (1933), 1–82.

[26] ———, *Sur le mouvement d'un liquide visqueux emplissant l'espace*, Acta Math. **63** (1934), 193–248.
[27] P. Maremonti, *Stabilità asintotica in media per moti fluidi viscosi in domini esterni*, Ann. Mat. Pura Appl. **97** (1985), 57–75.
[28] K. Masuda, *On the stability of incompressible viscous fluid motions past objects*, J. Math. Soc. Japan **27** (1975), 294–327.
[29] T. Miyakawa, *On nonstationary solutions of the Navier–Stokes equations in an exterior domain*, Hiroshima Math. J. **12** (1982), 115–140.
[30] C. W. Oseen, *Neuere Methoden und Ergebniss in der Hydrodynamik*, Adademische Verlagsgesellschaft m.b.H., Leipnig, 1927.
[31] Y. Shibata, *On the global existence of classical solutions of second order fully nonlinear hyperbolic equations with first order dissipation in the exterior domain*, Tsukuba J. Math. **7** (1983), 1–68.
[32] ———, *On an exterior intial boundary value problem for Navier-Stokes equations*, Quart. Appl. Math. **LVII** (1999), 117–155.
[33] M. Yamazaki, *The Navier-Stokes equations in the weak - L^n space with time-dependent external force*, To appear in Math. Ann..