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Bilinear estimates and critical Sobolev inequality in $BMO$, with applications to the Navier-Stokes and the Euler equations

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Bilinear estimates and critical Sobolev inequality in $BMO$, with applications to the Navier-Stokes and the Euler equations

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**Introduction.**

In this paper we prove that the $BMO$ norm of the velocity and the vorticity controls the blow-up phenomena of smooth solutions to the Navier-Stokes and the Euler equations. Our result is applied to the criterion on regularity of weak solutions to the Navier-Stokes equations.

We consider the Navier-Stokes and the Euler equations in $\mathbb{R}^n, n \geq 3$:

\begin{align*}
\text{(N-S)} & \quad \left\{ \begin{array}{l}
\frac{\partial u}{\partial t} - \Delta u + u \cdot \nabla u + \nabla p = 0, \\
\text{div } u = 0 \quad \text{in } x \in \mathbb{R}^n, \ t > 0, \\
\quad u \big|_{t=0} = a,
\end{array} \right.
\end{align*}

\begin{align*}
\text{(E)} & \quad \left\{ \begin{array}{l}
\frac{\partial u}{\partial t} + u \cdot \nabla u + \nabla p = 0, \\
\text{div } u = 0 \quad \text{in } x \in \mathbb{R}^n, \ t > 0, \\
\quad u \big|_{t=0} = a
\end{array} \right.
\end{align*}

where $u = (u_1(x,t), u_2(x,t), \ldots, u_n(x,t))$ and $p = p(x,t)$ denote the unknown velocity vector and the unknown pressure of the fluid at the point $(x,t) \in \mathbb{R}^n \times (0,\infty)$, respectively, while $a = (a_1(x), a_2(x), \ldots, a_n(x))$ is the given initial velocity vector.

It is proved by Fujita-Kato [10] that for every $a \in H_s^2 \equiv \{v \in H^s; \text{div } v = 0\}$ with $s > n/2 - 1$, there exist $T > 0$ and a unique solution $u(t)$ of (N-S) on $[0,T)$ in the class

$$u \in C([0,T); H^s) \cap C^1((0,T); H^s) \cap C((0,T); H^{s+2}).$$

Concerning the Euler equations, Kato-Lai [15] and Kato-Ponce [16] proved that for every $a \in W^{s,p}_\sigma$ for $s > n/p + 1, 1 < p < \infty$, there are $T > 0$ and a unique solution $u$ of (E) on the interval $[0,T)$ in the class

$$u \in C([0,T); W^{s,p}_\sigma) \cap C^1([0,T); W^{s-2,p}_\sigma),$$
where subindex $\sigma$ means the divergence free. It is an interesting question whether the solution $u(t)$ really blows up as $t \uparrow T$. Giga [11] showed that if the strong solution $u$ in $(CN)_s$ satisfies

$$(\text{Se}) \quad \int_0^T \|u(t)\|_{L^2}^2 \, dt < \infty \quad \text{for } 2/\kappa + n/r = 1 \text{ with } n < r \leq \infty,$$

then $u$ can be continued to the solution in the class $(CN)_s$ beyond $t = T$. Concerning the Euler equations, Beale-Kato-Majda [1] dealt with the vorticity $\omega = \text{rot} \, u$ and proved that under the condition

$$\int_0^T \|\omega(t)\|_{L^\infty} \, dt < \infty,$$

$u(t)$ can never break down its regularity at $t = T$. To prove this assertion, in [1] they made use of the logarithmic inequality such as

$$(0.1) \quad \|\nabla u\|_{L^\infty} \leq C \left(1 + \|\text{rot} \, u\|_{L^\infty} (1 + \log^+ \|u\|_{W^{s+1,p}}) + \|\text{rot} \, u\|_{L^2}\right), \quad sp > n$$

for all vector functions $u$ with $\text{div} \, u = 0$, where $\log^+ a = \log a$ if $a \geq 1$, $= 0$ if $0 < a < 1$.

The purpose of this paper is to extend these results to the marginal space $BMO$ which is larger than $L^\infty$.

## 1 Results.

Before stating our results, we introduce some function spaces. Let $C^{\infty}_{0,\sigma}$ denote the set of all $C^\infty$ vector functions $\phi = (\phi^1, \phi^2, \cdots, \phi^n)$ with compact support in $\mathbb{R}^n$, such that $\text{div} \, \phi = 0$. $L^r_\sigma$ is the closure of $C^{\infty}_{0,\sigma}$ with respect to the $L^r$-norm $\|\cdot\|_r$; $(\cdot, \cdot)$ denotes the duality pairing between $L^r$ and $L^{r'}$, where $1/r + 1/r' = 1$. $L^r$ stands for the usual (vector-valued) $L^r$-space over $\mathbb{R}^n$, $1 \leq r \leq \infty$. $H^s_\sigma$ denotes the closure of $C^{\infty}_{0,\sigma}$ with respect to the $H^s$-norm $\|\phi\|_{H^s} = \|(1 - \Delta)^{s/2}\phi\|_2$, \hspace{1cm} $s \geq 0$.

Our result on continuation of strong solutions of (N-S) now reads:

**Theorem 1** Let $s > n/2 - 1$ and let $a \in H^s_\sigma$. Suppose that $u$ is the strong solution of (N-S) in the class $(CN)_s$ on $(0, T)$. If

$$\int_{\varepsilon_0}^T \|u(t)\|_{BMO}^2 \, dt < \infty \quad \text{for some } 0 < \varepsilon_0 < T,$$

then $u$ can be continued to the strong solution in the class $(CN)_s$ on $(0, T')$ for some $T' > T$.

**Corollary 1** Let $u$ be the strong solution of (N-S) in the class $(CN)_s$ on $(0, T)$ for $s > n/2 - 1$. Suppose that $T$ is maximal, i.e., $u$ cannot be continued in the class $(CN)_s$ on $(0, T')$ for any $T' > T$. Then

$$\int_{\varepsilon}^T \|u(t)\|_{BMO}^2 \, dt = \infty \quad \text{for all } 0 < \varepsilon < T.$$
For the space $BMO$, we refer to Stein [24]. Since $s > n/2 - 1$, there holds $H^{s+2} \subset BMO$, and hence for every $u$ in the class $(\text{CN})_s$ on $(0, T)$, we have $u \in C((0, T); BMO)$.

We next consider a criterion on uniqueness and regularity of weak solutions to (N-S). Our definition of a weak solution is as follows.

**Definition 1.** Let $a \in L^2_{\sigma}$. A measurable function $u$ on $R^n \times (0, T)$ is called a weak solution of (N-S) on $(0, T)$ if
\begin{enumerate}[(i)]  
  \item $u \in L^\infty(0, T; L^2_{\sigma}) \cap L^2(0, T; H^1_{\sigma})$;  
  \item $u(t)$ is continuous on $[0, T]$ in the weak topology of $L^2_{\sigma}$;  
  \item \( (1.3) \quad \int_s^t \{-(u, \partial_t \Phi) + (\nabla u, \nabla \Phi) + (u \cdot \nabla u, \Phi) \} \, d\tau = -(u(t), \Phi(t)) + (u(s), \Phi(s)) \) for every $0 \leq s \leq t < T$ and every $\Phi \in H^1((s, t); H^1_{\sigma} \cap L^n)$.\end{enumerate}

Our result on weak solutions of (N-S) now reads:

**Theorem 2** (1) (uniqueness) Let $a \in L^2_{\sigma}$ and let $u, v$ be two weak solutions of (N-S) on $(0, T)$. Suppose that
\begin{equation}  
  u \in L^2(0, T; BMO)  
\end{equation}
and that $v$ satisfies the energy inequality
\begin{equation}  
  \|v(t)\|_2^2 + 2 \int_0^t \|\nabla v\|_2^2 \, d\tau \leq \|a\|_2^2, \quad 0 < t < T.  
\end{equation}
Then we have $u \equiv v$ on $[0, T]$.

(2) (regularity) Let $a \in L^2_{\sigma}$ and let $u$ be a weak solution with the additional property (1.4). Then for every $0 < \epsilon < T$, $u$ is actually a strong solution of (N-S) on $(\epsilon, T)$ in the class $(\text{CN})_s$ for $s > n/2 - 1$.

**Remark.** Theorem 2 may be regarded as an extension of Serrin’s criterion [22], [23] on uniqueness and regularity of weak solutions $u$ in the class
\begin{equation}  
  u \in L^\kappa(0, T; L^r) \quad \text{for} \quad 2/\kappa + n/r = 1 \quad \text{with} \quad n < \tau \leq \infty.  
\end{equation}
Our class (1.4) is larger than the marginal case $L^2(0, T; L^\infty)$ in (1.6). Moreover, by virtue of the estimate $\|u\|_{BMO} \leq C\|\nabla u\|_{M^n}$ of John-Nirenberg [13], we see that the weak solution $u$ with $\nabla u \in L^2(0, T; M^n)$ becomes regular, where $M^n$ denotes the Morrey space which is larger than $L^n$. See Beirão da Veiga [2].

We shall next investigate continuation of the strong solution in terms of the vorticity $\omega = \text{rot } u \equiv (\partial_j u^k - \partial_k u^j)_{1 \leq j, k \leq n}$ and the deformation tensor $\text{Def } u \equiv (\partial_j u^k + \partial_k u^j)_{1 \leq j, k \leq n}$. 
Theorem 3 Let $s > n/2 - 1$. Suppose that $u$ is the strong solution of \((N-S)\) in the class \((CN)_s\) on \((0,T)\). If either

\[
\int_{\epsilon_0}^{T} ||\omega(t)||_{BMO} dt < \infty \tag{1.7}
\]

or

\[
\int_{\epsilon_0}^{T} ||\text{Def} u(t)||_{BMO} dt < \infty \tag{1.8}
\]

holds for some $0 < \epsilon_0 < T$, then $u$ can be continued to the strong solution in the class \((CN)_s\) on \((0, T')\) for some $T' > T$.

Corollary 2 Suppose that $u$ is the strong solution of \((N-S)\) in the class \((CN)_s\) on \((0, T)\) for $s > n/2 - 1$. Assume that $T$ is maximal in the same sense as in Corollary 1. Then both

\[
\int_{\epsilon}^{T} ||\omega(t)||_{BMO} dt = \infty \tag{1.9}
\]

hold for all $0 < \epsilon < T$.

Theorem 3 yields the following regularity criterion on weak solutions of \((N-S)\) by means of \(\text{rot} u\) and \(\text{Def} u\).

Theorem 4 Let $a \in L^2_{\sigma}$. Suppose that $u$ is a weak solution of \((N-S)\) on \((0,T)\). If either

\[
\omega \in L^1(0,T;BMO) \quad \text{or} \quad \text{Def} u \in L^1(0,T;BMO) \tag{1.10}
\]

holds, then for every $0 < \epsilon < T$, $u$ is actually a strong solution of \((N-S)\) in the class \((CN)_s\) on \((\epsilon, T)\) for $s > n/2 - 1$.

Remark. Beirão da Veiga [2] proved the regularity criterion in the class $\nabla u \in L^\kappa(0,T;L^r)$ for $2/\kappa + n/r = 2$ with $1 < \kappa < \infty$, $n/2 < r < \infty$. Theorem 4 covers the borderline case $\kappa = 1$ and $r = \infty$.

Our result on \((E)\) reads as follows.

Theorem 5 Let $1 < p < \infty$, $s > n/p + 1$. Suppose that $u$ is the solution of \((E)\) in the class \((CE)_{s,p}\) on \((0,T)\). If either

\[
\int_{0}^{T} ||\omega(t)||_{BMO} dt(\equiv M_0) < \infty \tag{1.11}
\]

or

\[
\int_{0}^{T} ||\text{Def} u(t)||_{BMO} dt(\equiv M_1) < \infty \tag{1.12}
\]

holds, then $u$ can be continued to the solution in the class \((CE)_{s,p}\) on \((0, T')\) for some $T' > T$.

Corollary 3 Let $u$ be the solution of \((E)\) in the class \((CE)_{s,p}\) on \((0,T)\) for $1 < p < \infty$, $s > n/p + 1$. Assume that $T$ is maximal, i.e., $u$ cannot be continued to the solution in the class \((CE)_{s,p}\) on \((0, T')\) for any $T' > T$. Then both

\[
\int_{0}^{T} ||\text{rot} u(t)||_{BMO} dt = \infty \quad \text{and} \quad \int_{0}^{T} ||\text{Def} u(t)||_{BMO} dt = \infty
\]

hold.
2 Bilinear estimates and critical Sobolev inequality in $BMO$.

In this section we shall prepare some lemmas. In what follows we shall denote by $C$ various constants. In particular, $C = C(*, \cdots, *)$ denotes constants depending only on the quantities appearing in the parenthesis.

We first prove the following key estimate.

**Lemma 2.1 (Bilinear estimates)** Let $1 < r < \infty$. Then we have

\[(2.1) \quad \|f \cdot \nabla g\|_r \leq C(\|f\|_r \|(-\Delta)^{1/2} g\|_{BMO} + \|(-\Delta)^{1/2} f\|_{BMO} \|g\|_r)\]

for all $f, g \in W^{1,r}$ with $\nabla f, \nabla g \in BMO$ and $C = C(n, r)$.

**Proof of Lemma 2.1.** Here we prove only (2.2). The proof of (2.1) is similar to that of (2.2). Let $\Phi_1$ be a $C^\infty$-function on $[0, \infty)$ such that supp $\Phi_1 \subset [0, 1)$, $0 \leq \Phi_1 \leq 1$, $\Phi_1(t) \equiv 1$ for $0 \leq t \leq 1/2$, and let $\Phi_2 = 1 - \Phi_1$. Then we have

\[
\partial^\alpha f(x) \partial^\beta g(x) = C \int_{\mathbb{R}^n \times \mathbb{R}^n} e^{ix \cdot (\xi + \eta)} \xi^\alpha \eta^\beta \hat{f}(\xi) \hat{g}(\eta) d\xi d\eta
\]

Let $\alpha = (\alpha_1, \cdots, \alpha_n)$, $\beta = (\beta_1, \cdots, \beta_n)$ be multi-indices with $|\alpha| = \alpha_1 + \cdots + \alpha_n \geq 1$ and $|\beta| = \beta_1 + \cdots + \beta_n \geq 1$. Then

\[(2.2) \quad \|\partial^\alpha f \cdot \partial^\beta g\|_2 \leq C(\|f\|_{BMO} \|(-\Delta)^{|\alpha|+|\beta|/2} g\|_2 + \|(-\Delta)^{|\alpha|+|\beta|/2} f\|_2 \|g\|_{BMO})\]

for all $f, g \in BMO \cap H^{|\alpha|+|\beta|}$ with $C = C(n, \alpha, \beta)$, where $\partial^\alpha = \frac{\partial^{|\alpha|}}{\partial x_1^{\alpha_1} \cdots \partial x_n^{\alpha_n}}$.

The proof of this lemma is based on the following proposition due to Coifman-Meyer [6, Chapter V. Proposition 2].

**Proposition 2.1 (Coifman-Meyer)** Let $\sigma = \sigma(\xi, \eta) \in C^\infty(\mathbb{R}^n \times \mathbb{R}^n \setminus \{(0,0)\})$ satisfy

\[
|\partial_\xi^\alpha \partial_\eta^\beta \sigma(\xi, \eta)| \leq C(|\xi| + |\eta|)^{-|\alpha|-|\beta|}, \quad (\xi, \eta) \in \mathbb{R}^n \times \mathbb{R}^n \setminus \{(0,0)\}
\]

for all multi-indices $\alpha, \beta$ with $C = C(\alpha, \beta)$. Suppose that

\[\sigma(\xi, 0) = 0.\]

Then the bilinear operator $\sigma(D)(\cdot, \cdot)$ defined by

\[(2.3) \quad \sigma(D)(f, g)(x) \equiv \int_{\mathbb{R}^n \times \mathbb{R}^n} e^{ix \cdot (\xi + \eta)} \sigma(\xi, \eta) \hat{f}(\xi) \hat{g}(\eta) d\xi d\eta, \quad x \in \mathbb{R}^n\]

satisfies

\[(2.4) \quad \|\sigma(D)(f, g)\|_p \leq C\|f\|_p \|g\|_{BMO} \quad (1 < p < \infty)\]

with $C = C(n, p)$. 

**Proof of Proposition 2.1.** Here we prove only (2.4). The proof of (2.3) is similar to that of (2.2). Let $\Phi_1$ be a $C^\infty$-function on $[0, \infty)$ such that supp $\Phi_1 \subset [0, 1)$, $0 \leq \Phi_1 \leq 1$, $\Phi_1(t) \equiv 1$ for $0 \leq t \leq 1/2$, and let $\Phi_2 = 1 - \Phi_1$. Then we have
$$= \int_{\mathbb{R}^n \times \mathbb{R}^n} e^{ix \cdot (\xi + \eta)} \times \left( \frac{\xi^\alpha \eta^\beta}{|\eta|^{\alpha + \beta}} \Phi_1(|\xi|/|\eta|) \eta^\alpha + \frac{\xi^\alpha \eta^\beta}{|\xi|^{\alpha + \beta}} \Phi_2(|\xi|/|\eta|) \xi^\alpha \eta^\beta \right) d\xi d\eta$$

$$= C \left( \sigma_1(D)(f, (-\Delta)^{\frac{\alpha + \beta}{2}} g)(x) + \sigma_2(D)((-\Delta)^{\frac{\alpha + \beta}{2}} f, g)(x) \right),$$

where

$$\sigma_1(\xi, \eta) = \frac{\xi^\alpha \eta^\beta}{|\eta|^{\alpha + \beta}} \Phi_1(|\xi|/|\eta|) \eta^\alpha, \quad \sigma_2(\xi, \eta) = \frac{\xi^\alpha \eta^\beta}{|\xi|^{\alpha + \beta}} \Phi_2(|\xi|/|\eta|).$$

Since $|\alpha| \geq 1$ and $|\beta| \geq 1$, we see that

$$\sigma_1(0, \eta) = 0, \quad \sigma_2(\xi, 0) = 0$$

and that $\sigma_1$ and $\sigma_2$ satisfy the hypotheses of Proposition 2.1. Hence there holds

$$\| \sigma_1(D)(f, (-\Delta)^{\frac{\alpha + \beta}{2}} g) \|_2 \leq C \| f \|_{BMO} \| (-\Delta)^{\frac{\alpha + \beta}{2}} g \|_2,$$

$$\| \sigma_2(D)((-\Delta)^{\frac{\alpha + \beta}{2}} f, g) \|_2 \leq C \| (-\Delta)^{\frac{\alpha + \beta}{2}} f \|_2 \| g \|_{BMO},$$

which yields (2.2). This proves Lemma 2.1.

The next lemma plays an important role to show the energy identity of weak solutions in the class (1.4) and (1.10).

**Lemma 2.2**

(i) Let $w \in L^\infty(0, T; L^2_{\sigma}) \cap L^2(0, T; H^1_{\sigma})$ and $u \in L^2(0, T; H^1_{\sigma} \cap BMO)$. Then we have

$$\int_0^T (w \cdot \nabla u, u) d\tau = 0.$$

(ii) Let $w, u \in L^\infty(0, T; L^2_{\sigma}) \cap L^2(0, T; H^1_{\sigma})$. Suppose that either

$$\text{rot } w, \text{rot } u \in L^1(0, T; BMO)$$

or

$$\text{Def } w, \text{Def } u \in L^1(0, T; BMO)$$

holds. Then we have

$$\int_0^T (w \cdot \nabla u, u) d\tau = 0.$$

To prove (2.5), we use the estimate of Coifman-Lions-Meyer-Semmes [5]:

$$w \cdot \nabla u \in \mathcal{H}^1 \quad \text{with } \| w \cdot \nabla u \|_{\mathcal{H}^1} \leq C \| w \|_2 \| \nabla u \|_2,$$

where $\mathcal{H}^1$ denotes the Hardy space on $\mathbb{R}^n$. For detail, see [5].

To prove (2.6), we use (2.1) and the Biot-Savart law. Indeed, by the Biot-Savart law, we have the representation

$$\frac{\partial u}{\partial x_j} = R_j(R \times \omega), \quad j = 1, \ldots, n, \quad \text{where } \omega = \text{rot } u;$$

$$\frac{\partial u^l}{\partial x_j} = R_j(\sum_{k=1}^n R_k \text{Def } u_{kl}), \quad j, l = 1, \ldots, n, \quad \text{where } \text{Def } u_{kl} = \frac{\partial u^k}{\partial x_l} + \frac{\partial u^l}{\partial x_k}.$$
Here $R = (R_1, \cdots, R_n)$, and $R_j = \frac{\partial}{\partial x_j}(-\Delta)^{-\frac{1}{2}}$ denote the Riesz transforms. Since $R$ is a bounded operator in $BMO$, we have by (2.8), (2.9) and assumption that

\begin{equation}
(2.10) \quad \nabla u, \nabla w \in L^1(0, T; BMO).
\end{equation}

It follows from Lemma 2.1 (2.1) and (2.10) that $\int_0^T (w \cdot \nabla u, u)dt$ is well-defined. For details of the proof of Lemma 2.2 we refer to [17].

Using the usual mollifier argument, by Lemma 2.2, we have the following energy identity for weak solutions with (1.4) or (1.10).

**Lemma 2.3** Let $n \geq 3$ and let $a \in L^2_\sigma$. Suppose that $u$ is a weak solution of (N-S) on $(0, T)$ satisfying one of the additional conditions (1.4) and (1.10). Then $u$ fulfills the energy identity

\begin{equation}
(2.11) \quad \|u(t)\|_2^2 + 2 \int_s^t \|\nabla u\|_2^2 d\tau = \|u(s)\|_2^2 \quad \text{for all } 0 \leq s \leq t < T.
\end{equation}

Now we prove the following lemma which is an extension of (0.1).

**Lemma 2.4 (Critical Sobolev Inequality)** Let $1 < p < \infty$ and let $s > n/p$. There is a constant $C = C(n, p, s)$ such that the estimate

\begin{equation}
(2.12) \quad \|f\|_\infty \leq C (1 + \|f\|_{BMO}(1 + \log^+ \|f\|_{W^{s,p}}))
\end{equation}

holds for all $f \in W^{s,p}$.

**Remark.** Compared with (0.1), we do not need to add $\|f\|_{L^2}$ to the right hand side of (2.12). This makes it easier to derive an apriori estimate of solutions to the Euler equations than Beale-Kato-Majda [1].

**Proof of Lemma 2.4.**

We shall make use of the Littlewood-Paley decomposition; there exists a non-negative function $\varphi \in S (S; \text{the Schwartz class})$ such that $\text{supp} \varphi \subset \{2^{-1} \leq |\xi| \leq 2\}$ and such that $\sum_{k=-\infty}^\infty \varphi(2^{-k}\xi) = 1$ for $\xi \neq 0$. See Bergh-Löfström [3, Lemma 6.1.7]. Let us define $\phi_0$ and $\phi_1$

\begin{equation}
\phi_0(\xi) = \sum_{k=1}^\infty \varphi(2^k \xi) \quad \text{and} \quad \phi_1(\xi) = \sum_{k=-\infty}^{-1} \varphi(2^k \xi),
\end{equation}

respectively. Then we have that $\phi_0(\xi) = 1$ for $|\xi| \leq 1/2$, $\phi_0(\xi) = 0$ for $|\xi| \geq 1$ and that $\phi_1(\xi) = 0$ for $|\xi| \leq 1$, $\phi_1(\xi) = 1$ for $|\xi| \geq 2$. It is easy to see that for every positive integer $N$ there holds the identity

\begin{equation}
(2.13) \quad \phi_0(2^N \xi) + \sum_{k=-N}^{N} \varphi(2^{-k}\xi) + \phi_1(2^{-N}\xi) = 1, \quad \xi \neq 0.
\end{equation}

Since $C_0^\infty$ is dense in $W^{s,p}$ and since $W^{s,p}$ is continuously embedded in $BMO$, implied by $s > n/p$, it suffices to prove (2.12) for $f \in C_0^\infty$. For such $f$ we have the representation

\begin{equation}
f(x) = \int_{y \in \mathbb{R}^n} K(x-y) \cdot \nabla f(y) dy \quad \text{with} \quad K(y) = \frac{1}{n \omega_n} \frac{y}{|y|^n},
\end{equation}
for all $x \in \mathbb{R}^n$, where $\omega_n$ denotes the volume of the unit ball in $\mathbb{R}^n$. By (2.13) we decompose $f$ into three parts:

$$
f(x) = \int_{y \in \mathbb{R}^n} K(x - y) \times \left( \phi_0(2^N(x - y)) + \sum_{k = -N}^{N} \varphi(2^{-k}(x - y)) + \phi_1(2^{-N}(x - y)) \right) \cdot \nabla f(y) dy
$$

for all $x \in \mathbb{R}^n$.

We can show that

$$|f_0(x)| \leq C 2^{-\beta N} \|f\|_{W^{s,p}}$$

for all $x \in \mathbb{R}^n$, where $\beta = \beta(n, p, s)$ is a positive constant. For detail, see [18].

By integration by parts we have

$$g(x) = \sum_{k = -N}^{N} (\text{d} \text{iv} \Psi)_{2^{k}} \ast f(x), \quad x \in \mathbb{R}^n,$$

where $\Psi(x) = K(x) \varphi(x)$ and $\psi_t(x) = t^{-n} \psi(x/t)$ for $t > 0$. Since $\Psi \in S$ with the property that

$$\int_{\mathbb{R}^n} \text{div} \Psi(x) dx = 0,$$

it follows from Stein [24, Chap. IV, 4.3.3] that

$$\|g\|_{\infty} \leq \sum_{k = -N}^{N} \|(\text{d} \text{iv} \Psi)_{2^{k}} \ast f\|_{\infty}
\leq \sum_{k = -N}^{N} \sup_{t > 0} \|(\text{d} \text{iv} \Psi)t \ast f\|_{\infty}
\leq CN \|f\|_{BMO},$$

(2.16)

where $C = C(n)$ is independent of $N$.

Integrating by parts, we have by a direct calculation

$$|f_1(x)| = \left| \int_{y \in \mathbb{R}^n} \text{div}_y \left( K(x - y) \phi_1(2^{-N}(x - y)) \right) f(y) dy \right|
\leq C 2^{-N \cdot \frac{n}{p}} \|f\|_p$$

(2.17)

for all $x \in \mathbb{R}^n$, where $C = C(n, p)$ is independent of $N$.

Now it follows from (2.14) and (2.15)-(2.17) that

$$\|f\|_{\infty} \leq C(2^{-\gamma N} \|f\|_{W^{s,p}} + N \|f\|_{BMO})$$

(2.18)

with $\gamma = \text{Min.}\{\beta, n/p\}$, where $C = C(n, s, p)$ is independent of $N$ and $f$. If $\|f\|_{W^{s,p}} \leq 1$, then we may take $N = 1$; otherwise, we take $N$ so large that the first term of the right hand
side of (2.18) is dominated by 1, i.e., \( N \equiv \left[ \frac{\log ||f||_{W^{s,p}}}{\gamma \log 2} \right] + 1 \) ([; Gauss symbol) and (2.18) becomes
\[
||f||_{\infty} \leq C \left\{ 1 + ||f||_{BMO} \left( \frac{\log ||f||_{W^{s,p}}}{\gamma \log 2} + 1 \right) \right\}.
\]
In both cases, (2.12) holds. This proves Lemma 2.4.

3 Proof of Theorems 1-4

3.1 Proof of Theorem 1.

It is proved by Kato [14] and Giga [11] that, for the initial data \( a \in H^s \) with \( s > n/2 - 1 \), the local existence time interval \( T \) of the strong solution \( u \) of (N-S) in the class \((\text{CN})_s\) can be estimated from below as
\[
T \geq \frac{C}{||a||_{H^s(n/2-1)}^3},
\]
where \( C = C(n, s) \). Actually, for \( a \in L^r \) with \( r > n \), Giga [11, Theorem 1 (ii)] gave \( T \) in such a way that
\[
T = \frac{C}{||a||_{L^r}^{2r/(r-n)}},
\]
so from the continuous embedding \( H^s \subset L^r \) for \( 1/r = 1/2 - s/n \), we obtain (3.1). Hence by the standard argument of continuation of local solutions, it suffices to prove the following apriori estimate
\[
\sup_{\epsilon_0 < t < T} \|u(t)\|_{H^{[s]+1}} \leq \|u(\epsilon_0)\|_{H^{[s]+1}} \exp \left( C \int_{\epsilon_0}^{T} \|u\|_{BMO}^2 dt \right),
\]
where \( C = C(n, s) \) is independent of \( T \).

Let \( \alpha = (\alpha_1, \ldots, \alpha_n) \) be a multi-index with \( |\alpha| \equiv \alpha_1 + \cdots + \alpha_n \leq [s] + 1 \), and let
\[
v = \partial^\alpha u = \frac{\partial^{|\alpha|} u}{\partial x_1^{\alpha_1} \cdots \partial x_n^{\alpha_n}}.
\]
Applying \( \partial^\alpha \) to (N-S), we have for \( v \) the equation
\[
\frac{\partial v}{\partial t} - \Delta v + u \cdot \nabla v + \nabla q = F, \quad \epsilon_0 < t < T,
\]
where \( q = \partial^\alpha p \) and
\[
F = - \sum_{|\beta| \leq |\alpha| - 1} a \alpha C \beta \partial^{\alpha - \beta} u \cdot \nabla (\partial^\beta u).
\]
Taking the inner product in \( L^2 \) between (3.4) and \( v \), and then integrating the result identity on the time interval \( (\epsilon_0, t) \), we obtain
\[
\|v(t)\|^2_2 + 2 \int_{\epsilon_0}^{t} \|\nabla v\|^2_2 d\tau \leq \|v(\epsilon_0)\|^2_2 + 2 \int_{\epsilon_0}^{t} \|F\|_2 \|v\|_2 d\tau.
\]
On the other hand, by (2.2), we have
\[
\|F\|_2 \leq C\|u\|_{BMO}(\Delta)^{\frac{|\alpha|+1}{2}} u_2,
\]
from which and (3.6) it follows that
\[ \|\partial^\alpha u(t)\|_2^2 + 2\int_{\epsilon_0}^{t} \|\nabla(\partial^\alpha u)\|_2^2 \, d\tau \]
\[ \leq \|\partial^\alpha u(\epsilon_0)\|_2^2 + \int_{\epsilon_0}^{t} \|(-\Delta)^{\frac{|\alpha|+1}{2}} u\|_2^2 \, d\tau + C \int_{\epsilon_0}^{t} \|u\|_{BMO}^2 \|\partial^\alpha u\|_2^2 \, d\tau \]
with \( C \) independent of \( t \). Summing over \( \alpha \) with \( 0 \leq |\alpha| \leq [s] + 1 \), we have
\[ \|u(t)\|_{H^{[s]+1}}^2 \leq \|u(\epsilon_0)\|_{H^{[s]+1}}^2 + C \int_{\epsilon_0}^{t} \|u\|_{BMO}^2 \|u\|_{H^{[s]+1}}^2 \, d\tau \]
for all \( \epsilon_0 \leq t < T \). Now the Gronwall inequality yields (3.3). This proves Theorem 1.

### 3.2 Proof of Theorem 2.

(1) Let us first prove uniqueness. We follow the argument of Masuda [20, Theorems 2, 3]. We can show that

(3.7) \[ \int_0^t \{2(\nabla u, \nabla v) + (v \cdot \nabla v, u) - (u \cdot \nabla v, u)\} \, d\tau = -(u(t), v(t)) + \|a\|_2^2. \]

See Masuda [20, p.640 (4.4)]. By Lemma 2.3, \( u \) satisfies the energy identity

(3.8) \[ \|u(t)\|_2^2 + 2\int_0^t \|\nabla u\|_2^2 \, d\tau = \|a\|_2^2. \]

Addition of (3.7) (multiplied by \(-2\)), (3.8) and (1.5) yields

(3.9) \[ \|w(t)\|_2^2 + 2\int_0^t \|\nabla w\|_2^2 \, d\tau \leq 2\int_0^t (w \cdot \nabla v, u) \, d\tau = 2\int_0^t (w \cdot \nabla w, u) \, d\tau, \]

where \( w = v - u \). In the last identity, we have used (2.5). By (2.7) we have

\[ \text{RHS of (3.9)} \leq C \int_0^t \|w \cdot \nabla w\|_{H^1} \|u\|_{BMO} \, d\tau \]
\[ \leq C \int_0^t \|w\|_2 \|\nabla w\|_2 \|u\|_{BMO} \, d\tau \]
\[ \leq \int_0^t \|\nabla w\|_2^2 \, d\tau + C \int_0^t \|w\|_2^2 \|u\|_{BMO}^2 \, d\tau. \]

Hence by (3.9)
\[ \|w(t)\|_2^2 \leq C \int_0^t \|w\|_2^2 \|u\|_{BMO}^2 \, d\tau, \quad 0 \leq t < T. \]

Since \( u \in L^2(0,T;BMO) \), the Gronwall inequality yields
\[ \|w(t)\|_2^2 = 0, \quad 0 \leq t < T \]
from which we get the desired uniqueness.
(2) We next prove regularity. Since \( u \in L^2(0,T;H^1 \sigma \cap BMO) \), for every \( 0 < \varepsilon < T \), there is \( 0 < \delta < \varepsilon \) such that \( u(\delta) \in H^1 \sigma \cap BMO \subset L^2 \sigma \cap L^p \sigma \) for \( n < r < \infty \). Hence it follows from the local existence theorem of Kato [14] and Giga [11] that there are \( T_* > \delta \) and a unique solution \( \tilde{u} \) on \( [\delta, T_*) \) with \( \tilde{u}|_{t=\delta} = u(\delta) \), such that

\[
\tilde{u} \in C([\delta, T_*); H^1 \sigma \cap BMO) \subset L^2 \sigma \cap L^r \sigma \quad \text{for} \quad n < r < \infty.
\]

Since \( u \) satisfies the energy identity

\[
\|u(t)\|_2^2 + 2 \int_0^t \|\nabla u\|_2^2 \, dt = \|u(0)\|_2^2, \quad \delta \leq t < T,
\]

implied by Lemma 2.3, we have by the uniqueness criterion of Serrin-Masuda [23], [20]

\[
u(\delta) \equiv \tilde{u} \quad \text{on} \quad [\delta, T_*).
\]

By (3.10) and (3.12), we may regard \( u \) as a strong solution in the class \((\mathrm{CN})_s\) on \((\delta', T_*)\) for \( \delta < \delta' < \varepsilon \).

In fact, there holds \( T_* = T \). Suppose that \( T_* < T \). Then there exists \( T_0 < T \) such that \( u \) is a strong solution in the class \((\mathrm{CN})_s\) on \((\delta', T_0)\), but cannot be continued in the class \((\mathrm{CN})_s\) on \((\delta', T)\) for any \( T > T_0 \). By assumption, we have

\[
\int_{T_0}^{T} \|u\|_{BMO}^2 \, dt \leq \int_{0}^{T} \|u\|_{BMO}^2 \, dt < \infty.
\]

This contradicts Corollary 1, so we get \( T_* = T \). This proves Theorem 2.

3.3 Proof of Theorems 3-4.

Proof of Theorem 3:

On account of (3.2), it suffices to prove

\[
\sup_{\varepsilon_0 < t < T} \|u(t)\|_r \leq \|u(\varepsilon_0)\|_r \exp \left( C \int_{\varepsilon_0}^T \|\nabla u\|_{BMO} \, dt \right), \quad r > n.
\]

In the same way as in (2.10), we see that the hypothesis (1.7) or (1.8) yields

\[
\int_{\varepsilon_0}^T \|\nabla u\|_{BMO} \, dt < \infty.
\]

Since \( u \in C([\varepsilon_0, T); H^{r+2}) \subset C([\varepsilon_0, T); W^{1,r}) \cap C^1((\varepsilon_0, T); W^{1,r}) \cap C^1((\varepsilon_0, T); W^{3,r}) \) for all \( 2 \leq r < \infty \) and has the integral representation:

\[
u(t) = e^{(t-\varepsilon_0)\Delta} u(\varepsilon_0) - \int_{\varepsilon_0}^t e^{(t-s)\Delta} P(u \cdot \nabla u)(s) \, ds, \quad \varepsilon_0 < t < T.
\]

See Kato [14]. Here \( e^{t\Delta} \) is the well-known heat operator and \( P = \{P_{kl}\}_{k,l=1,\ldots,n} \) is the Helmholtz projection defined by \( P_{kl} = \delta_{kl} + R_k R_l \).
Since \( \|e^{t\Delta}\|_{B(L^r,L^r)} \leq 1 \) for all \( t > 0 \), it follows from Lemma 2.1(i) and (3.16) that

\[
\|u(t)\|_r \leq \|u(\epsilon_0)\|_r + C \int_{\epsilon_0}^{t} \|u \cdot \nabla u\|_r d\tau,
\]

\( \epsilon_0 < t < T \).

From this and the Gronwall inequality, we obtain the desired apriori estimate (3.14), which proves Theorem 3.

Proof of Theorem 4:

The proof of Theorem 4 is parallel to that of Theorem 2.

4 Proof of Theorem 5.

We follow the argument of Beale-Kato-Majda [1]. It is proved by Kato-Lai [15] and Kato-Ponce [16] that for the given initial data \( a \in W^{s,p} \) for \( s > 1 + n/p \), the time interval \( T \) of the existence of the solution \( u \) to (E) in the class \( (CE)_{s,p} \) depends only on \( \|a\|_{W^{s,p}} \). Hence by the standard argument of continuation of local solutions, it suffices to establish an apriori estimate for \( u \) in \( W^{s,p} \) in terms of \( a, T, M_0 \) or \( a, T, M_1 \) according to (1.11) or (1.12). Indeed, we shall show that the solution \( u(t) \) in the class \( (CE)_{s,p} \) on \( (0, T) \) is subject to the following estimate:

\[
(4.17) \quad \sup_{0 < t < T} \|u(t)\|_{W^{s,p}} \leq (\|a\|_{W^{s,p}} + e)^{\alpha_j} \exp(C\int_{0}^{t} \|\nabla u(\tau)\|_\infty d\tau),
\]

where \( C = C(n,p,s) \) is a constant independent of \( a \) and \( T \).

We shall first prove (4.17) under (1.11). It follows from the commutator estimate in \( L^p \) given by Kato-Ponce [16, Proposition 4.2] that

\[
(4.18) \quad \|u(t)\|_{W^{s,p}} \leq \|a\|_{W^{s,p}} \exp \left( C \int_{0}^{t} \|\nabla u(\tau)\|_\infty d\tau \right), \quad 0 < t < T,
\]

where \( C = C(n,p,s) \).

By the Biot-Savard law (2.8), we have

\[
(4.19) \quad \|\nabla u\|_{BMO} \leq C\|\omega\|_{BMO}
\]

with \( C = C(n) \). Hence it follows from (4.19) and Lemma 2.4 that

\[
(4.20) \quad \|\nabla u(t)\|_\infty \leq C(1 + \|\omega(t)\|_{BMO}(1 + \log^+ \|u(t)\|_{W^{s,p}}))
\]

for all \( 0 < t < T \) with \( C = C(n,p,s) \). Substituting (4.20) to (4.18), we have

\[
\|u(t)\|_{W^{s,p}} + e \leq (\|a\|_{W^{s,p}} + e) \exp \left( C \int_{0}^{t} \{1 + \|\omega(\tau)\|_{BMO} \log(\|u(\tau)\|_{W^{s,p}} + e)\} d\tau \right)
\]
for all $0 < t < T$. Defining $z(t) \equiv \log(\|u(t)\|_{W^{s,p}} + e)$, we obtain from the above estimate

$$z(t) \leq z(0) + CT + C \int_0^t \|\omega(\tau)\|_{BMO} d\tau, \quad 0 < t < T.$$ 

Now (1.11) and the Gronwall inequality yield

$$z(t) \leq (z(0) + CT) \exp\left( C \int_0^t \|\omega(\tau)\|_{BMO} d\tau \right) \leq (z(0) + CT) \alpha_0$$

for all $0 < t < T$ with $C = C(n, p, s)$, which implies (4.17) for $j = 0$.

Similarly we prove (4.17) for $j = 1$ under (1.12). This proves Theorem 5.

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参考文献


