数理解析研究所講究録
1146 巻 2000 年 1-15

On thermal convection equations of Oberbeck-Boussinesq type with the dissipation function

Yoshiyuki KAGEI (隠居 良行)
Graduate School of Mathematics
Kyushu University,
Fukuoka 812-8581, JAPAN

1. Introduction

We study the stability of the motionelss state and bifurcation of cellular patterns in the Rayleigh-Bénard convection under the effect of the dissipative heating.

The Oberbeck-Boussinesq equations are frequently used as model equations in the mathematical analysis of convection phenomena such as the Rayleigh-Bénard convection problem. Many interesting and usueful mathmatical results have been obtained through the Oberbeck-Boussinesq equations, and Rajagopal, Růžička and Srinivasa [7] gave a justification for the derivaton of the Oberbeck-Boussinesq equations from the point of view of continuum mechanics. However, there are some phenomena such as the earth's upper mantle convetcion, convection in fast rotating configurations and etc., in which the Oberbeck-Boussinesq equations seem inappropriate due to the fact that the effect of dissipative heating is not taken into account in the equations.

Our purpose here is to study the model equations including the effect of dissipative heating, which was derived in [3], in the context of the Rayleigh-Bénard convection. We consider convection phenomena in the infinite fluid layer $\{(x_1, x_2, x_3) \in \mathbb{R}^3 \ ; \ 0 < x_3 < 1\}$, where x_3-direction is taken opposite to the gravity and temperatures at the lower and upper boundaries $\{x_3 = 0, 1\}$ are prescribed by constants θ^b and θ^t, respectively, with $\theta^b > \theta^t$. Then the model equations derived in [3] take the form

(1.1)

$$\begin{aligned} \operatorname{div} \mathbf{v} &= 0, \\ \partial_t \mathbf{v} - \Delta \mathbf{v} - \lambda \theta \mathbf{e}_3 + \nabla p + \mathbf{v} \cdot \nabla \mathbf{v} &= 0, \\ \partial_t \theta - \tfrac{1}{\mathrm{Pr}} \Delta \theta - \tfrac{\lambda}{\mathrm{Pr}} v_3 + \tfrac{\lambda \zeta}{\mathrm{Pr}} (\Theta - x_3) v_3 + \zeta \theta v_3 + \mathbf{v} \cdot \nabla \theta &= \tfrac{2\zeta}{\lambda} \mathbf{D}(\mathbf{v}) \cdot \mathbf{D}(\mathbf{v}). \end{aligned}$$

Here the unknown $\{\mathbf{v}, p, \theta\}$, $\mathbf{v} = (v_1, v_2, v_3)$, denotes the deviation of the fluid velocity, pressure and temperature from the motionless state $\{\bar{\mathbf{v}}, \bar{p}, \bar{\theta}\} =$

$\{0, -x_3 + \frac{\varepsilon^3}{2}x_3(1-x_3), \frac{1}{2} - x_3\}$; $\mathbf{e}_3 = (0,0,1)$; $\Theta = \frac{\theta^b+\theta^t}{2(\theta^b-\theta^t)} + \frac{1}{2}$; $\lambda > 0$ is defined by $\lambda^2 = \mathrm{R}$; R is the Rayleigh number ; Pr and ζ are the Prandtl and dissipation numbers, respectively ; and $\varepsilon > 0$ is a small non-dimensional parameter. The function $2\mathbf{D}(\mathbf{v}) \cdot \mathbf{D}(\mathbf{v})$ denotes the dissipation function : $2\mathbf{D}(\mathbf{v}) \cdot \mathbf{D}(\mathbf{v}) = \frac{1}{2}\sum_{i,j=1}^{3}(\partial_{x_j}v_i + \partial_{x_i}v_j)^2$.

In (1.1) the effect of the dissipative heating is controlled by the parameter ζ. If one sets $\zeta = 0$ in (1.1), one formally obtains the usual Oberbeck-Boussinesq equations.

The boundary conditions on $\{x_3 = 0, 1\}$ are given by

$$\mathbf{v} = \mathbf{0} \quad \text{and} \quad \theta = 0 \quad \text{on} \quad \{x_3 = 0, 1\}.$$

We require $\{\mathbf{v}, p, \theta\}$ to be $\frac{2\pi}{l_j}$-periodic in x_j-direction for given $l_j > 0$ $(j = 1, 2)$.

As a first step of the mathematical analysis of (1.1), we consider the stability of the motionless state. As is well known, in the usaul Oberbeck-Boussinesq case $(\zeta = 0)$, there exists a critical Rayleigh number ${\lambda_c}^2$ (depending on l_1 and l_2) such that if $\lambda < \lambda_c$, then the motionless state is unconditionally stable, while if $\lambda > \lambda_c$, then the motionless state is unstable ([2, 4, 8, 9]). We will see, in section 2, that in case $\zeta > 0$ but small the motionless state is still stable even when λ is slightly beyond λ_c ([3]).

In section 3 we consider the bifurcation problem. In case $\zeta = 0$ it is known that various types of stationary solutions with cellular patterns bifurcate from the critical value λ_c supercritically. (See [4] and references therein). We will consider stationary problem of (1.1) under the slip boundary conditions for $\mathbf{v}$ on $\{x_3 = 0, 1\}$ and show that some transcritical bifurcation branches exist when $\zeta > 0$, in particular, solutions of hexagonal patterns bifurcate transcritically. This is in contrast to the usual Oberbeck-Boussinesq case $(\zeta = 0)$ where only supercritical bifurcations can occur.

2. Stability of the motionless state

We investigate the stability of the motionless state in the Rayleigh-Bénard convection, i.e., the stability of the trivial solution of (1.1). We consider the initial boundary value problem for (1.1) under the boundary conditions described above and initial condition

$$\mathbf{v}|_{t=0} = \mathbf{v}_0, \quad \theta|_{t=0} = \theta_0.$$

Notation : We set $\Omega = \mathbb{T}_{l_1,l_2} \times (0,1)$, $\mathbb{T}_{l_1,l_2} = \mathbb{R}^2/(\frac{2\pi}{l_1}\mathbb{Z} \times \frac{2\pi}{l_2}\mathbb{Z})$; $(\cdot,\cdot)$ denotes the scalar product of $L^2(\Omega)$; $H^m(\Omega)$ denotes the m-th order L^2-Sobolev space on Ω.

In the case of the Oberbeck-Boussinesq equation ($\zeta = 0$) the stability of the motionless state is known to be controlled by the critical Rayleigh number ${\lambda_c}^2 > 0$ which is given by

$$(2.1)\qquad \frac{1}{\lambda_c} \equiv \sup\left\{\frac{2(\mathbf{v}\cdot\mathbf{e}_3, \theta)}{\|\nabla\mathbf{v}\|_2^2 + \|\nabla\theta\|_2^2}\ ;\ \{\mathbf{v},\theta\} \in H_0^1(\Omega)^4 - \{0\},\ \operatorname{div}\mathbf{v} = 0\right\}.$$

The motionless state is unconditionally stable if $\lambda < \lambda_c$ while unstable if $\lambda > \lambda_c$ [2, 4, 8, 9].

In case $\zeta > 0$ the motionless state is (conditionally) asymptotically stable even when λ is slightly beyond λ_c for sufficiently small $\zeta > 0$, namely, we have the following

Theorem 2.1 ([3]). (i) *For each* $\{\mathbf{v}_0, \theta_0\} \in H_0^1(\Omega)^3 \times L^2(\Omega)$ *with* $\operatorname{div}\mathbf{v}_0 = 0$ *there exist* $T > 0$ *and a unique solution* $\{\mathbf{v}(t), \theta(t)\}$ *of* (1.1) *on* $[0,T]$ *in the class*

$$\mathbf{v} \in C([0,T];(H_0^1)^3) \cap L^2(0,T;(H^2)^3),\quad \theta \in C([0,T];L^2) \cap L^2(0,T;H_0^1).$$

(ii) *There exist* $\zeta_0 > 0$ *and* $\lambda_c(\zeta)$ *such that if* $0 \le \zeta \le \zeta_0$ *and* $\lambda < \lambda_c(\zeta)$, *then the motionless state is asymptotically stable, namely, there exists* $\delta > 0$ *such that for each* $\{\mathbf{v}_0, \theta_0\} \in H_0^1(\Omega)^3 \times L^2(\Omega)$ *with* $\operatorname{div}\mathbf{v}_0 = 0$ *and* $\|\mathbf{v}_0\|_{H^1} + \|\theta_0\|_{L^2} < \delta$, *the solution* $\{\mathbf{v}(t), \theta(t)\}$ *exists on* $[0,\infty)$ *and satisfies*

$$\|\mathbf{v}(t)\|_{H^1} + \|\theta(t)\|_{L^2} \le Ce^{-\gamma t}(\|\mathbf{v}_0\|_{H^1} + \|\theta_0\|_{L^2})$$

for some constants C, $\gamma > 0$. *If* $\lambda > \lambda_c(\zeta)$, *then the motionless state is unstable.*

The number $\lambda_c(\zeta)$ *satisfies*

$$\lambda_c(0) = \lambda_c \ \ \text{and}\ \ \lambda_c(\zeta) > \lambda_c \ \ \text{for}\ \ 0 < \zeta \le \zeta_0.$$

Proof. Following [3] we here give an outline of the proof of (ii). See [3] for details.

To prove the assertion (ii) we consider the eigenvalue problem linearized at the motionless state :

$$(2.2)\qquad -\sigma\mathbf{u} + \mathcal{L}\mathbf{u} = 0,$$

where $\mathbf{u} = \{\mathbf{v}, \theta\}$,

$$\mathcal{L}\mathbf{u} = \mathcal{L}(\lambda, \zeta)\mathbf{u} \equiv \begin{pmatrix} A\mathbf{v} - \lambda P(\theta \mathbf{b}) \\ -\frac{1}{\mathrm{Pr}}\Delta\theta + \frac{\lambda}{\mathrm{Pr}}(\zeta(\Theta - x_3) - 1)v_3 \end{pmatrix},$$

A is the Stokes operator $-P\Delta$, P is the orthogonal projector from $L^2(\Omega)^3$ to H and H is the L^2-closure of the set of all smooth solenoidal vector fields in Ω vanishing near $\{x_3 = 0, 1\}$.

Since $\mathcal{L}$ has compact resolvent, the spectrum $\sigma(\mathcal{L})$ of $\mathcal{L}$ consists of discrete eigenvalues $\{\sigma_n\}_{n\geq 1}$ with $\mathrm{Re}\,\sigma_1 \leq \mathrm{Re}\,\sigma_2 \leq \cdots \leq \mathrm{Re}\,\sigma_n \leq \cdots \to +\infty$.

The principle of linearized stability impiles that the motionless stable is stable if $\mathrm{Re}\,\sigma_1 > 0$ while unstable if $\mathrm{Re}\,\sigma_1 < 0$. Therefore the assertion (ii) follows from the next proposition.

We denote the eigenvalues σ_j of $\mathcal{L}$ by $\sigma_j(\lambda, \zeta)$.

Proposition 2.2. *There exist $\zeta_0 > 0$ and $\lambda_c(\zeta) \geq \lambda_c$ such that if $0 \leq \zeta \leq \zeta_0$ and $\lambda < \lambda_c(\zeta)$, then $\sigma_1(\lambda, \zeta) > 0$. Moreover, if $0 \leq \zeta \leq \zeta_0$ and $\lambda > \lambda_c(\zeta)$, then $\sigma_1(\lambda, \zeta) < 0$. Here the number $\lambda_c(\zeta)$ satisfies*

$$\lambda_c(0) = \lambda_c \quad \text{and} \quad \lambda_c(\zeta) > \lambda_c \;\text{ for }\; 0 < \zeta \leq \zeta_0.$$

Proof. We consider the eigenvalue problem (2.2):

$$-\sigma\mathbf{u} + \mathcal{L}(\lambda, \zeta)\mathbf{u} = 0,$$

$$\mathcal{L}(\lambda, \zeta)\mathbf{u} \equiv \begin{pmatrix} A\mathbf{v} - \lambda P(\theta \mathbf{b}) \\ -\frac{1}{\mathrm{Pr}}\Delta\theta + \frac{\lambda}{\mathrm{Pr}}(\zeta(\hat{\Theta} - x_3) - 1)v_3 \end{pmatrix}.$$

It is known that in case $\zeta = 0$, all eigenvalues $\{\sigma_n(\lambda) \equiv \sigma_n(\lambda, 0)\}_{n\geq 1}$ are real, the smallest eigenvalue has even multiplicity, say $2m$ ($m \in \mathbb{N}$), and

$$\sigma_0(\lambda) \equiv \sigma_1(\lambda) = \cdots = \sigma_{2m}(\lambda) < \sigma_{2m+1}(\lambda) \leq \cdots \leq \sigma_n(\lambda) \leq \cdots \to +\infty.$$

Furthermore,

$$(\mathcal{L}(\lambda, 0)\mathbf{u}, \mathbf{u}) \geq \sigma_0(\lambda)\|\mathbf{u}\|^2 \tag{2.3}$$

for $\mathbf{u} \in D(A) \times (H^2(\Omega) \cap H_0^1(\Omega))$, where $D(A)$ denotes the domain of A. Here and in the following we denote the scalar product of $H \times L^2(\Omega)$ by $(\cdot, \cdot)$ which is defined as, for $\mathbf{u}_j = \{\mathbf{v}_j, \theta_j\} \in H \times L^2(\Omega)$ $(j = 1, 2)$,

$$(\mathbf{u}_1, \mathbf{u}_2) \equiv (\mathbf{v}_1, \mathbf{v}_2)_{L^2(\Omega)} + \mathrm{Pr}\,(\theta_1, \theta_2)_{L^2(\Omega)} \text{ and } \|\mathbf{u}\| \equiv \sqrt{(\mathbf{u}, \mathbf{u})}.$$

There also holds that $\sigma_0(\lambda) > 0$ (resp. $\sigma_0(\lambda) < 0$) if and only if $\lambda < \lambda_0 \equiv \lambda_c$ (resp. $\lambda > \lambda_0$) while $\sigma_0(\lambda) = 0$ if and only if $\lambda = \lambda_0$, and there exists $\gamma_0 = \gamma_0(l_1, l_2, \mathrm{Pr}) > 0$ such that if $j \geq 2m+1$ and $\lambda \leq \lambda_0$ then $\sigma_j(\lambda) \geq \gamma_0$. If $1 \leq j \leq 2m$ each $\sigma_j(\lambda)$ is continuous in λ. In particular, for any $\varepsilon > 0$ there exists $\delta(\varepsilon) > 0$ such that if $\lambda < \lambda_0 - \varepsilon$, then $\sigma_0(\lambda) \geq \delta(\varepsilon)$. ([4, 8, 9].)

We now consider the case $0 < \zeta \ll 1$. We write (2.2) as

$$-\sigma \mathbf{u} + \mathcal{L}_0 \mathbf{u} + (\lambda - \lambda_0)\mathcal{M}_1 \mathbf{u} + \zeta \mathcal{M}_2 \mathbf{u} + \mathcal{M}_3(\lambda, \zeta)\mathbf{u} = 0,$$

where

$$\mathcal{L}_0 = \mathcal{L}(\lambda_0, 0), \quad \mathcal{M}_1 \mathbf{u} = \begin{pmatrix} -P(\theta \mathbf{b}) \\ -\frac{1}{\mathrm{Pr}} v_3 \end{pmatrix}, \quad \mathcal{M}_2 \mathbf{u} = \begin{pmatrix} 0 \\ \frac{\lambda_0}{\mathrm{Pr}}(\Theta - x_3) v_3 \end{pmatrix}$$

and

$$\mathcal{M}_3(\lambda, \zeta)\mathbf{u} = \begin{pmatrix} 0 \\ \frac{\lambda - \lambda_0}{\mathrm{Pr}} \zeta (\Theta - x_3) v_3 \end{pmatrix}.$$

We first consider the case $\lambda < \lambda_0 - \varepsilon$ for some $\varepsilon > 0$.

Proposition 2.3. *For any $\varepsilon > 0$ there exists $\delta(\varepsilon) > 0$ and $\zeta_1(\varepsilon) > 0$ with*

$$\sigma(\mathcal{L}(\lambda, \zeta)) \subset \{\sigma\,;\, \mathrm{Re}\,\sigma \geq \delta(\varepsilon)/2\}$$

if $\lambda < \lambda_0 - \varepsilon$ and $0 \leq \zeta \leq \zeta_1(\varepsilon)$.

Proof. Since $\|(\Theta - x_3) v_3\|_2 \leq C\,\|\mathbf{u}\|$, we see from (2.3) that

$$\begin{aligned} \mathrm{Re}\,(\mathcal{L}(\lambda, \zeta)\mathbf{u}, \mathbf{u}) &= (\mathcal{L}(\lambda, 0)\mathbf{u}, \mathbf{u}) + \mathrm{Re}\,(\zeta \mathcal{M}_2 \mathbf{u}, \mathbf{u}) + \mathrm{Re}\,(\mathcal{M}_3(\lambda, \zeta)\mathbf{u}, \mathbf{u}) \\ &\geq (\sigma_0 - C\lambda_0 \zeta)\|\mathbf{u}\|^2. \end{aligned}$$

Now recall that for any $\varepsilon > 0$ there exists $\delta = \delta(\varepsilon) > 0$ such that if $\lambda < \lambda_0 - \varepsilon$, then $\sigma_0 \geq \delta(\varepsilon)$. Thus, if $\zeta \leq \frac{\delta(\varepsilon)}{2C\lambda_0}$ and $\lambda < \lambda_0 - \varepsilon$, then

$$\mathrm{Re}\,(\mathcal{L}(\lambda, \zeta)\mathbf{u}, \mathbf{u}) \geq \delta(\varepsilon)\tfrac{1}{2}\|\mathbf{u}\|^2,$$

which implies that $\sigma(\mathcal{L}(\lambda, \zeta)) \subset \{\sigma\,;\, \mathrm{Re}\,\sigma \geq \frac{1}{2}\delta(\varepsilon)\}$ for $\lambda < \lambda_0 - \varepsilon$ and $0 \leq \zeta \leq \frac{\delta(\varepsilon)}{2C\lambda_0}$. This shows Proposition 2.3.

We next investigate $\mathcal{L}(\lambda, \zeta)$ for $|\lambda - \lambda_0| \leq \varepsilon \ll 1$ and $0 < \zeta \ll 1$.

Proposition 2.4. (i) *There exist* $\varepsilon_2 > 0$ *and* $\zeta_2 > 0$ *such that*

$$\sigma(\mathcal{L}(\lambda,\zeta)) \subset \{\sigma\,;\, |\sigma| \le \tfrac{1}{4}\gamma_0\} \cup \{\sigma\,;\, \mathrm{Re}\,\sigma \ge \tfrac{3}{4}\gamma_0\} \tag{2.4}$$

if $|\lambda-\lambda_0| \le \varepsilon_2$ *and* $0 \le \zeta \le \zeta_2$.

(ii) *There exist* $0 < \varepsilon_3 \le \varepsilon_2$ *and* $0 < \zeta_3 \le \zeta_2$ *such that the eigenvalues of* $\mathcal{L}(\lambda,\zeta)$ *in* $\{\sigma\,;\, |\sigma| \le \frac{1}{4}\gamma_0\}$ *have the form*

$$\sigma = \sigma^{(1,0)}(\lambda-\lambda_0) + \sigma^{(0,1)}\zeta + O(|\lambda-\lambda_0|^2+\zeta^2) \tag{2.5}$$

with constants $\sigma^{(1,0)} < 0$ *and* $\sigma^{(0,1)} > 0$, *if* $|\lambda-\lambda_0| \le \varepsilon_3$ *and* $0 \le \zeta \le \zeta_3$. *Moreover, there exists* $\lambda_c = \lambda_c(\zeta) > 0$ *satisfying*

$$\lambda_c(0) = \lambda_0 \quad \text{and} \quad \lambda_c(\zeta) > \lambda_0 \quad \text{for} \quad 0 < \zeta \le \zeta_3$$

and it holds $\sigma_1(\lambda,\zeta) > 0$ *if* $\lambda < \lambda_c(\zeta)$ *and* $\sigma_1(\lambda,\zeta) < 0$ *if* $\lambda > \lambda_c(\zeta)$, *provided that* $|\lambda-\lambda_0| < \varepsilon_3$ *and* $0 \le \zeta \le \zeta_3$.

Proof. We first observe

$$\|\mathcal{M}_j\mathbf{u}\|_2 \le C\|\mathbf{u}\| \quad (j=1,2,3).$$

Since $\mathcal{L}_0$ is self-adjoint, we obtain for some constant $a = a(\lambda_0, \hat{\Theta}, \mathrm{Pr}\,) > 0$,

$$\begin{aligned}&\|((\lambda-\lambda_0)\mathcal{M}_1 + \zeta\mathcal{M}_2 + \mathcal{M}_3(\lambda,\zeta))(-\mu+\mathcal{L}_0)^{-1}\mathbf{u}\| \\ &\le \tfrac{a(|\lambda-\lambda_0|+\zeta)}{\min_{k\ge1}|-\mu+\sigma_k|}\|\mathbf{u}\| \le \tfrac{1}{2}\|\mathbf{u}\|\end{aligned}$$

provided that $\mu \in \Sigma \equiv \{\sigma\,;\, |\sigma| > \frac{1}{4}\gamma_0\} \cap \{\sigma\,;\, \mathrm{Re}\,\sigma < \frac{3}{4}\gamma_0\}$, $|\lambda-\lambda_0| \le \varepsilon_2$ and $0 \le \zeta \le \zeta_2$ for some small $\varepsilon_2 > 0$ and $\zeta_2 > 0$. This inequality immediately implies that Σ is included in the resolvent set of $\mathcal{L}(\lambda,\zeta)$ and (2.4) follows.

To prove (2.5) we note that the problem (2.2) is equivalent to

$$\left\{\begin{array}{l} -\sigma\mathbf{v} - \Delta\mathbf{v} - \lambda\theta\mathbf{b} + \nabla p = 0, \\ -\sigma\theta - \frac{1}{\mathrm{Pr}}\Delta\theta + \frac{\lambda}{\mathrm{Pr}}(\zeta(\Theta - x_3) - 1)v_3 = 0, \\ \mathrm{div}\,\mathbf{v} = 0 \end{array}\right. \tag{2.6}$$

with boundary conditions under consideration.

To solve (2.6) we expand $\mathbf{v}$, θ and ∇p into Fourier series in x_1 and x_2, and so we assume $\mathbf{v}$, θ and ∇p to have the form $e^{2\pi i(\frac{k_1}{l_1}x_1+\frac{k_2}{l_2}x_2)}h(x_3)$, where

$(k_1, k_2) \in \mathbb{Z}^2$. We first consider the case $(k_1, k_2) = (0,0)$, namely, $v_j = v_j(x_3)$ $(j = 1,2,3)$, $\theta = \theta(x_3)$. Due to $\operatorname{div} \mathbf{v} = 0$ we have $\frac{d}{dx_3} v_3 = 0$. This, together with $\mathbf{v} = 0$ on $\{x_3 = 0, 1\}$, yields $v_3 \equiv 0$. We then obtain

$$\begin{cases} -\sigma \|v_j\|^2_{L^2(0,1)} + \|\frac{d}{dx_3} v_j\|^2_{L^2(0,1)} = 0 \quad (j = 1,2), \\ -\sigma \|\theta\|^2_{L^2(0,1)} + \dfrac{1}{\mathrm{Pr}} \|\frac{d}{dx_3}\theta\|^2_{L^2(0,1)} = 0. \end{cases}$$

This implies that

$$\sigma \geq a\pi^2 = a \inf \left\{ \frac{\|\frac{d}{dx_3} h\|^2_{L^2(0,1)}}{\|h\|^2_{L^2(0,1)}} ; h \in H^1_0(0,1),\, h \neq 0 \right\},$$

where $a = \min(1, \mathrm{Pr}^{-1})$. Therefore, we see that $\sigma \in \{\sigma;\, \mathrm{Re}\,\sigma \geq \frac{3}{4}\gamma_0\}$.

We next consider $(k_1, k_2) \neq (0,0)$. This is the case where there really occurs $\sigma \in \{\sigma; |\sigma| \leq \frac{1}{4}\gamma_0\}$. Taking curl curl of $(2.6)_1$, we obtain

$$(2.7) \qquad \begin{cases} \sigma \Delta v_3 + \Delta^2 v_3 + \lambda \Delta_2 \theta = 0, \\ -\sigma\theta - \frac{1}{\mathrm{Pr}} \Delta\theta + \frac{\lambda}{\mathrm{Pr}} (\zeta(\Theta - x_3) - 1) v_3 = 0 \end{cases}$$

with boundary conditions $v_3 = \partial_3 v_3 = \theta = 0$ at $x_3 = 0, 1$ and the periodic boundary conditions in x_1 and x_2. Here $\Delta_2 = \partial_{11} + \partial_{22}$.

We now substitute $v_3 = e^{2\pi i (\frac{k_1}{l_1} x_1 + \frac{k_2}{l_2} x_2)} f(x_3)$, $\theta = e^{2\pi i (\frac{k_1}{l_1} x_1 + \frac{k_2}{l_2} x_2)} g(x_3)$ for $(k_1, k_2) \neq (0,0)$ into (2.7). Then we find the eigenvalue problem :

$$(2.8) \qquad \begin{cases} -\sigma D_\omega f + {D_\omega}^2 f - \lambda \omega^2 g = 0 \\ -\sigma g + \frac{1}{\mathrm{Pr}} D_\omega g + \frac{\lambda}{\mathrm{Pr}} (\zeta(\Theta - x_3) - 1) f = 0 & (0 < x_3 < 1), \\ f = \frac{d}{dx_3} f = g = 0 & (x_3 = 0, 1), \end{cases}$$

where $\omega^2 \equiv (\frac{2\pi k_1}{l_1})^2 + (\frac{2\pi k_2}{l_2})^2 > 0$, $D_\omega \equiv (-\frac{d^2}{dx_3^2} + \omega^2)$ and ${D_\omega}^2 \equiv (\frac{d^2}{dx_3^2} - \omega^2)^2$.

It is easily verified that the eigenvalues and eigenfunctions of (2.6) for $(k_1, k_2) \neq (0,0)$ can be obtained from those of (2.8) with suitable $\omega^2 > 0$ and vice versa, since $\omega^2 > 0$. We write (2.8) as

$$(2.9) \qquad -\sigma M \mathbf{f} + L(\lambda, \zeta)\mathbf{f} = 0, \quad \mathbf{f} = \{f, g\}.$$

Here

$$M \equiv \begin{pmatrix} D_\omega & 0 \\ 0 & 1 \end{pmatrix}, \quad L(\lambda, \zeta) \equiv \begin{pmatrix} {D_\omega}^2 & -\lambda\omega^2 \\ \frac{\lambda}{\mathrm{Pr}}(\zeta(\Theta - x_3) - 1) & \frac{1}{\mathrm{Pr}} D_\omega \end{pmatrix}$$

and the operators D_ω and ${D_\omega}^2$ are defined as above for $g \in H^2(0,1) \cap H_0^1(0,1)$ and $f \in \{f \in H^4(0,1)\,;\, f = \frac{d}{dx_3} f = 0 \text{ at } x_3 = 0, 1\}$, respectively.

The eigenvalues $\sigma_j(\lambda_0)$ of $\mathcal{L}_0$ are given by the eigenvalues of the eigenvalue problem (2.9) with $\lambda = \lambda_0$ and $\zeta = 0$, and moreover, the eigenvalues of $\mathcal{L}(\lambda, \zeta)$ in $\{\sigma\,;\, |\sigma| \leq \frac{1}{4}\gamma_0\}$ are given by those of $L(\lambda, \zeta)$ in $\{\sigma\,;\, |\sigma| \leq \frac{1}{4}\gamma_0\}$. In particular, $\sigma(L(0,0)) \cap \{\sigma\,;\, |\sigma| \leq \frac{1}{4}\gamma_0\} = \{\sigma_0(\lambda_0) = 0\}$. The following lemma summarizes the results in [6, page 38].

Lemma 2.5. (i) *The eigenvalue* $\sigma_0(\lambda_0) = 0$ *of* $L^{(0,0)} \equiv L(0,0)$ *is simple.*

(ii) *One can choose an eigenfunction* $\mathbf{f}_0 = \{f_0, g_0\}$ *of* $L^{(0,0)}$ *associated with* $\sigma_0(\lambda_0) = 0$ *in such a way that* $f_0(x_3) > 0$ *and* $g_0(x_3) > 0$ *for* $0 < x_3 < 1$.

Since $\sigma_0(\lambda_0)$ is simple by Lemma 2.5 (i), there exists only one eigenvalue $\sigma = \sigma(\lambda, \zeta)$ of $L(\lambda, \zeta)$ in $\{\sigma\,;\, |\sigma| \leq \frac{\gamma_0}{4}\}$ when $|\lambda - \lambda_0|$ and ζ are sufficiently small. Furthermore, due to the simplicity of $\sigma_0(\lambda_0)$, one can see that $\sigma(\lambda, \zeta)$ is analytic in λ and ζ near $\lambda = \lambda_0$ and $\zeta = 0$ and it is expanded as

$$\sigma(\lambda, \zeta) = \sum_{j,k \geq 0}^{\infty} \sigma^{(j,k)} (\lambda - \lambda_0)^j \zeta^k \quad \text{with } \sigma^{(0,0)} = \sigma(\lambda_0) = 0. \tag{2.10}$$

We denote by $\mathbf{f}(\lambda, \zeta)$ the eigenfunction associated with $\sigma(\lambda, \zeta)$ satisfiying $\mathbf{f}(0,0) = \mathbf{f}_0$. Then

$$\mathbf{f}(\lambda, \zeta) = \sum_{j,k \geq 0}^{\infty} (\lambda - \lambda_0)^j \zeta^k \mathbf{f}^{(j,k)} \tag{2.11}$$

with $\mathbf{f}^{(0,0)} = \mathbf{f}_0$. Substituting (2.10) and (2.11) into (2.9) we obtain

$$L^{(0,0)} \mathbf{f}_0 = 0,$$

$$-\sigma^{(1,0)} M \mathbf{f}_0 + L^{(0,0)} \mathbf{f}^{(1,0)} + L^{(1,0)} \mathbf{f}_0 = 0, \tag{2.12}$$

$$-\sigma^{(0,1)} M \mathbf{f}_0 + L^{(0,0)} \mathbf{f}^{(0,1)} + L^{(0,1)} \mathbf{f}_0 = 0 \tag{2.13}$$

and so on. Here $L(\lambda,\zeta) = \sum_{0\le j,k\le 1}(\lambda-\lambda_0)^j\zeta^k L^{(j,k)}$ with $L^{(0,0)} = L(0,0)$,

$$L^{(1,0)}\mathbf{f} = \left\{-\omega^2 g, -\frac{1}{\mathrm{Pr}}f\right\}, \quad L^{(0,1)}\mathbf{f} = \left\{0, \frac{\lambda_0}{\mathrm{Pr}}(\hat{\Theta}-x_3)f\right\}$$

and $L^{(1,1)} = \lambda_0^{-1}L^{(0,1)}$. To compute $\sigma^{(j,k)}$ we define $\langle\cdot,\cdot\rangle$ by

$$\langle \mathbf{f}_1, \mathbf{f}_2\rangle = \frac{1}{\omega^2}\int_0^1 f_1(x_3)\overline{f_2(x_3)}dx_3 + \mathrm{Pr}\int_0^1 g_1(x_3)\overline{g_2(x_3)}dx_3$$

for $\mathbf{f}_j = \{f_j, g_j\} \in L^2(0,1)^2$ $(j=1,2)$. Here $\overline{f}$ denotes the complex conjugate of f. Note that $\langle L^{(0,0)}\mathbf{f}_1, \mathbf{f}_2\rangle = \langle \mathbf{f}_1, L^{(0,0)}\mathbf{f}_2\rangle$ and $\langle M\mathbf{f}, \mathbf{f}\rangle > 0$ for $\mathbf{f} \neq 0$.

Taking $\langle\cdot,\cdot\rangle$ of (2.12) and (2.13) with $\mathbf{f}_0$ respectively, we obtain

$$\sigma^{(1,0)} = \frac{\langle L^{(1,0)}\mathbf{f}_0, \mathbf{f}_0\rangle}{\langle M\mathbf{f}_0, \mathbf{f}_0\rangle} \quad \text{and} \quad \sigma^{(0,1)} = \frac{\langle L^{(0,1)}\mathbf{f}_0, \mathbf{f}_0\rangle}{\langle M\mathbf{f}_0, \mathbf{f}_0\rangle}$$

respectively. The coefficient $\sigma^{(1,0)}$ must satisfy $\sigma^{(1,0)} < 0$, since $\sigma_0(\lambda) > 0$ if and only if $\lambda < \lambda_0$, and $\sigma_0(\lambda) < 0$ if and only if $\lambda > \lambda_0$. Since f_0, $g_0 > 0$ by Lemma 2.5 (ii) and since $\Theta > 1 \ge x_3$ for $0 \le x_3 \le 1$, we see that

$$\langle L^{(0,1)}\mathbf{f}_0, \mathbf{f}_0\rangle = \int_0^1 \lambda_0(\Theta - x_3)f_0(x_3)g_0(x_3)dx_3 > 0.$$

Thus, $\sigma^{(0,1)} > 0$, and we have obtained (2.5). Now we define $\lambda_0(\zeta)$ by $\sigma(\lambda_0(\zeta),\zeta) = 0$. We then have

$$\lambda_0(\zeta) = \lambda_0 - \frac{\sigma^{(0,1)}}{\sigma^{(1,0)}}\zeta + O(\zeta^2). \tag{2.14}$$

Since $\lambda_0(\zeta)$ also depends on ω^2, we denote $\lambda_0(\zeta)$ by $\lambda_0(\zeta;\omega^2)$. Then the critical number $\lambda_c(\zeta)$ is given by

$$\lambda_c(\zeta) = \inf_{(k_1,k_2)\in\mathbb{Z}^2\setminus(0,0)} \lambda_0\left(\zeta; \left(\frac{2\pi k_1}{l_1}\right)^2 + \left(\frac{2\pi k_2}{l_2}\right)^2\right). \tag{2.15}$$

This completes the proof.

Proposition 2.2 now follows from Propositions 2.3 and 2.4 by taking $\varepsilon = \varepsilon_3$ in Proposition 2.3 and $\zeta_0 = \min\{\zeta_1(\varepsilon_3), \zeta_3\}$. This completes the proof of Proposition 2.2.

3. Remarks on bifurcation problem

In this section we consider bifurcation problem for (1.1). Due to a technical reason, we here consider (1.1) under the slip boundary conditions for $\mathbf{v}$ on $\{x_3 = 0, 1\}$ instead of the no-slip boundary conditions, i.e., we consider

$$\partial_{x_3} v_1 = \partial_{x_3} v_2 = v_3 = 0 \quad \text{on} \quad \{x_3 = 0, 1\}.$$

The boundary conditions for θ are the same as in sections 1 and 2, and we also impose the same periodic boundary conditions in x_1 and x_2-variables as in sections 1 and 2. We also require

$$\int_\Omega v_1(\mathbf{x}) d\mathbf{x} = \int_\Omega v_2(\mathbf{x}) d\mathbf{x} = 0.$$

Under these boundary conditions one can also obtain similar critical numbers $\lambda_c(\zeta)$ for the stability of the motionless state. In case $\zeta = 0$ it is known that nontrivial solution branches of various cellular patterns such as rolls and hexagones bifurcate at λ_c. (See [4] and references therein.) Due to the unconditional stability of the motionless state, only supercritical bifurcations can occur when $\zeta = 0$.

We will show that, in contrast to the case of $\zeta = 0$, some transcritical bifurcation branches exist when $\zeta > 0$. In particular, hexagonal solutions bifurcate at $\lambda_c(\zeta)$ transcritically when $\zeta > 0$.

Notation. In this section we denote the spatial variable $\mathbf{x}$ and the fluid velocity $\mathbf{v}$ by

$$\mathbf{x} = (x_1, x_2, x_3) = (x, y, z) \quad \text{and} \quad \mathbf{v} = (v_1, v_2, v_3) = (u, v, w)$$

respectively. We also write the periods l_1 and l_2 as

$$l_1 = \frac{2\pi}{\alpha} \quad \text{and} \quad l_2 = \frac{2\pi}{\beta}.$$

When $\zeta = 0$, the usual critical Rayleigh number ${\lambda_c}^2$ under the slip boundary conditions is given by a similar formula to (2.1). But in this case it has an explicit formula :

$${\lambda_c}^2 = \inf_{(k,m)\in\mathbb{Z}^2} \frac{({\omega_{k,m}}^2 + \pi^2)^3}{{\omega_{k,m}}^2}, \quad {\omega_{k,m}}^2 = (\alpha k)^2 + (\beta m)^2.$$

Note that $\frac{(\omega^2+\pi^2)^3}{\omega^2}$ attains its mimimum value at $\omega = \omega_c = \pi/\sqrt{2}$. By a similar argument in section 2, one can obtain the critical number $\lambda_c(\zeta)$ for sufficiently small $\zeta > 0$, which is given by an analogue of (2.15) :

$$\lambda_c(\zeta) = \lambda_c(\zeta;\omega^2) = \inf_{(k,m)\in\mathbb{Z}^2} \lambda_0(\zeta;\omega_{k,m}{}^2), \tag{3.1}$$

where $\omega^2 = \alpha^2 + \beta^2$ and $\omega_{k,m}{}^2 = (\alpha k)^2 + (\beta m)^2$. Here the function $\lambda_0(\zeta;\omega^2)$ is given by an analogue of (2.14) :

$$\lambda_0(\zeta;\omega^2) = \sqrt{\frac{(\omega^2+\pi^2)^3}{\omega^2}} - \frac{\sigma^{(0,1)}}{\sigma^{(1,0)}}\zeta + O(\zeta^2).$$

$\lambda_c(\zeta) = \lambda_c(\zeta;\omega^2)$ attains its minimum in ω at $\omega_c(\zeta) = \omega_c + O(\zeta)$.

3.1 Two-dimensional case.

We first consider the two-dimensional problem ; this means that the unknowns $\mathbf{v}$, θ (and p) depend only on x and z but not on y, and $v(x,z) \equiv 0$.

In this case the critical number $\lambda_c(\zeta)$ in (3.1) may be written as

$$\lambda_c(\zeta) = \lambda_c(\zeta;\alpha^2) = \inf_{k\in\mathbb{Z}} \lambda_0(\zeta;(\alpha k)^2). \tag{3.2}$$

We now take α in such a way that the infimum in (3.2) is attained at both $k=1$ and $k=2$. (This really occurs. See [1, 8] for the case $\zeta = 0$.)

For this choice of α one sees that $\dim\ker\mathcal{L}_{\lambda_c(\zeta)} = 4$. We restrict ourselves to the subspace of functions which have the Fourier expansions of the form :

$$\begin{cases} u = \sum_{k,n} u_{k,n} \sin\alpha k x \cos n\pi z, \\ w = \sum_{k,n} w_{k,n} \cos\alpha k x \sin n\pi z, \\ \theta = \sum_{k,n} \theta_{k,n} \cos\alpha k x \sin n\pi z. \end{cases}$$

Then if $\mathcal{L}_{\lambda_c(\zeta)}$ is restricted on this space, we have $\dim\ker\mathcal{L}_{\lambda_c(\zeta)} = 2$, and $\ker\mathcal{L}_{\lambda_c(\zeta)} = \mathrm{span}\{\mathbf{u}_0^1, \mathbf{u}_0^2\}$, where

$$\mathbf{u}_0^j = \begin{pmatrix} 0 \\ w^j \\ \theta^j \end{pmatrix} \cos\alpha j x \sin\pi z + O(\zeta) \quad (j=1,2)$$

with some constants w^j and θ^j.

We look for nontrivial stationary solutions for λ near $\lambda_c(\zeta)$ by the Lyapunov-Schmidt method. To do so, we write $\mathbf{u}$ as

$$\mathbf{u} = A_1\mathbf{u}_0^1 + A_2\mathbf{u}_0^2 + \Phi, \quad A_j \in \mathbb{R}, \quad (\Phi, \mathbf{u}_0^{j*}) = 0 \quad (j = 1, 2),$$

where $\mathbf{u}_0^{j*}$ are functions in $\ker \mathcal{L}_{\lambda_c(\zeta)}{}^*$ satisfying $(\mathbf{u}_0^j, \mathbf{u}_0^{k*}) = \delta_{j,k}$. The Lyapunov-Schmidt reduction then yields

$$(3.3) \qquad \begin{cases} p_0(\lambda - \lambda_c(\zeta))A_1 + \zeta(p_1 + \Pr p_2)A_1A_2 + O(|A|^3) = 0, \\ p_0(\lambda - \lambda_c(\zeta))A_2 + \zeta q_1 {A_1}^2 + O(|A|^3) = 0, \end{cases}$$

where $p_0 = O(1) < 0$, $p_1 = O(1) < 0$, $p_2 = O(1) > 0$ and $q_1 = O(1) > 0$ as $\zeta \to 0$.

From (3.3) we obtain the following

Theorem 3.1. (i) *(Usual roll solutions) There exist nontrivial solution branches* $\{\{A_1, 0\}, \lambda - \lambda_c(\zeta) = \mu_1 {A_1}^2\}$ *and* $\{\{0, A_2\}, \lambda - \lambda_c(\zeta) = \mu_2 {A_2}^2\}$, *where* μ_j $(j = 1, 2)$ *are positive constants. The solutions* $\mathbf{u}_j$ *corresponding to these branches have the forms :*

$$\mathbf{u}_j = A_j\mathbf{u}_0^j + O(|A_j|^2) \quad (j = 1, 2).$$

These are the usual roll solutions.

(ii) *(Mixed solutions)* (a) *(Existence)There exists* $\Pr_0 > 0$ *such that if* $\Pr > \Pr_0$, *then there exist two nontrivial solution branches of the forms :*

$$\begin{cases} A_1 = \varepsilon, \quad A_2 = \pm a_2\varepsilon + O(|\varepsilon|^2), \\ \lambda - \lambda_c(\zeta) = \mp\mu_3\varepsilon + O(|\varepsilon|^2), \end{cases}$$

where $a_2 = O(1) > 0$ *and* $\mu_3 = O(1) > 0$ *as* $\varepsilon \to 0$. (Fig. 1). *The solutions* $\mathbf{u}_{(\pm)}$ *corresponding to these branches have the forms*

$$\mathbf{u}_{(\pm)} = \varepsilon(\mathbf{u}_0^1 \pm a_2\mathbf{u}_0^2) + O(\varepsilon^2).$$

(b) *(No existence) If* $\Pr < \Pr_0$, *then there exist no small stationary solutions except for the trivial solution* $\mathbf{u} = 0$ *and the usual roll solutions* $\mathbf{u}_j$ $(j = 1, 2)$ *obtained in (i).*

Remark. In case $\zeta = 0$ the analysis of mixed solutions was given in details in [1].

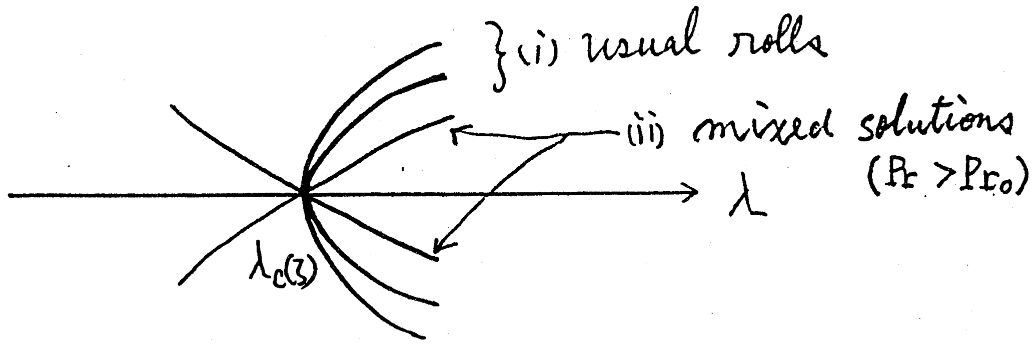

Fig. 1

3.2 Hexagonal solutions

We next consider the bifurcation problem of solutions of hexagonal patterns. To obtain hexagonal solutions we require $\beta = \sqrt{3}\alpha$ and also $2\alpha = \omega_c(\zeta)$. We restrict ourselves to the subspace of functions invariant under $\frac{2\pi}{3}$-rotation in (x, y). We further require that $\mathbf{u}$ has the Fouier expansions of the form:

$$(3.4)\quad \begin{cases} u = \displaystyle\sum_{k,m,n} u_{kmn} \sin \alpha k x \cos \sqrt{3}\alpha m y \cos n\pi z, \\ v = \displaystyle\sum_{k,m,n} v_{kmn} \cos \alpha k x \sin \sqrt{3}\alpha m y \cos n\pi z, \\ w = \displaystyle\sum_{k,m,n} w_{kmn} \cos \alpha k x \cos \sqrt{3}\alpha m y \sin n\pi z, \\ \theta = \displaystyle\sum_{k,m,n} \theta_{kmn} \cos \alpha k x \cos \sqrt{3}\alpha m y \sin n\pi z. \end{cases}$$

The requirement of $\frac{2\pi}{3}$-rotation invariance restricts the form of functions in (3.4), for example, θ has the form

$$\begin{aligned} \theta = \sum_{\substack{k,m,n \\ k+m=even}} \theta_{kmn} \{ & \cos \alpha k x \cos \sqrt{3}\alpha m y \\ & + \cos \{\alpha(\tfrac{1}{2}k - \tfrac{3}{2}m)x\} \cos \{\sqrt{3}\alpha(\tfrac{1}{2}k + \tfrac{1}{2}m)y\} \\ & + \cos \{\alpha(\tfrac{1}{2}k + \tfrac{3}{2}m)x\} \cos \{\sqrt{3}\alpha(\tfrac{1}{2}k - \tfrac{1}{2}m)y\}\} \sin n\pi z. \end{aligned}$$

(See [4, 5].)

In this space we have $\dim\ker \mathcal{L}_{\lambda_c(\zeta)} = 1$. We take a nontrivial vecter $\mathbf{u}_0$ from $\ker \mathcal{L}_{\lambda_c(\zeta)}$, whose w-component w_0 has, say, the form

$$w_0 = \{2\cos\alpha x \cos\sqrt{3}\alpha y + \cos 2\alpha x\}\sin\pi z + O(\zeta).$$

Similarly as in secton 3.1 we look for nontrivial stationary solutions for λ near $\lambda_c(\zeta)$ by the Lyapunov-Schmidt method. We write $\mathbf{u}$ as

$$\mathbf{u} = A\mathbf{u}_0 + \Phi, \quad A \in \mathbb{R}, \quad (\Phi, \mathbf{u}_0^*) = 0,$$

where $\mathbf{u}_0^*$ is a function in $\ker \mathcal{L}_{\lambda_c(\zeta)}{}^*$ satisfying $(\mathbf{u}_0, \mathbf{u}_0^*) = 1$. The Lyapunov-Schmidt reduction then yields

$$p_0(\lambda - \lambda_c(\zeta))A + \zeta p_1 A^2 + p_2 A^3 + O(|A|^4) = 0, \tag{3.5}$$

where $p_0 = O(1) < 0$, $p_1 = p_1(\mathrm{Pr}) = O(1)$ and $p_2 = O(1) > 0$ as $\zeta \to 0$. Here $p_1 = p_1(\mathrm{Pr})$ changes signs at some $\mathrm{Pr} = \mathrm{Pr}_1$.

From (3.5) we obtain the following

Theorem 3.2. *There exists* $\mathrm{Pr}_1 > 0$ *such that*
(i) *if* $\mathrm{Pr} \neq \mathrm{Pr}_1$, *then there exists a hexagonal solutions branch bifurcating at* $\lambda_c(\zeta)$ *transcritically*
and
(ii) *if* $\mathrm{Pr} = \mathrm{Pr}_1$, *there exists a hexagonal solutions branch bifurcating at* $\lambda_c(\zeta)$ *supercritically.* (Fig. 2).

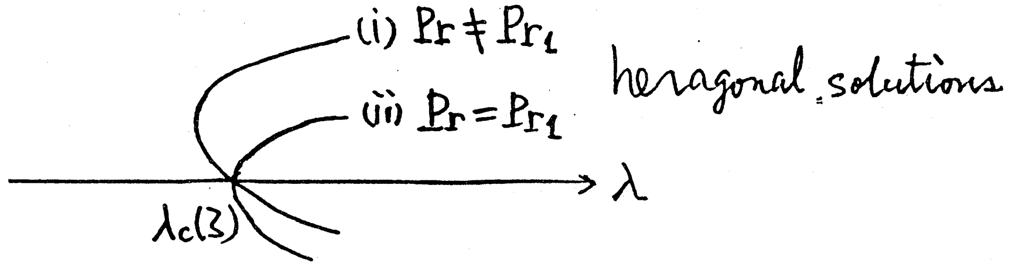

Fig. 2

References

[1] Busse, F. H., Or, A. C. : Subharmonic and asymmetric convection rolls, Z. Angew. Math. Phys. **37**, 608–623 (1986).

[2] Joseph, D. D. : On the stability of the Boussinesq Equations, Arch. Rational Mech. Anal. **20**, 59–71 (1965).

[3] Kagei, Y., Růžička, M., Thäter, G.: Natural convection with dissipative heating. Preprint, Universität Bonn SFB 256, no. 602 (1999).

[4] Kirchgäßner, K., Kielhöfer, H. : Stability and bifurcation in fluid dynamics, Rocky Mountain J. Math. **3**, 275–318 (1973).

[5] Nishida, T. : Partial differential equations in hydrodynamics, The world of nonlinear analysis, Suugaku-Seminar **38**(6), 25–29 (1999), Nippon Hyoron Sha, Tokyo (in Japanese).

[6] Rabinowitz, P. : Existence and nonuniqueness of rectangular solutions of the Bénard problem, Arch. Rational Mech. Anal. **29**, 32–57 (1968).

[7] Rajagopal, K. R., Růžička, M., Srinivasa, A. R. : On the Oberbeck-Boussinesq Approximation, Math. Models Methods Appl. Sci. **6**, 1157–1167 (1996).

[8] Schmitt, B. J., von Wahl, W. : Monotonicity and boundedness in the Boussinesq-equations, Eur. J. Mech. B/Fluids **12**, 245–270 (1993).

[9] von Wahl, W. : The Boussinesq-Equations in terms of poloidal and toroidal fields and the mean flow, Bayreuther Math. Schriften **40**, 203–290 (1992).