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Kyoto University
On thermal convection equations of Oberbeck-Boussinesq type with the dissipation function

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1. Introduction

We study the stability of the motionless state and bifurcation of cellular patterns in the Rayleigh-Bénard convection under the effect of the dissipative heating.

The Oberbeck-Boussinesq equations are frequently used as model equations in the mathematical analysis of convection phenomena such as the Rayleigh-Bénard convection problem. Many interesting and useful mathematical results have been obtained through the Oberbeck-Boussinesq equations, and Rajagopal, Růžička and Srinivasa [7] gave a justification for the derivation of the Oberbeck-Boussinesq equations from the point of view of continuum mechanics. However, there are some phenomena such as the earth’s upper mantle convection, convection in fast rotating configurations and etc., in which the Oberbeck-Boussinesq equations seem inappropriate due to the fact that the effect of dissipative heating is not taken into account in the equations.

Our purpose here is to study the model equations including the effect of dissipative heating, which was derived in [3], in the context of the Rayleigh-Bénard convection. We consider convection phenomena in the infinite fluid layer \( \{(x_1, x_2, x_3) \in \mathbb{R}^3 ; 0 < x_3 < 1\} \), where \( x_3 \)-direction is taken opposite to the gravity and temperatures at the lower and upper boundaries \( \{x_3 = 0, 1\} \) are prescribed by constants \( \theta^b \) and \( \theta^t \), respectively, with \( \theta^b > \theta^t \). Then the model equations derived in [3] take the form

\[
\text{div} \mathbf{v} = 0,
\]

\[
\partial_t \mathbf{v} - \Delta \mathbf{v} - \lambda \mathbf{e}_3 + \nabla p + \mathbf{v} \cdot \nabla \mathbf{v} = 0,
\]

\[
\partial_t \theta - \frac{1}{Pr} \Delta \theta - \frac{\Delta}{Pr} \mathbf{v}_3 + \frac{\kappa}{Pr} (\theta - x_3) \mathbf{v}_3 + \zeta \theta \mathbf{v}_3 + \mathbf{v} \cdot \nabla \theta = \frac{2}{\lambda} \mathbf{D}(\mathbf{v}) \cdot \mathbf{D}(\mathbf{v}).
\]

Here the unknown \( \{\mathbf{v}, p, \theta\} \), \( \mathbf{v} = (v_1, v_2, v_3) \), denotes the deviation of the fluid velocity, pressure and temperature from the motionless state \( \{\bar{v}, \bar{p}, \bar{\theta}\} = \)}
\{0, -x_3 + \frac{e^3}{2}x_3(1 - x_3), \frac{1}{2} - x_3\}; e_3 = (0, 0, 1); \Theta = \frac{\Theta^{b} + \Theta^{t}}{2(\Theta^{b} - \Theta^{t})} + \frac{1}{2}; \lambda > 0 is defined by \lambda^2 = R; R is the Rayleigh number; Pr and \zeta are the Prandtl and dissipation numbers, respectively; and \epsilon > 0 is a small non-dimensional parameter. The function \(2D(v) \cdot D(v)\) denotes the dissipation function: 

\[2D(v) \cdot D(v) = \frac{1}{2} \sum_{i,j=1}^{3} (\partial x_j v_i + \partial x_i v_j)^2.\]

In (1.1) the effect of the dissipative heating is controlled by the parameter \(\zeta\). If one sets \(\zeta = 0\) in (1.1), one formally obtains the usual Oberbeck-Boussinesq equations.

The boundary conditions on \(\{x_3 = 0, 1\}\) are given by

\[v = 0 \text{ and } \theta = 0 \text{ on } \{x_3 = 0, 1\}.
\]

We require \(\{v, p, \theta\}\) to be \(\frac{2\pi}{l_j}\)-periodic in \(x_j\)-direction for given \(l_j > 0\) (\(j = 1, 2\)).

As a first step of the mathematical analysis of (1.1), we consider the stability of the motionless state. As is well known, in the usual Oberbeck-Boussinesq case \((\zeta = 0)\), there exists a critical Rayleigh number \(\lambda_c^2\) (depending on \(l_1\) and \(l_2\)) such that if \(\lambda < \lambda_c\), then the motionless state is unconditionally stable, while if \(\lambda > \lambda_c\), then the motionless state is unstable ([2, 4, 8, 9]). We will see, in section 2, that in case \(\zeta > 0\) but small the motionless state is still stable even when \(\lambda\) is slightly beyond \(\lambda_c\) ([3]).

In section 3 we consider the bifurcation problem. In case \(\zeta = 0\) it is known that various types of stationary solutions with cellular patterns bifurcate from the critical value \(\lambda_c\) supercritically. (See [4] and references therein.) We will consider stationary problem of (1.1) under the slip boundary conditions for \(v\) on \(\{x_3 = 0, 1\}\) and show that some transcritical bifurcation branches exist when \(\zeta > 0\), in particular, solutions of hexagonal patterns bifurcate transcritically. This is in contrast to the usual Oberbeck-Boussinesq case \((\zeta = 0)\) where only supercritical bifurcations can occur.

2. Stability of the motionless state

We investigate the stability of the motionless state in the Rayleigh-Bénard convection, i.e., the stability of the trivial solution of (1.1). We consider the initial boundary value problem for (1.1) under the boundary conditions described above and initial condition

\[v|_{t=0} = v_0, \quad \theta|_{t=0} = \theta_0.\]
Notation: We set $\Omega = \mathbb{T}_{l_1, l_2} \times (0, 1)$, $\mathbb{T}_{l_1, l_2} = \mathbb{R}^2 / (\frac{2\pi}{l_1} \mathbb{Z} \times \frac{2\pi}{l_2} \mathbb{Z})$; $(\cdot, \cdot)$ denotes the scalar product of $L^2(\Omega)$; $H^m(\Omega)$ denotes the $m$-th order $L^2$-Sobolev space on $\Omega$.

In the case of the Oberbeck-Boussinesq equation ($\zeta = 0$) the stability of the motionless state is known to be controlled by the critical Rayleigh number $\lambda_c > 0$ which is given by

$$\lambda_c(\zeta) = \sup \left\{ \frac{2(\mathbf{v} \cdot \mathbf{e}_3, \theta)}{||\nabla \mathbf{v}||^2 + ||\nabla \theta||^2}; \{\mathbf{v}, \theta\} \in H^1_0(\Omega)^4 - \{0\}, \text{div} \mathbf{v} = 0 \right\}.$$  

The motionless state is unconditionally stable if $\lambda < \lambda_c$ while unstable if $\lambda > \lambda_c$ [2, 4, 8, 9].

In case $\zeta > 0$ the motionless state is (conditionally) asymptotically stable even when $\lambda$ is slightly beyond $\lambda_c$ for sufficiently small $\zeta > 0$, namely, we have the following

Theorem 2.1 ([3]). (i) For each $\{\mathbf{v}_0, \theta_0\} \in H^1_0(\Omega)^3 \times L^2(\Omega)$ with $\text{div} \mathbf{v}_0 = 0$ there exist $T > 0$ and a unique solution $\{\mathbf{v}(t), \theta(t)\}$ of (1.1) on $[0, T]$ in the class

$$\mathbf{v} \in C([0, T]; (H^1_0)^3) \cap L^2(0, T; (H^2)^3), \quad \theta \in C([0, T]; L^2) \cap L^2(0, T; H^1_0).$$

(ii) There exist $\zeta_0 > 0$ and $\lambda_0(\zeta) < \lambda_c(\zeta)$ such that if $0 \leq \zeta \leq \zeta_0$ and $\lambda < \lambda_0(\zeta)$, then the motionless state is asymptotically stable, namely, there exists $\delta > 0$ such that for each $\{\mathbf{v}_0, \theta_0\} \in H^1_0(\Omega)^3 \times L^2(\Omega)$ with $\text{div} \mathbf{v}_0 = 0$ and $||\mathbf{v}_0||_{H^1} + ||\theta_0||_{L^2} < \delta$, the solution $\{\mathbf{v}(t), \theta(t)\}$ exists on $[0, \infty)$ and satisfies

$$||\mathbf{v}(t)||_{H^1} + ||\theta(t)||_{L^2} \leq Ce^{-\gamma t}(||\mathbf{v}_0||_{H^1} + ||\theta_0||_{L^2})$$

for some constants $C, \gamma > 0$. If $\lambda > \lambda_0(\zeta)$, then the motionless state is unstable.

The number $\lambda_c(\zeta)$ satisfies

$$\lambda_c(0) = \lambda_c \quad \text{and} \quad \lambda_c(\zeta) > \lambda_c \quad \text{for} \quad 0 < \zeta \leq \zeta_0.$$


To prove the assertion (ii) we consider the eigenvalue problem linearized at the motionless state:

$$-\sigma \mathbf{u} + \mathcal{L} \mathbf{u} = 0,$$
where \( \mathbf{u} = \{ \mathbf{v}, \theta \} \),

\[
\mathcal{L} \mathbf{u} = \mathcal{L}(\lambda, \zeta) \mathbf{u} \equiv \left( -\frac{1}{\Pr} \Delta \theta + \frac{\lambda}{\Pr} (\zeta(\Theta - x_3) - 1)v_3 \right),
\]

\( A \) is the Stokes operator \(-P\Delta\), \( P \) is the orthogonal projector from \( L^2(\Omega)^3 \) to \( H \) and \( H \) is the \( L^2 \)-closure of the set of all smooth solenoidal vector fields in \( \Omega \) vanishing near \( \{x_3 = 0, 1\} \).

Since \( \mathcal{L} \) has compact resolvent, the spectrum \( \sigma(\mathcal{L}) \) of \( \mathcal{L} \) consists of discrete eigenvalues \( \{\sigma_n\}_{n \geq 1} \) with \( \text{Re}\sigma_1 \leq \text{Re}\sigma_2 \leq \cdots \leq \text{Re}\sigma_n \leq \cdots \rightarrow +\infty \).

The principle of linearized stability implies that the motionless stable is stable if \( \text{Re}\sigma_1 > 0 \) while unstable if \( \text{Re}\sigma_1 < 0 \). Therefore the assertion (ii) follows from the next proposition.

We denote the eigenvalues \( \sigma_j \) of \( \mathcal{L} \) by \( \sigma_j(\lambda, \zeta) \).

**Proposition 2.2.** There exist \( \zeta_0 > 0 \) and \( \lambda_c(\zeta) \geq \lambda_c \) such that if \( 0 \leq \zeta \leq \zeta_0 \) and \( \lambda < \lambda_c(\zeta) \), then \( \sigma_1(\lambda, \zeta) > 0 \). Moreover, if \( 0 \leq \zeta \leq \zeta_0 \) and \( \lambda > \lambda_c(\zeta) \), then \( \sigma_1(\lambda, \zeta) < 0 \). Here the number \( \lambda_c(\zeta) \) satisfies

\[
\lambda_c(0) = \lambda_c \quad \text{and} \quad \lambda_c(\zeta) > \lambda_c \quad \text{for} \quad 0 < \zeta \leq \zeta_0.
\]

**Proof.** We consider the eigenvalue problem (2.2):

\[
-\sigma \mathbf{u} + \mathcal{L}(\lambda, \zeta) \mathbf{u} = 0,
\]

\[
\mathcal{L}(\lambda, \zeta) \mathbf{u} \equiv \left( -\frac{1}{\Pr} \Delta \theta + \frac{\lambda}{\Pr} (\zeta(\Theta - x_3) - 1)v_3 \right).
\]

It is known that in case \( \zeta = 0 \), all eigenvalues \( \{\sigma_n(\lambda) \equiv \sigma_n(\lambda, 0)\}_{n \geq 1} \) are real, the smallest eigenvalue has even multiplicity, say \( 2m \) (\( m \in \mathbb{N} \)), and

\[
\sigma_0(\lambda) \equiv \sigma_1(\lambda) = \cdots = \sigma_{2m}(\lambda) < \sigma_{2m+1}(\lambda) \leq \cdots \leq \sigma_n(\lambda) \leq \cdots \rightarrow +\infty.
\]

Furthermore,

(2.3) \( (\mathcal{L}(\lambda, 0) \mathbf{u}, \mathbf{u}) \geq \sigma_0(\lambda)\| \mathbf{u} \|^2 \)

for \( \mathbf{u} \in D(A) \times (H^2(\Omega) \cap H_0^1(\Omega)) \), where \( D(A) \) denotes the domain of \( A \).

Here and in the following we denote the scalar product of \( H \times L^2(\Omega) \) by \( (\cdot, \cdot) \) which is defined as, for \( \mathbf{u}_j = \{ \mathbf{v}_j, \theta_j \} \in H \times L^2(\Omega) \) (\( j = 1, 2 \)),

\[
(\mathbf{u}_1, \mathbf{u}_2) \equiv (\mathbf{v}_1, \mathbf{v}_2)_{L^2(\Omega)} + \Pr (\theta_1, \theta_2)_{L^2(\Omega)} \quad \text{and} \quad \| \mathbf{u} \| \equiv \sqrt{(\mathbf{u}, \mathbf{u})}.
\]
There also holds that $\sigma_0(\lambda) > 0$ (resp. $\sigma_0(\lambda) < 0$) if and only if $\lambda < \lambda_0 \equiv \lambda_c$ (resp. $\lambda > \lambda_0$) while $\sigma_0(\lambda) = 0$ if and only if $\lambda = \lambda_0$, and there exists $\gamma_0 = \gamma_0(l_1, l_2, Pr) > 0$ such that if $j \geq 2m + 1$ and $\lambda \leq \lambda_0$ then $\sigma_j(\lambda) \geq \gamma_0$. If $1 \leq j \leq 2m$ each $\sigma_j(\lambda)$ is continuous in $\lambda$. In particular, for any $\varepsilon > 0$ there exists $\delta(\varepsilon) > 0$ such that if $\lambda < \lambda_0 - \varepsilon$, then $\sigma_0(\lambda) \geq \delta(\varepsilon)$. ([4, 8, 9].)

We now consider the case $0 < \zeta \ll 1$. We write (2.2) as

$$-\sigma u + L_0 u + (\lambda - \lambda_0) M_1 u + \zeta M_2 u + M_3(\lambda, \zeta) u = 0,$$

where

$$L_0 = L(\lambda_0, 0), \quad M_1 u = \begin{pmatrix} -P(\theta b) \\ -\frac{1}{Pr} v_3 \end{pmatrix}, \quad M_2 u = \begin{pmatrix} 0 \\ \frac{\lambda_0}{Pr} (\Theta - x_3)v_3 \end{pmatrix}$$

and

$$M_3(\lambda, \zeta) u = \begin{pmatrix} 0 \\ \frac{\lambda - \lambda_0}{Pr} \zeta (\Theta - x_3)v_3 \end{pmatrix}.$$ 

We first consider the case $\lambda < \lambda_0 - \varepsilon$ for some $\varepsilon > 0$.

**Proposition 2.3.** For any $\varepsilon > 0$ there exists $\delta(\varepsilon) > 0$ and $\zeta_1(\varepsilon) > 0$ with

$$\sigma(L(\lambda, \zeta)) \subset \{\sigma; \text{Re}\sigma \geq \delta(\varepsilon)/2\}$$

if $\lambda < \lambda_0 - \varepsilon$ and $0 \leq \zeta \leq \zeta_1(\varepsilon)$.

**Proof.** Since $\|(\Theta - x_3)v_3\|_2 \leq C\|u\|$, we see from (2.3) that

$$\text{Re}(L(\lambda, \zeta) u, u) = (L(\lambda, 0) u, u) + \text{Re}(\zeta M_2 u, u) + \text{Re}(M_3(\lambda, \zeta) u, u)$$

$$\geq (\sigma_0 - C\lambda_0 \zeta)\|u\|^2.$$

Now recall that for any $\varepsilon > 0$ there exists $\delta = \delta(\varepsilon) > 0$ such that if $\lambda < \lambda_0 - \varepsilon$, then $\sigma_0 \geq \delta(\varepsilon)$. Thus, if $\zeta \leq \frac{\delta(e)}{2C\lambda_0}$ and $\lambda < \lambda_0 - \varepsilon$, then

$$\text{Re}(L(\lambda, \zeta) u, u) \geq \delta(\varepsilon)\frac{1}{2}\|u\|^2,$$

which implies that $\sigma(L(\lambda, \zeta)) \subset \{\sigma; \text{Re}\sigma \geq \frac{1}{2} \delta(\varepsilon)\}$ for $\lambda < \lambda_0 - \varepsilon$ and $0 \leq \zeta \leq \frac{\delta(e)}{2C\lambda_0}$. This shows Proposition 2.3.

We next investigate $L(\lambda, \zeta)$ for $|\lambda - \lambda_0| \leq \varepsilon \ll 1$ and $0 < \zeta \ll 1$. 

Proposition 2.4. (i) There exist $\epsilon_2 > 0$ and $\zeta_2 > 0$ such that

$$\sigma(\mathcal{L}(\lambda, \zeta)) \subset \{\sigma; |\sigma| \leq \frac{1}{4}\gamma_0\} \cup \{\sigma; \text{Re}\sigma \geq \frac{3}{4}\gamma_0\}$$

if $|\lambda - \lambda_0| \leq \epsilon_2$ and $0 \leq \zeta \leq \zeta_2$.

(ii) There exist $0 < \epsilon_3 \leq \epsilon_2$ and $0 < \zeta_3 \leq \zeta_2$ such that the eigenvalues of $\mathcal{L}(\lambda, \zeta)$ in $\{\sigma; |\sigma| \leq \frac{1}{4}\gamma_0\}$ have the form

$$\sigma = \sigma^{(1,0)}(\lambda - \lambda_0) + \sigma^{(0,1)}\zeta + O(|\lambda - \lambda_0|^2 + \zeta^2)$$

with constants $\sigma^{(1,0)} < 0$ and $\sigma^{(0,1)} > 0$, if $|\lambda - \lambda_0| \leq \epsilon_3$ and $0 \leq \zeta \leq \zeta_3$.

Moreover, there exists $\lambda_c = \lambda_c(\zeta) > 0$ satisfying

$$\lambda_c(0) = \lambda_0 \quad \text{and} \quad \lambda_c(\zeta) > \lambda_0 \quad \text{for} \quad 0 < \zeta \leq \zeta_3$$

and it holds $\sigma_1(\lambda, \zeta) > 0$ if $\lambda < \lambda_c(\zeta)$ and $\sigma_1(\lambda, \zeta) < 0$ if $\lambda > \lambda_c(\zeta)$, provided that $|\lambda - \lambda_0| < \epsilon_3$ and $0 \leq \zeta \leq \zeta_3$.

Proof. We first observe

$$\|\mathcal{M}_j\mathrm{u}\|_2 \leq C\|\mathrm{u}\| \quad (j = 1, 2, 3).$$

Since $\mathcal{L}_0$ is self-adjoint, we obtain for some constant $a = a(\lambda_0, \hat{\Theta}, \text{Pr}) > 0$,

$$\|((\lambda - \lambda_0)\mathcal{M}_1 + \zeta\mathcal{M}_2 + \mathcal{M}_3(\lambda, \zeta))(-\mu + \mathcal{L}_0)^{-1}\mathrm{u}\| \leq \frac{1}{2}\|\mathrm{u}\|$$

provided that $\mu \in \Sigma \equiv \{\sigma; |\sigma| > \frac{1}{4}\gamma_0\} \cap \{\sigma; \text{Re}\sigma < \frac{3}{4}\gamma_0\}, |\lambda - \lambda_0| \leq \epsilon_2$ and $0 \leq \zeta \leq \zeta_2$ for some small $\epsilon_2 > 0$ and $\zeta_2 > 0$. This inequality immediately implies that $\Sigma$ is included in the resolvent set of $\mathcal{L}(\lambda, \zeta)$ and (2.4) follows.

To prove (2.5) we note that the problem (2.2) is equivalent to

$$\begin{align*}
-\sigma \mathbf{v} - \Delta \mathbf{v} - \lambda \theta \mathbf{b} + \nabla p &= 0, \\
-\sigma \theta - \frac{1}{\text{Pr}} \Delta \theta + \frac{1}{\text{Pr}} (\zeta(\Theta - x_3) - 1)v_3 &= 0, \\
\text{div} \mathbf{v} &= 0
\end{align*}$$

with boundary conditions under consideration.

To solve (2.6) we expand $\mathbf{v}$, $\theta$ and $\nabla p$ into Fourier series in $x_1$ and $x_2$, and so we assume $\mathbf{v}$, $\theta$ and $\nabla p$ to have the form $e^{2\pi i i \frac{x_1}{l_1} + \frac{x_2}{l_2}} h(x_3)$, where
$(k_1, k_2) \in \Z^2$. We first consider the case $(k_1, k_2) = (0, 0)$, namely, $v_j = v_j(x_3)$ $(j = 1, 2, 3)$, $\theta = \theta(x_3)$. Due to $\text{div} \, v = 0$ we have $\frac{d}{dx_3} v_3 = 0$. This, together with $v = 0$ on $\{x_3 = 0, 1\}$, yields $v_3 \equiv 0$. We then obtain

$$\begin{cases}
-\sigma \|v_j\|_{L^2(0,1)}^2 + \|\frac{d}{dx_3} v_j\|_{L^2(0,1)}^2 &= 0 \quad (j = 1, 2), \\
-\sigma \|\theta\|_{L^2(0,1)}^2 + \frac{1}{\Pr} \|\frac{d}{dx_3} \theta\|_{L^2(0,1)}^2 &= 0.
\end{cases}$$

This implies that

$$\sigma \geq a \pi^2 = a \inf \left\{ \frac{\|\frac{d}{dx_3} h\|_{L^2(0,1)}^2}{\|h\|_{L^2(0,1)}^2} ; h \in H_0^1(0,1), \ h \neq 0 \right\},$$

where $a = \min(1, \Pr^{-1})$. Therefore, we see that $\sigma \in \{\sigma ; \Re \sigma \geq \frac{3}{4} \gamma_0\}$.

We next consider $(k_1, k_2) \neq (0, 0)$. This is the case where there really occurs $\sigma \in \{\sigma ; |\sigma| \leq \frac{1}{4} \gamma_0\}$. Taking curl curl of (2.6)$_1$, we obtain

*(2.7)*

$$\begin{cases}
\sigma \Delta v_3 + \Delta^2 v_3 + \lambda \Delta_2 \theta = 0, \\
-\sigma \theta - \frac{1}{\Pr} \Delta \theta + \frac{\lambda}{\Pr} (\zeta(\Theta - x_3) - 1) v_3 = 0
\end{cases}$$

with boundary conditions $v_3 = \partial_3 v_3 = \theta = 0$ at $x_3 = 0, 1$ and the periodic boundary conditions in $x_1$ and $x_2$. Here $\Delta_2 = \partial_{11} + \partial_{22}$.

We now substitute $v_3 = e^{2\pi i (\frac{k_1 x_1}{l_1} + \frac{k_2 x_2}{l_2})} f(x_3), \ \theta = e^{2\pi i (\frac{k_1 x_1}{l_1} + \frac{k_2 x_2}{l_2})} g(x_3)$ for $(k_1, k_2) \neq (0, 0)$ into (2.7). Then we find the eigenvalue problem:

*(2.8)*

$$\begin{cases}
-\sigma D_\omega f + D_\omega^2 f - \lambda \omega^2 g = 0 \\
-\sigma g + \frac{1}{\Pr} D_\omega g + \frac{\lambda}{\Pr} (\zeta(\Theta - x_3) - 1) f = 0 \quad (0 < x_3 < 1), \\
f = \frac{d}{dx_3} f = g = 0 \quad (x_3 = 0, 1),
\end{cases}$$

where $\omega^2 \equiv (\frac{2\pi k_1}{l_1})^2 + (\frac{2\pi k_2}{l_2})^2 > 0$, $D_\omega \equiv (-\frac{d^2}{dx_3^2} + \omega^2)$ and $D_\omega^2 \equiv (\frac{d^2}{dx_3^2} - \omega^2)^2$.

It is easily verified that the eigenvalues and eigenfunctions of (2.6) for $(k_1, k_2) \neq (0, 0)$ can be obtained from those of (2.8) with suitable $\omega^2 > 0$ and vice versa, since $\omega^2 > 0$. We write (2.8) as

*(2.9)*

$$-\sigma M f + L(\lambda, \zeta)f = 0, \quad f = \{f, g\}.$$
Here
\[
M \equiv \begin{pmatrix} D_\omega^2 & 0 \\ 0 & 1 \end{pmatrix}, \quad L(\lambda, \zeta) \equiv \begin{pmatrix} \frac{1}{\partial \zeta} (\zeta (\Theta - x_3) - 1) & -\lambda \omega^2 \\ \frac{1}{\partial \zeta} D_\omega \end{pmatrix}
\]
and the operators \( D_\omega \) and \( D_\omega^2 \) are defined as above for \( g \in H^2(0,1) \cap H_0^1(0,1) \) and \( f \in \{ f \in H^4(0,1); f = \frac{d}{dx_3} f = 0 \text{ at } x_3 = 0, 1 \} \), respectively.

The eigenvalues \( \sigma_j(\lambda_0) \) of \( L_0 \) are given by the eigenvalues of the eigenvalue problem (2.9) with \( \lambda = \lambda_0 \) and \( \zeta = 0 \), and moreover, the eigenvalues of \( L(\lambda, \zeta) \) in \( \{ \sigma; |\sigma| \leq \frac{1}{4} \gamma_0 \} \) are given by those of \( L(\lambda_0, 0) \) in \( \{ \sigma; |\sigma| \leq \frac{1}{4} \gamma_0 \} \). In particular, \( \sigma(L(0,0)) \cap \{ \sigma; |\sigma| \leq \frac{1}{4} \gamma_0 \} = \{ \sigma_0(\lambda_0) = 0 \} \).

The following lemma summarizes the results in [6, page 38].

**Lemma 2.5.** (i) The eigenvalue \( \sigma_0(\lambda_0) = 0 \) of \( L^{(0,0)}(\lambda_0, 0) \equiv L(0,0) \) is simple.

(ii) One can choose an eigenfunction \( \mathbf{f}_0 = \{ f_0, g_0 \} \) of \( L^{(0,0)} \) associated with \( \sigma_0(\lambda_0) = 0 \) in such a way that \( f_0(x_3) > 0 \) and \( g_0(x_3) > 0 \) for \( 0 < x_3 < 1 \).

Since \( \sigma_0(\lambda_0) \) is simple by Lemma 2.5 (i), there exists only one eigenvalue \( \sigma = \sigma(\lambda, \zeta) \) of \( L(\lambda, \zeta) \) in \( \{ \sigma; |\sigma| \leq \frac{1}{4} \gamma_0 \} \) when \( |\lambda - \lambda_0| \) and \( \zeta \) are sufficiently small. Furthermore, due to the simplicity of \( \sigma_0(\lambda_0) \), one can see that \( \sigma(\lambda, \zeta) \) is analytic in \( \lambda \) and \( \zeta \) near \( \lambda = \lambda_0 \) and \( \zeta = 0 \) and it is expanded as

\[
\sigma(\lambda, \zeta) = \sum_{j,k \geq 0}^{\infty} \sigma^{(j,k)}(\lambda - \lambda_0)^j \zeta^k \quad \text{with } \sigma^{(0,0)} = \sigma(\lambda_0) = 0.
\]

We denote by \( \mathbf{f}(\lambda, \zeta) \) the eigenfunction associated with \( \sigma(\lambda, \zeta) \) satisfying \( \mathbf{f}(0,0) = \mathbf{f}_0 \). Then

\[
\mathbf{f}(\lambda, \zeta) = \sum_{j,k \geq 0}^{\infty} (\lambda - \lambda_0)^j \zeta^k \mathbf{f}^{(j,k)}
\]
with \( \mathbf{f}^{(0,0)} = \mathbf{f}_0 \). Substituting (2.10) and (2.11) into (2.9) we obtain

\[
L^{(0,0)} \mathbf{f}_0 = 0,
\]

(2.12) \[-\sigma^{(1,0)} M \mathbf{f}_0 + L^{(0,0)} \mathbf{f}^{(1,0)} + L^{(1,0)} \mathbf{f}_0 = 0,\]

(2.13) \[-\sigma^{(0,1)} M \mathbf{f}_0 + L^{(0,0)} \mathbf{f}^{(0,1)} + L^{(0,1)} \mathbf{f}_0 = 0.\]
and so on. Here $L(\lambda, \zeta) = \sum_{0 \leq j,k \leq 1} (\lambda - \lambda_0)^j \zeta^k L(j,k)$ with $L^{(0,0)} = L(0,0)$,

$$L^{(1,0)f} = \left\{ -\omega^2 g, -\frac{1}{Pr} f \right\}, \quad L^{(0,1)f} = \left\{ 0, \frac{\lambda_0}{Pr} (\hat{\Theta} - x_3) f \right\}$$

and $L^{(1,1)} = \lambda_0^{-1} L^{(0,0)}$. To compute $\sigma^{(i,k)}$ we define $\langle \cdot , \cdot \rangle$ by

$$\langle f_1, f_2 \rangle = \frac{1}{\omega^2} \int_0^1 f_1(x_3) \overline{f_2(x_3)} dx_3 + Pr \int_0^1 g_1(x_3) \overline{g_2(x_3)} dx_3$$

for $f_j = \{ f_j, g_j \} \in L^2(0,1)^2 \ (j = 1, 2)$. Here $\overline{f}$ denotes the complex conjugate of $f$. Note that $\langle L^{(0,0)} f_1, f_2 \rangle = \langle f_1, L^{(0,0)} f_2 \rangle$ and $\langle M f, f \rangle > 0$ for $f \neq 0$.

Taking $\langle \cdot, \cdot \rangle$ of (2.12) and (2.13) with $f_0$ respectively, we obtain

$$\langle L^{(1,0)} f_0, f_0 \rangle = \int_0^1 \lambda_0 (\Theta - x_3) f_0(x_3) g_0(x_3) dx_3 > 0.$$

Thus, $\sigma^{(0,1)} > 0$, and we have obtained (2.5). Now we define $\lambda_0(\zeta)$ by $\sigma(\lambda_0(\zeta), \zeta) = 0$. We then have

$$\lambda_0(\zeta) = \lambda_0 - \frac{\sigma^{(0,1)}}{\sigma^{(1,0)}} \zeta + O(\zeta^2). \quad (2.14)$$

Since $\lambda_0(\zeta)$ also depends on $\omega^2$, we denote $\lambda_0(\zeta)$ by $\lambda_0(\zeta; \omega^2)$. Then the critical number $\lambda_c(\zeta)$ is given by

$$\lambda_c(\zeta) = \inf_{(k_1, k_2) \in \mathbb{Z}^2 \setminus (0,0)} \lambda_0 \left( \zeta; \left( \frac{2\pi k_1}{l_1} \right)^2 + \left( \frac{2\pi k_2}{l_2} \right)^2 \right). \quad (2.15)$$

This completes the proof.

Proposition 2.2 now follows from Propositions 2.3 and 2.4 by taking $\epsilon = \varepsilon_3$ in Proposition 2.3 and $\zeta_0 = \min \{ \zeta_1(\varepsilon_3), \zeta_3 \}$. This completes the proof of Proposition 2.2.
3. Remarks on bifurcation problem

In this section we consider bifurcation problem for (1.1). Due to a technical reason, we here consider (1.1) under the slip boundary conditions for \( v \) on \( \{ x_3 = 0, 1 \} \) instead of the no-slip boundary conditions, i.e., we consider

\[
\partial_{x_3} v_1 = \partial_{x_3} v_2 = v_3 = 0 \quad \text{on} \quad \{ x_3 = 0, 1 \}.
\]

The boundary conditions for \( \theta \) are the same as in sections 1 and 2, and we also impose the same periodic boundary conditions in \( x_1 \) and \( x_2 \)-variables as in sections 1 and 2. We also require

\[
\int_{\Omega} v_1(x) dx = \int_{\Omega} v_2(x) dx = 0.
\]

Under these boundary conditions one can also obtain similar critical numbers \( \lambda_c(\zeta) \) for the stability of the motionless state. In case \( \zeta = 0 \) it is known that nontrivial solution branches of various cellular patterns such as rolls and hexagones bifurcate at \( \lambda_c \). (See [4] and references therein.) Due to the unconditional stability of the motionless state, only supercritical bifurcations can occur when \( \zeta = 0 \).

We will show that, in contrast to the case of \( \zeta = 0 \), some transcritical bifurcation branches exist when \( \zeta > 0 \). In particular, hexagonal solutions bifurcate at \( \lambda_c(\zeta) \) transcritically when \( \zeta > 0 \).

Notation. In this section we denote the spatial variable \( x \) and the fluid velocity \( v \) by

\[
x = (x_1, x_2, x_3) = (x, y, z) \quad \text{and} \quad v = (v_1, v_2, v_3) = (u, v, w)
\]

respectively. We also write the periods \( l_1 \) and \( l_2 \) as

\[
l_1 = \frac{2\pi}{\alpha} \quad \text{and} \quad l_2 = \frac{2\pi}{\beta}.
\]

When \( \zeta = 0 \), the usual critical Rayleigh number \( \lambda_c^2 \) under the slip boundary conditions is given by a similar formula to (2.1). But in this case it has an explicit formula:

\[
\lambda_c^2 = \inf_{(k, m) \in \mathbb{Z}^2} \frac{(\omega_{k,m}^2 + \pi^2)^3}{\omega_{k,m}^2}, \quad \omega_{k,m}^2 = (\alpha k)^2 + (\beta m)^2.
\]
Note that $(\omega^2 + \pi^2)^3/\omega^4$ attains its minimum value at $\omega = \omega_c = \pi/\sqrt{2}$. By a similar argument in section 2, one can obtain the critical number $\lambda_c(\zeta)$ for sufficiently small $\zeta > 0$, which is given by an analogue of (2.15):

\[(3.1) \quad \lambda_c(\zeta) = \lambda_c(\zeta; \omega^2) = \inf_{(k,m) \in \mathbb{Z}^2} \lambda_0(\zeta; \omega_{k,m}^2),\]

where $\omega^2 = \alpha^2 + \beta^2$ and $\omega_{k,m}^2 = (\alpha k)^2 + (\beta m)^2$. Here the function $\lambda_0(\zeta; \omega^2)$ is given by an analogue of (2.14):

$$\lambda_0(\zeta; \omega^2) = \sqrt{\frac{(\omega^2 + \pi^2)^3}{\omega^4}} - \frac{\sigma^{(0,1)}}{\sigma^{(1,0)}} \zeta + O(\zeta^2).$$

$\lambda_c(\zeta) = \lambda_c(\zeta; \omega^2)$ attains its minimum in $\omega$ at $\omega_c(\zeta) = \omega_c + O(\zeta)$.

### 3.1 Two-dimensional case.

We first consider the two-dimensional problem; this means that the unknowns $v$, $\theta$ (and $p$) depend only on $x$ and $z$ but not on $y$, and $v(x, z) \equiv 0$.

In this case the critical number $\lambda_c(\zeta)$ in (3.1) may be written as

\[(3.2) \quad \lambda_c(\zeta) = \lambda_c(\zeta; \alpha^2) = \inf_{k \in \mathbb{Z}} \lambda_0(\zeta; (\alpha k)^2).\]

We now take $\alpha$ in such a way that the infimum in (3.2) is attained at both $k = 1$ and $k = 2$. (This really occurs. See [1, 8] for the case $\zeta = 0$.)

For this choice of $\alpha$ one sees that $\dim \ker \mathcal{L}_{\lambda_c(\zeta)} = 4$. We restrict ourselves to the subspace of functions which have the Fourier expansions of the form:

$$\begin{align*}
  u &= \sum_{k,n} u_{k,n} \sin \alpha kx \cos n\pi z, \\
  w &= \sum_{k,n} w_{k,n} \cos \alpha kx \sin n\pi z, \\
  \theta &= \sum_{k,n} \theta_{k,n} \cos \alpha kx \sin n\pi z.
\end{align*}$$

Then if $\mathcal{L}_{\lambda_c(\zeta)}$ is restricted on this space, we have $\dim \ker \mathcal{L}_{\lambda_c(\zeta)} = 2$, and $\ker \mathcal{L}_{\lambda_c(\zeta)} = \text{span}\{u_0^1, u_0^2\}$, where

$$u_0^j = \begin{pmatrix} 0 \\ w_j^j \\ \theta_j \\ \omega_j \\ \alpha \end{pmatrix} \cos \alpha jx \sin \pi z + O(\zeta) \quad (j = 1, 2)$$
with some constants $w^j$ and $\theta^j$.

We look for nontrivial stationary solutions for $\lambda$ near $\lambda_c(\zeta)$ by the Lyapunov-Schmidt method. To do so, we write $u$ as

$$u = A_1u_0^1 + A_2u_0^2 + \Phi, \quad A_j \in \mathbb{R}, \quad (\Phi, u_0^{j*}) = 0 \quad (j = 1, 2),$$

where $u_0^{j*}$ are functions in $\ker L_{\lambda_c(\zeta)}$ satisfying $(u_0^j, u_0^{k*}) = \delta_{j,k}$. The Lyapunov-Schmidt reduction then yields

$$(3.3) \left\{ \begin{array}{l} p_0(\lambda - \lambda_c(\zeta))A_1 + \zeta(p_1 + Pr p_2)A_1A_2 + O(|A|^3) = 0, \\ p_0(\lambda - \lambda_c(\zeta))A_2 + \zeta q_1 A_1^2 + O(|A|^3) = 0, \end{array} \right.$$ \noindent where $p_0 = O(1) < 0$, $p_1 = O(1) < 0$, $p_2 = O(1) > 0$ and $q_1 = O(1) > 0$ as $\zeta \to 0$.

From (3.3) we obtain the following

**Theorem 3.1.** (i) *(Usual roll solutions)* There exist nontrivial solution branches $\{\{A_1, 0\}, \lambda - \lambda_c(\zeta) = \mu_1 A_1^2\}$ and $\{\{0, A_2\}, \lambda - \lambda_c(\zeta) = \mu_2 A_2^2\}$, where $\mu_j$ are positive constants. The solutions $u_j$ corresponding to these branches have the forms:

$u_j = A_j u_0^j + O(|A_j|^2) \quad (j = 1, 2).$

These are the usual roll solutions.

(ii) *(Mixed solutions)* (a) *(Existence)* There exists $Pr_0 > 0$ such that if $Pr > Pr_0$, then there exist two nontrivial solution branches of the forms:

$$(a) \left\{ \begin{array}{l} A_1 = \varepsilon, \quad A_2 = \pm a_2 \varepsilon + O(|\varepsilon|^2), \\ \lambda - \lambda_c(\zeta) = \mp \mu_3 \varepsilon + O(|\varepsilon|^2), \end{array} \right.$$ \noindent where $a_2 = O(1) > 0$ and $\mu_3 = O(1) > 0$ as $\varepsilon \to 0$. (Fig. 1). The solutions $u_{(\pm)}$ corresponding to these branches have the forms

$u_{(\pm)} = \varepsilon(u_0^1 \pm a_2 u_0^2) + O(\varepsilon^2).$

(b) *(No existence)* If $Pr < Pr_0$, then there exist no small stationary solutions except for the trivial solution $u = 0$ and the usual roll solutions $u_j$ ($j = 1, 2$) obtained in (i).
Remark. In case $\zeta = 0$ the analysis of mixed solutions was given in details in [1].

3.2 Hexagonal solutions

We next consider the bifurcation problem of solutions of hexagonal patterns. To obtain hexagonal solutions we require $\beta = \sqrt{3}\alpha$ and also $2\alpha = \omega_c(\zeta)$. We restrict ourselves to the subspace of functions invariant under $\frac{2\pi}{3}$-rotation in $(x, y)$. We further require that $u$ has the Fourier expansions of the form:

$$
\begin{align*}
  u &= \sum_{k,m,n} u_{kmn} \sin \alpha k x \cos \sqrt{3} \alpha m y \cos n\pi z, \\
  v &= \sum_{k,m,n} v_{kmn} \cos \alpha k x \sin \sqrt{3} \alpha m y \cos n\pi z, \\
  w &= \sum_{k,m,n} w_{kmn} \cos \alpha k x \cos \sqrt{3} \alpha m y \sin n\pi z, \\
  \theta &= \sum_{k,m,n} \theta_{kmn} \cos \alpha k x \cos \sqrt{3} \alpha m y \sin n\pi z.
\end{align*}
$$

(3.4)

The requirement of $\frac{2\pi}{3}$-rotation invariance restricts the form of functions in (3.4), for example, $\theta$ has the form

$$
\begin{align*}
  \theta &= \sum_{k,m,n \atop k + m = \text{even}} \theta_{kmn} \{ \cos \alpha k x \cos \sqrt{3} \alpha m y \\
  &\quad \quad + \cos \{ \alpha (\frac{1}{2} k - \frac{3}{2} m) x \} \cos \{ \sqrt{3} \alpha (\frac{1}{2} k + \frac{1}{2} m) y \} \\
  &\quad \quad + \cos \{ \alpha (\frac{1}{2} k + \frac{3}{2} m) x \} \cos \{ \sqrt{3} \alpha (\frac{1}{2} k - \frac{1}{2} m) y \} \} \sin n\pi z.
\end{align*}
$$

(See [4, 5].)
In this space we have \( \dim \ker \mathcal{L}_{\lambda_c(\zeta)} = 1 \). We take a nontrivial vector \( u_0 \) from \( \ker \mathcal{L}_{\lambda_c(\zeta)} \), whose \( w \)-component \( w_0 \) has, say, the form

\[
w_0 = \{2 \cos \alpha x \cos \sqrt{3} \alpha y + \cos 2 \alpha x\} \sin \pi z + O(\zeta).
\]

Similarly as in secton 3.1 we look for nontrivial stationary solutions for \( \lambda \) near \( \lambda_c(\zeta) \) by the Lyapunov-Schmidt method. We write \( u \) as

\[ u = Au_0 + \Phi, \quad A \in \mathbb{R}, \quad (\Phi, u_0^*) = 0, \]

where \( u_0^* \) is a function in \( \ker \mathcal{L}_{\lambda_c(\zeta)} \) satisfying \( (u_0, u_0^*) = 1 \). The Lyapunov-Schmidt reduction then yields

\[
p_0(\lambda - \lambda_c(\zeta))A + \zeta p_1 A^2 + p_2 (A^3 + O(|A|^4) = 0,
\]

where \( p_0 = O(1) < 0 \), \( p_1 = p_1(\Pr) = O(1) \) and \( p_2 = O(1) > 0 \) as \( \zeta \to 0 \). Here \( p_1 = p_1(\Pr) \) changes signs at some \( \Pr = \Pr_1 \).

From (3.5) we obtain the following

**Theorem 3.2.** There exists \( \Pr_1 > 0 \) such that

(i) if \( \Pr \neq \Pr_1 \), then there exists a hexagonal solutions branch bifurcating at \( \lambda_c(\zeta) \) transcritically

and

(ii) if \( \Pr = \Pr_1 \), there exists a hexagonal solutions branch bifurcating at \( \lambda_c(\zeta) \) supercritically. (Fig. 2).
References


