Superconvergence and Nonsuperconvergence of the Shortley-Weller Approximation for Dirichlet Problems

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1. Introduction

Consider the Dirichlet problem

\[-\Delta u = f(x, y) \quad \text{in } \Omega, \quad (1.1)\]
\[u = g(x, y) \quad \text{on } \Gamma = \partial \Omega, \quad (1.2)\]

where \( \Omega \) is a bounded domain of \( \mathbb{R}^2 \) and \( f, g \) are given functions. We assume the (unique) existence of a solution \( u \) for (1.1)-(1.2).

It was recently shown by Yamamoto [8] and Matsunaga-Yamamoto [7] that the Shortley-Weller approximation applied to (1.1)-(1.2) had a superconvergence property and numerical examples illustrating this fact were also given there.

To state the result, we construct a net over \( \overline{\Omega} = \Omega \cup \Gamma \) by the grid points \( P_{ij} = (x_i, y_j) \) in \( \overline{\Omega} \) with the mesh size \( h \). The set of the grid points is denoted by \( \Omega_h \). We denote by \( \mathcal{P}_{\Gamma} \) the set of points \( P_{ij} \) such that at least one of \( (x_i \pm h, y_j), (x_i, y_j \pm h) \) does not belong to \( \Omega \) and put \( \mathcal{P}_0 = \Omega_h \backslash \mathcal{P}_{\Gamma} \). Furthermore, we denote by \( \Gamma_h \) the set of points of intersection of grid lines with \( \Gamma \) and \( \mathcal{S}_h(\kappa) \) by the set of points \( P_{ij} \in \Omega_h \) which satisfy \( \text{dist}(P_{ij}, \Gamma) \leq \kappa h \), where \( \kappa \) is a constant with \( \kappa > 1 \), which is arbitrary chosen independently of \( h \). We define the neighbors of \( P \in \Omega_h \)
to be four points in $\Omega_h \cup \Gamma_h$ which are adjacent to $P$ and on horizontal and vertical grid lines through $P$. As is shown in Figures 1.1 and 1.2, these points are denoted by $P_E, P_W, P_S, P_N$ and their distance to $P$ by $h_E, h_W, h_S, h_N$. Note that at least one of $P_E, P_W, P_S, P_N$ is on $\Gamma$ if and only if $P \in \mathcal{P}_\Gamma$ and that all of them are in $\Omega$ if and only if $P \in \mathcal{P}_0$, in which case we have $h_E = h_W = h_S = h_N = h$.

We denote by $U(P)$ an approximate solution at $P \in \Omega_h$. Then the Shortley-Weller (S-W) approximation $-\Delta_h^{(SW)}$ for $-\Delta$ at $P$ is defined by

$$-\Delta_h^{(SW)}U(P) = \left(\frac{2}{h_Eh_W} + \frac{2}{h_Sh_N}\right)U(P) - \frac{2}{h_E(h_E+h_W)}U(P_E) - \frac{2}{h_W(h_E+h_W)}U(P_W) - \frac{2}{h_S(h_S+h_N)}U(P_S) - \frac{2}{h_N(h_S+h_N)}U(P_N),$$

which includes the usual centered five point formula

$$-\Delta_h U(P) = \frac{1}{h^2}[4U(P) - U(P_E) - U(P_W) - U(P_S) - U(P_N)]$$

as a special case $h_E = h_W = h_S = h_N = h$. Hence, if $P \in \mathcal{P}_0$, then the S-W approximation means the centered five point approximation.

As is easily seen, if $u \in C^{3,1}(\overline{\Omega})$, then the local truncation error $\tau^{(SW)}(P) \equiv -[\Delta_h^{(SW)}u(P) - \Delta u(P)]$ of the S-W formula at $P$ is estimated by

$$|\tau^{(SW)}(P)| \leq \begin{cases} \frac{2h^2}{3} = O(h^2) & \text{if } h_E = h_W = h_S = h_N = h \\ \frac{2M_3}{3}h = O(h) & \text{otherwise,} \end{cases}$$
where \( L \) is a Lipschitz constant common to all third order derivatives \( \partial^3/\partial x^i\partial y^{3-i}, \) \( 0 \leq i \leq 3 \) and

\[
M_3 = \sup_{P \in \Omega} \left\{ \left| \frac{\partial^3 u(P)}{\partial x^i \partial y^{3-i}} \right| : i = 0, 1, 2, 3 \right\}.
\]

Then the following result holds for the S-W approximation.

**Theorem 1.1** (Superconvergence of the S-W approximation [8], [7])

Let \( \Omega \) be a bounded convex domain with a piecewise \( C^{2,\alpha} \) boundary. If \( u \in C^{l+2,\alpha}(\overline{\Omega}), \) \( l = 0 \) or \( 1, \) \( \alpha \in (0,1] \), then

\[
|u(P) - U(P)| \leq \begin{cases} 
O(h^{l+1+\alpha}) & P \in S_h(\kappa) \\
O(h^{l+\alpha}) & \text{otherwise.}
\end{cases}
\]

This implies that if \( u \in C^{3,1}(\overline{\Omega}) \), then we have

\[ u(P) - U(P) = O(h^3) \quad \text{at } P \in S_h(\kappa) \]

even if \( \tau^{(SW)}(P) = O(h) \) and \( u(P) - U(P) = O(h^2) \) at other grid points.

Theorem 1.1 is a refinement of the following result due to Bramble-Hubbard [1]:

**Theorem 1.2.** If \( u \in C^4(\overline{\Omega}), \) then

\[
|u(P) - U(P)| \leq \frac{M_4}{96} d^2 h^2 + \frac{2M_3}{3} h^3 = O(h^2) \quad \forall P \in \Omega_h,
\]

where

\[
M_4 = \sup_{P \in \Omega} \left\{ \left| \frac{\partial^4 u(P)}{\partial x^i \partial y^{4-i}} \right| : i = 0, 1, 2, 3, 4 \right\}
\]

and \( d \) denotes the diameter of the smallest circle containing \( \Omega \).

It is also known by Matsunaga’s numerical experiments [4] that even if \( u \in C^4(\overline{\Omega}), \) the Bramble and the Collatz approximations do not have the superconvergence property like Theorem 1.1, although both have \( O(h^2) \) accuracy at every \( P \in \Omega_h \).

Now, we are interested in the behavior of the S-W approximate solution for the case \( u \notin C^{l+2,\alpha}(\overline{\Omega}) \). Has the S-W approximation any superconvergence property for such a case? The purpose of this paper is to answer
this question: Three examples with $\Omega = (0, 1) \times (0, 1)$ are given in § 2, which show three kinds of different behavior: (i) nonsuperconvergence at any point of $\Omega_h$, (ii) superconvergence near a part of $\Gamma$ and (iii) superconvergence in a neighborhood of a point of $\Gamma$. Furthermore, in § 3, we shall give two theorems by which the above phenomena can be illustrated.

2. Numerical Examples

In this section, we give three examples in which the S-W approximations applied to (1.1)-(1.2) show different behaviors.

Example 2.1. Let $f$ and $g$ be chosen so that the function

$$u = \sqrt{x(1-x)} + \sqrt{y(1-y)}$$

is the solution of (1.1)-(1.2). Observe that $u \in C(\overline{\Omega}) \cap C^\infty(\Omega)$, but $u \notin H^1(\Omega)$. Then, as is shown in Table 2.1, we see

$$u(P) - U(P) = O(h^{1/2}) \quad \forall P \in \Omega_h \quad (2.1)$$

and nonsuperconvergence occurs at any point in $\Omega_h$.

<table>
<thead>
<tr>
<th>$\alpha = 0.4$</th>
<th>$\alpha = 0.5$</th>
<th>$\alpha = 0.6$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$h = 1.0e-01$</td>
<td>2.942673010e-01</td>
<td>3.704605813e-01</td>
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<tr>
<td>$h = 5.0e-02$</td>
<td>2.730515389e-01</td>
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<td>3.658538669e-01</td>
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<tr>
<td>$h = 5.0e-03$</td>
<td>2.151573948e-01</td>
<td>3.654763480e-01</td>
</tr>
</tbody>
</table>

The local truncation error $\tau(Q)$ approaches to infinity as

$$u^{(4)}(Q) = O(h^{1/2-4})$$

if $Q$ is close to the boundary $\Gamma$.
$Q$ approaches to $\Gamma$. The distribution of errors $|u(P) - U(P)|$ in the case $h = 1.0e-002$ is shown in Figure 2.1.

Example 2.2. Let $f$ and $g$ be chosen so that the function

$$u = \sqrt{x} + y$$

is the solution of (1.1)-(1.2). Then

$$|u(P) - U(P)| = \begin{cases} O(h^{3/2}) & \text{near } \{(1, y)|0 \leq y \leq 1\} \\ O(h^{1/2}) & \text{otherwise.} \end{cases}$$

(2.2)

The results are shown in Table 2.2 and Figure 2.2 for $h = 1.0e-002$.

| max $|u(P) - U(P)|/h^\alpha$ $(P \in \Omega_h)$ | $h = 1.0e-001$ | $h = 5.0e-002$ | $h = 2.5e-002$ | $h = 1.0e-002$ |
|-----------------------------------------------|----------------|----------------|----------------|----------------|
| $\alpha = 0.4$ | 2.179020589e-001 | 2.299787622e-001 | 2.271939190e-001 | 2.166254535e-001 |
| $\alpha = 0.5$ | 2.743224392e-001 | 3.103063992e-001 | 3.285509309e-001 | 3.433282065e-001 |
| $\alpha = 0.6$ | 3.453514897e-001 | 4.186911020e-001 | 4.751258955e-001 | 5.441385373e-001 |

Table 2.2

In this case, a superconvergence occurs near the side $x = 1$ of $\Gamma$. 
Example 2.3. Let $f$ and $g$ be chosen so that the function $u = \sqrt{x} + \sqrt{y}$ is the solution of (1.1)-(1.2). Then

$$|u(P) - U(P)| = \begin{cases} O(h^{3/2}) & \text{near the corner (1,1),} \\ O(h^{1/2}) & \text{otherwise.} \end{cases} \quad (2.3)$$

The superconvergence occurs only near the corner (1,1). (See Figure 2.3 for the case $h = 1.0 \times 002$).

In the above examples, observe that the S-W approximation works well, although

$$\max_{P \in \Omega_h} |\tau^{(SW)}(P)| \to +\infty \quad \text{as } h \to 0.$$  

This is a nice feature of the finite difference method.

3. Convergence Theorems

It is possible to give mathematical proofs for the error estimates (2.1)-(2.3). We can first prove the following results for the two-point boundary value problem

$$-u''(x) = \varphi(x), \quad 0 < x < 1 \quad (3.1)$$
$$u(0) = \alpha, \quad u(1) = \beta, \quad (3.2)$$

where $\varphi$ is a given function and $\alpha, \beta$ are given constants.

**Theorem 3.1.** Let $d(x) = \min(x, 1-x), \quad 0 < x < 1$. If $0 < p < 1$, and the solution $u(x)$ of (3.1)-(3.2) belongs to $C^4(0,1)$ and satisfies

$$\sup_{x \in (0,1)} \frac{d(x)^k |u^{(k)}(x)|}{d(x)^p} < \infty, \quad k = 0, 1, 2, 3, 4,$$

then

$$|u_i - U_i| = O(h^p) \quad \forall i,$$

where $\{U_i\}$ is the finite difference solution for (3.1)-(3.2) and $u_i = u(x_i)$, $x_i = ih, \quad i = 0, 1, 2, \ldots, n + 1, \quad h = 1/(n + 1)$. That is, superconvergence does not occur at any $x_i \in \Omega_h$. 

Theorem 3.2. If the solution $u(x)$ of (3.1)-(3.2) satisfies

$$\sup_{x\in(0,1)} \frac{x^k|u^{(k)}(x)|}{x^p} < \infty, \quad k = 0, 1, 2, 3, 4 \tag{3.3}$$

with some constant $p \in (0, 1)$, then

$$|u_i - U_i| \leq \begin{cases} O(h^{p+1}) & \text{near } x = 1 \\ O(h^p) & \text{otherwise.} \end{cases}$$

That is, superconvergence occurs near $x = 1$.

Theorems 3.1 and 3.2 can be derived with the use of the fact (e.g. Yamamoto-Ikebe [9]) that the inverse of the $n \times n$ tridiagonal matrix

$$A = \begin{pmatrix} 2 & -1 & & & \\ -1 & 2 & -1 & & \\ & \ddots & \ddots & \ddots & \\ & & -1 & 2 & -1 \\ & & & -1 & 2 \end{pmatrix}$$

is given by

$$A^{-1} = (\alpha_{ij}), \quad \alpha_{ij} = \begin{cases} i(1 - \frac{j}{n+1}) & (i \leq j) \\ j(1 - \frac{i}{n+1}) & (i > j) \end{cases}$$

so that

$$h\alpha_{ij} = \begin{cases} x_i(1 - x_j) & (i \leq j) \\ x_j(1 - x_i) & (i > j). \end{cases}$$

Now, consider the Dirichlet problem

$$-\Delta u = F_1(x) + F_2(y) \quad \text{in } \Omega = (0, 1) \times (0, 1), \tag{3.4}$$

$$u = G_1(x) + G_2(y) \quad \text{on } \Gamma \tag{3.5}$$

and $\{U_{ij}\}$ be the S-W approximation with the equal mesh size $h_E = h_W = h_S = h_N = h$ at every $P \in \Omega_h$. Let $\{U_i^{(1)}\}$ and $\{U_i^{(2)}\}$ be the usual finite difference solution for the two-point boundary value problems

$$-u''(x) = F_1(x), \quad 0 < x < 1,$$

$$u(0) = G_1(0), \ u(1) = G_1(1).$$
and
\[-u''(y) = F_2(y), \quad 0 < y < 1\]
\[u(0) = G_2(0), \quad u(1) = G_2(1),\]
respectively. Then, by the uniqueness of the S-W approximate solution applied to (3.4)-(3.5), we have
\[U_{ij} = U^{(1)}_i + U^{(2)}_j, \quad \forall i, j.\]
Hence, all the phenomena stated in § 2 can now be illustrated with the use of Theorems 3.1 and 3.2 with \(p = 1/2\).

Note: Proofs of Theorems 3.1 and 3.2 will be given elsewhere.

References


