Title
Superconvergence and Nonsuperconvergence of the Shortley-Weller Approximation for Dirichlet Problems (Self-validating numerical methods and related topics)

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1. Introduction

Consider the Dirichlet problem

\[-\Delta u = f(x, y) \quad \text{in } \Omega, \tag{1.1}\]
\[u = g(x, y) \quad \text{on } \Gamma = \partial \Omega, \tag{1.2}\]

where $\Omega$ is a bounded domain of $\mathbb{R}^2$ and $f, g$ are given functions. We assume the (unique) existence of a solution $u$ for (1.1)-(1.2).

It was recently shown by Yamamoto [8] and Matsunaga-Yamamoto [7] that the Shortley-Weller approximation applied to (1.1)-(1.2) had a superconvergence property and numerical examples illustrating this fact were also given there.

To state the result, we construct a net over $\overline{\Omega} = \Omega \cup \Gamma$ by the grid points $P_{ij} = (x_i, y_j)$ in $\overline{\Omega}$ with the mesh size $h$. The set of the grid points is denoted by $\Omega_h$. We denote by $\mathcal{P}_\Gamma$ the set of points $P_{ij}$ such that at least one of $(x_i \pm h, y_j), (x_i, y_j \pm h)$ does not belong to $\Omega$ and put $\mathcal{P}_0 = \Omega_h \setminus \mathcal{P}_\Gamma$. Furthermore, we denote by $\Gamma_h$ the set of points of intersection of grid lines with $\Gamma$ and $\mathcal{S}_h(\kappa)$ by the set of points $P_{ij} \in \Omega_h$ which satisfy $\text{dist}(P_{ij}, \Gamma) \leq \kappa h$, where $\kappa$ is a constant with $\kappa > 1$, which is arbitrary chosen independently of $h$. We define the neighbors of $P \in \Omega_h$
to be four points in $\Omega_h \cup \Gamma_h$ which are adjacent to $P$ and on horizontal and vertical grid lines through $P$. As is shown in Figures 1.1 and 1.2, these points are denoted by $P_E, P_W, P_S, P_N$ and their distance to $P$ by $h_E, h_W, h_S, h_N$. Note that at least one of $P_E, P_W, P_S, P_N$ is on $\Gamma$ if and only if $P \in \mathcal{P}_\Gamma$ and that all of them are in $\Omega$ if and only if $P \in \mathcal{P}_0$, in which case we have $h_E = h_W = h_S = h_N = h$.

We denote by $U(P)$ an approximate solution at $P \in \Omega_h$. Then the Shortley-Weller (S-W) approximation $-\Delta_h^{(SW)}$ for $-\Delta$ at $P$ is defined by

$$
-\Delta_h^{(SW)}U(P) = \left( \frac{2}{h_E h_W} + \frac{2}{h_S h_N} \right) U(P) - \frac{2}{h_E (h_E + h_W)} U(P_E) - \frac{2}{h_W (h_E + h_W)} U(P_W) - \frac{2}{h_S (h_S + h_N)} U(P_S) - \frac{2}{h_N (h_S + h_N)} U(P_N),
$$

which includes the usual centered five point formula

$$
-\Delta_h U(P) = \frac{1}{h^2} \left[ 4 U(P) - U(P_E) - U(P_W) - U(P_S) - U(P_N) \right]
$$

as a special case $h_E = h_W = h_S = h_N = h$. Hence, if $P \in \mathcal{P}_0$, then the S-W approximation means the centered five point approximation.

As is easily seen, if $u \in C^{3,1}(\overline{\Omega})$, then the local truncation error $\tau^{(SW)}(P) \equiv -[\Delta_h^{(SW)} u(P) - \Delta u(P)]$ of the S-W formula at $P$ is estimated by

$$
|\tau^{(SW)}(P)| \leq \begin{cases} \\
\frac{2}{3} h^2 = O(h^2) & \text{if } h_E = h_W = h_S = h_N = h \\
\frac{2}{3} h^2 = O(h) & \text{otherwise,}
\end{cases}
$$

\begin{figure}[h]
\centering
\begin{tikzpicture}
\draw[step=0.5cm,very thin,gray] (-1.5,-1.5) grid (1.5,1.5);
\node at (0,0) [circle,fill,inner sep=1.5pt] {};\node at (0,0) [below left] {$P$};\node at (0,1) [circle,fill,inner sep=1.5pt] {};\node at (0,1) [above left] {$P_N$};\node at (1,0) [circle,fill,inner sep=1.5pt] {};\node at (1,0) [above right] {$P_E$};\node at (0,-1) [circle,fill,inner sep=1.5pt] {};\node at (0,-1) [below right] {$P_W$};\node at (-1,0) [circle,fill,inner sep=1.5pt] {};\node at (-1,0) [below left] {$P_S$};\node at (1,1) [circle,fill,inner sep=1.5pt] {};\node at (1,1) [above right] {$P_N$};\node at (-1,-1) [circle,fill,inner sep=1.5pt] {};\node at (-1,-1) [below left] {$P_S$};\node at (-1,1) [circle,fill,inner sep=1.5pt] {};\node at (-1,1) [below right] {$P_W$};\node at (1,-1) [circle,fill,inner sep=1.5pt] {};\node at (1,-1) [above right] {$P_E$};\node at (1.5,1.5) [left] {$\Gamma$};
\end{tikzpicture}
\caption{Figure 1.1}
\end{figure}

\begin{figure}[h]
\centering
\begin{tikzpicture}
\draw[step=0.5cm,very thin,gray] (-1.5,-1.5) grid (1.5,1.5);
\node at (0,0) [circle,fill,inner sep=1.5pt] {};\node at (0,0) [below left] {$P$};\node at (0,1) [circle,fill,inner sep=1.5pt] {};\node at (0,1) [above left] {$P_N$};\node at (1,0) [circle,fill,inner sep=1.5pt] {};\node at (1,0) [above right] {$P_E$};\node at (0,-1) [circle,fill,inner sep=1.5pt] {};\node at (0,-1) [below right] {$P_W$};\node at (-1,0) [circle,fill,inner sep=1.5pt] {};\node at (-1,0) [below left] {$P_S$};\node at (1,1) [circle,fill,inner sep=1.5pt] {};\node at (1,1) [above right] {$P_N$};\node at (-1,-1) [circle,fill,inner sep=1.5pt] {};\node at (-1,-1) [below left] {$P_S$};\node at (-1,1) [circle,fill,inner sep=1.5pt] {};\node at (-1,1) [below right] {$P_W$};\node at (1,-1) [circle,fill,inner sep=1.5pt] {};\node at (1,-1) [above right] {$P_E$};\node at (1.5,1.5) [left] {$\Gamma$};
\end{tikzpicture}
\caption{Figure 1.2}
\end{figure}
where $L$ is a Lipschitz constant common to all third order derivatives $\partial^3/\partial x^i\partial y^{3-i}$, $0 \leq i \leq 3$ and

$$M_3 = \sup_{P \in \Omega} \{ |\partial^3 u(P) / \partial x^i \partial y^{3-i}| \mid i = 0, 1, 2, 3 \}. $$

Then the following result holds for the S-W approximation.

**Theorem 1.1** (Superconvergence of the S-W approximation [8], [7])

Let $\Omega$ be a bounded convex domain with a piecewise $C^{2,\alpha}$ boundary. If $u \in C^{l+2,\alpha}(\overline{\Omega}), l = 0$ or 1, $\alpha \in (0,1]$, then

$$|u(P) - U(P)| \leq \begin{cases} O(h^{l+1+\alpha}) & P \in S_h(\kappa) \\ O(h^{l+\alpha}) & \text{otherwise.} \end{cases}$$

This implies that if $u \in C^{3,1}(\overline{\Omega})$, then we have

$$u(P) - U(P) = O(h^3) \text{ at } P \in S_h(\kappa)$$

even if $\tau^{(SW)}(P) = O(h)$ and $u(P) - U(P) = O(h^2)$ at other grid points.

Theorem 1.1 is a refinement of the following result due to Bramble-Hubbard [1]:

**Theorem 1.2.** If $u \in C^4(\overline{\Omega})$, then

$$|u(P) - U(P)| \leq \frac{M_4}{96} d^2 h^2 + \frac{2M_3}{3} h^3 = O(h^2) \quad \forall P \in \Omega_h,$$

where

$$M_4 = \sup_{P \in \Omega} \{ |\partial^4 u(P) / \partial x^i \partial y^{4-i}| \mid i = 0, 1, 2, 3, 4 \}$$

and $d$ denotes the diameter of the smallest circle containing $\Omega$.

It is also known by Matsunaga’s numerical experiments [4] that even if $u \in C^4(\overline{\Omega})$, the Bramble and the Collatz approximations do not have the superconvergence property like Theorem 1.1, although both have $O(h^2)$ accuracy at every $P \in \Omega_h$.

Now, we are interested in the behavior of the S-W approximate solution for the case $u \notin C^{l+2,\alpha}(\overline{\Omega})$. Has the S-W approximation any superconvergence property for such a case? The purpose of this paper is to answer
this question: Three examples with $\Omega = (0, 1) \times (0, 1)$ are given in § 2, which show three kinds of different behavior: (i) nonsuperconvergence at any point of $\Omega_h$, (ii) superconvergence near a part of $\Gamma$ and (iii) superconvergence in a neighborhood of a point of $\Gamma$. Furthermore, in § 3, we shall give two theorems by which the above phenomena can be illustrated.

2. Numerical Examples

In this section, we give three examples in which the S-W approximations applied to (1.1)-(1.2) show different behaviors.

Example 2.1. Let $f$ and $g$ be chosen so that the function

$$u = \sqrt{x(1-x)} + \sqrt{y(1-y)}$$

is the solution of (1.1)-(1.2). Observe that $u \in C(\overline{\Omega}) \cap C^\infty(\Omega)$, but $u \notin H^1(\Omega)$. Then, as is shown in Table 2.1, we see

$$u(P) - U(P) = O(h^{1/2}) \quad \forall P \in \Omega_h$$

and nonsuperconvergence occurs at any point in $\Omega_h$.

<table>
<thead>
<tr>
<th>$\alpha$</th>
<th>$h = 1.0e-01$</th>
<th>$h = 5.0e-02$</th>
<th>$h = 2.0e-02$</th>
<th>$h = 1.0e-02$</th>
<th>$h = 5.0e-03$</th>
</tr>
</thead>
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<td>0.4</td>
<td>2.9426730102e-01</td>
<td>2.730515389e-01</td>
<td>2.478926488e-01</td>
<td>2.308381843e-01</td>
<td>2.151573948e-01</td>
</tr>
<tr>
<td>0.5</td>
<td>3.704605831e-01</td>
<td>3.684237579e-01</td>
<td>3.665731474e-01</td>
<td>3.658538669e-01</td>
<td>3.654763480e-01</td>
</tr>
<tr>
<td>0.6</td>
<td>4.663822421e-01</td>
<td>4.971078573e-01</td>
<td>5.420728410e-01</td>
<td>5.798393031e-01</td>
<td>6.208151064e-01</td>
</tr>
</tbody>
</table>

Table 2.1

It should also be remarked that $u^{(4)}(Q) = O(h^{1/2-4})$ if $Q$ is close to the boundary $\Gamma$ and the local truncation error $\tau(Q)$ approaches to infinity as
$Q$ approaches to $\Gamma$. The distribution of errors $|u(P) - U(P)|$ in the case $h = 1.0e-002$ is shown in Figure 2.1.

Example 2.2. Let $f$ and $g$ be chosen so that the function

$$u = \sqrt{x} + y$$

is the solution of (1.1)-(1.2). Then

$$|u(P) - U(P)| = \begin{cases} O(h^{3/2}) & \text{near } \{(1, y)|0 \leq y \leq 1\} \\ O(h^{1/2}) & \text{otherwise.} \end{cases}$$  \hspace{1cm} (2.2)

The results are shown in Table 2.2 and Figure 2.2 for $h = 1.0e-002$.

| $\max |u(P) - U(P)|/h^\alpha$ (P $\in \Omega_h$) | $\alpha = 0.4$ | $\alpha = 0.5$ | $\alpha = 0.6$ |
|---|---|---|---|
| $h = 1.0e-001$ | 2.179020589e-001 | 2.743224392e-001 | 3.453514897e-001 |
| $h = 5.0e-002$ | 2.29787622e-001 | 3.103063992e-001 | 4.186911020e-001 |
| $h = 2.5e-002$ | 2.27193910e-001 | 3.285509309e-001 | 4.751258955e-001 |
| $h = 2.0e-002$ | 2.246577933e-001 | 3.322144274e-001 | 4.912646216e-001 |
| $h = 1.0e-002$ | 2.166254535e-001 | 3.433282065e-001 | 5.441385373e-001 |

| $\max |u(P) - U(P)|/h^\alpha$ (P $\in S_h(2)$ and away from $x = 1$) | $\alpha = 0.4$ | $\alpha = 0.5$ | $\alpha = 0.6$ |
|---|---|---|---|
| $h = 1.0e-001$ | 2.179020589e-001 | 2.743224392e-001 | 3.453514897e-001 |
| $h = 5.0e-002$ | 2.29787622e-001 | 3.103063992e-001 | 4.186911020e-001 |
| $h = 2.5e-002$ | 2.27193910e-001 | 3.285509309e-001 | 4.751258955e-001 |
| $h = 2.0e-002$ | 2.246577933e-001 | 3.322144274e-001 | 4.912646216e-001 |
| $h = 1.0e-002$ | 2.142439333e-001 | 3.395537514e-001 | 5.385564290e-001 |

| $\max |u(P) - U(P)|/h^\alpha$ (P $\in S_h(2)$ and near $x = 1$) | $\alpha = 1.4$ | $\alpha = 1.5$ | $\alpha = 1.6$ |
|---|---|---|---|
| $h = 1.0e-001$ | 1.035198677e-001 | 1.303237921e-001 | 1.640679366e-001 |
| $h = 5.0e-002$ | 9.450738537e-002 | 1.275171941e-001 | 1.720567627e-001 |
| $h = 2.5e-002$ | 8.756602147e-002 | 1.266314609e-001 | 1.831249910e-001 |
| $h = 2.0e-002$ | 8.554801688e-002 | 1.265047833e-001 | 1.870699143e-001 |
| $h = 1.0e-002$ | 7.969656240e-002 | 1.263105392e-001 | 2.001887137e-001 |

Table 2.2

In this case, a superconvergence occurs near the side $x = 1$ of $\Gamma$. 
Example 2.3. Let $f$ and $g$ be chosen so that the function $u = \sqrt{x} + \sqrt{y}$ is the solution of (1.1)-(1.2). Then

$$|u(P) - U(P)| = \begin{cases} O(h^{3/2}) & \text{near the corner (1,1),} \\ O(h^{1/2}) & \text{otherwise.} \end{cases} \quad (2.3)$$

The superconvergence occurs only near the corner (1,1). (See Figure 2.3 for the case $h = 1.0e-002$).

In the above examples, observe that the S-W approximation works well, although

$$\max_{P \in \Omega_h} |\tau^{(SW)}(P)| \to +\infty \quad \text{as } h \to 0.$$ 

This is a nice feature of the finite difference method.

3. Convergence Theorems

It is possible to give mathematical proofs for the error estimates (2.1)-(2.3). We can first prove the following results for the two-point boundary value problem

$$-u''(x) = \varphi(x), \quad 0 < x < 1 \quad (3.1)$$

$$u(0) = \alpha, \ u(1) = \beta \quad (3.2)$$

where $\varphi$ is a given function and $\alpha, \beta$ are given constants.

**Theorem 3.1.** Let $d(x) = \min(x, 1-x), \ 0 < x < 1$. If $0 < p < 1$, and the solution $u(x)$ of (3.1)-(3.2) belongs to $C^4(0,1)$ and satisfies

$$\sup_{x \in (0,1)} \frac{d(x)^k |u^{(k)}(x)|}{d(x)^p} < \infty, \ k = 0, 1, 2, 3, 4,$$

then

$$|u_i - U_i| = O(h^p) \quad \forall i,$$

where $\{U_i\}$ is the finite difference solution for (3.1)-(3.2) and $u_i = u(x_i)$, $x_i = ih$, $i = 0, 1, 2, \ldots, n + 1$, $h = 1/(n + 1)$. That is, superconvergence does not occur at any $x_i \in \Omega_h$. 

Theorem 3.2. If the solution $u(x)$ of (3.1)-(3.2) satisfies
\[ \sup_{x \in (0,1)} \frac{x^k |u^{(k)}(x)|}{x^p} < \infty, \quad k = 0, 1, 2, 3, 4 \] (3.3)
with some constant $p \in (0,1)$, then
\[ |u_i - U_i| \leq \begin{cases} O(h^{p+1}) & \text{near } x = 1 \\ O(h^p) & \text{otherwise.} \end{cases} \]
That is, superconvergence occurs near $x = 1$.

Theorems 3.1 and 3.2 can be derived with the use of the fact (e.g. Yamamoto-Ikebe [9]) that the inverse of the $n \times n$ tridiagonal matrix
\[ A = \begin{pmatrix} 2 & -1 & & \\ -1 & 2 & -1 & \\ & \ddots & \ddots & \\ & & -1 & 2 & -1 \\ & & & -1 & 2 \end{pmatrix} \]
is given by
\[ A^{-1} = (\alpha_{ij}), \quad \alpha_{ij} = \begin{cases} i(1 - \frac{j}{n+1}) & (i \leq j) \\ j(1 - \frac{i}{n+1}) & (i > j) \end{cases} \]
so that
\[ h\alpha_{ij} = \begin{cases} x_i(1 - x_j) & (i \leq j) \\ x_j(1 - x_i) & (i > j) \end{cases} \]
Now, consider the Dirichlet problem
\[ -\Delta u = F_1(x) + F_2(y) \quad \text{in } \Omega = (0,1) \times (0,1), \] (3.4)
\[ u = G_1(x) + G_2(y) \quad \text{on } \Gamma \] (3.5)
and \{U_{ij}\} be the S-W approximation with the equal mesh size $h_E = h_W = h_S = h_N = h$ at every $P \in \Omega_h$. Let \{U_i^{(1)}\} and \{U_i^{(2)}\} be the usual finite difference solution for the two-point boundary value problems
\[ -u''(x) = F_1(x), \quad 0 < x < 1 \]
\[ u(0) = G_1(0), \ u(1) = G_1(1) \]
and
\[-u''(y) = F_2(y), \quad 0 < y < 1\]
\[u(0) = G_2(0), \quad u(1) = G_2(1),\]
respectively. Then, by the uniqueness of the S-W approximate solution applied to (3.4)-(3.5), we have
\[U_{ij} = U_i^{(1)} + U_j^{(2)}, \quad \forall i, j.\]
Hence, all the phenomena stated in § 2 can now be illustrated with the use of Theorems 3.1 and 3.2 with \(p = 1/2\).

Note: Proofs of Theorems 3.1 and 3.2 will be given elsewhere.

References


