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On isotypies between blocks of finite groups

Atumi Watanabe (渡辺アツミ)

Department of Mathematics, Faculty of Science
Kumamoto University, Kumamoto, Japan

1. Introduction

In this report we give two examples of isotypies between blocks of finite groups, one is obtained from the naturally Morita equivalence in normal subgroups, and another is obtained from the Isaacs character correspondence. At first we recall the definition of isotypies between blocks. Let $(\mathcal{K}, \mathcal{R}, \mathcal{F})$ be a p-modular system such that $\mathcal{K}$ is algebraically closed and G be a finite group. Let B be a block of G with defect group D and (D, B_D) be a maximal B-subpair of G . We denote by $\mathbf{Br}_B(G)$ the Brauer category of B. $\mathbf{Br}_B(G)$ is the category whose objects are B-subpairs of G and whose morphisms are defined in the following way : For B-subpairs (Q, b) and (R, b') Mor$((Q, b), (R, b')$)is the set of all cosets $gC_G(Q)$ of G such that ${}^g(Q, b) \subseteq (R, b')$ (see [B-O], §1). We denote by $\mathbf{Br}_{B,D}(G)$ the full subcategory of $\mathbf{Br}_B(G)$ whose objects are the B-subpairs (Q, b) such that $(Q, b) \subseteq (D, B_D)$. We note that for any $Q \leq D$ there exists a unique block b such that $(Q, b) \subseteq (D, B_D)$, and we set $b = B_Q$.

Let $\mathrm{CF}(G, \mathcal{K})$ be the $\mathcal{K}$-vector space of $\mathcal{K}$-valued class functions on G and let $\mathrm{CF}(G, B, \mathcal{K})$ be the subspace of $\mathrm{CF}(G, \mathcal{K})$ of class functions α such that α is a $\mathcal{K}$-linear combination of χ's in $\mathrm{Irr}(B)$. Let $\mathrm{CF}_{p'}(G, B, \mathcal{K})$ be the subspace of $\mathrm{CF}(G, B, \mathcal{K})$ of class functions vanishing on the p-singular elements of G. For a B-Brauer element $(x, \mathbf{b})$ of G, the decomposition map

$$d_G^{(x,\mathbf{b})} : \mathrm{CF}(G, B, \mathcal{K}) \to \mathrm{CF}_{p'}(C_G(x), \mathbf{b}, \mathcal{K})$$

is defined by $d_G^{(x,\mathbf{b})}(\alpha)(y) = \alpha(xye_{\mathbf{b}})$ for any p'-element y of $C_G(x)$, where $e_{\mathbf{b}}$ is the block idempotents of $\mathcal{R}C_G(x)$ corresponding to $\mathbf{b}$.

Let H be a second finite group and B' be a block of H with D as a defect group. Let (D, B'_D) be a maximal B'-subpair of H and for any subgroup Q of D let (Q, B'_Q) be the B'-subpair of H such that $(Q, B'_Q) \subseteq (D, B'_D)$.

Definition. ([B], 4.6) With the above notations (G, B) and (H, B') are *isotypic* if the following conditions hold :

(i) The inclusion of D into G and H induces an equivalence of the Brauer categories $\mathbf{Br}_{B,D}(G)$ and $\mathbf{Br}_{B',D}(H)$.

(ii) There exists a family of perfect isometries

$$\{R^Q : \mathcal{R}_{\mathcal{K}}(C_G(Q), B_Q) \to \mathcal{R}_{\mathcal{K}}(C_H(Q), B'_Q)\}_{\{Q(cyclic)\leq D\}}$$

such that for any $x \in D$

$$(*) \qquad d_H^{(x, B'_{\langle x\rangle})} \circ R^{\langle 1\rangle} = R_{p'}^{\langle x\rangle} \circ d_G^{(x, B_{\langle x\rangle})},$$

where $R_{p'}^{\langle x\rangle}$ is the $\mathcal{K}$-linear map from $\mathrm{CF}_{p'}(C_G(x), B_{\langle x\rangle}, \mathcal{K})$ onto $\mathrm{CF}_{p'}(C_H(x), B'_{\langle x\rangle}, \mathcal{K})$ induced by $R^{\langle x\rangle}$ and we regard $R^{\langle 1\rangle}$ as a $\mathcal{K}$-linear map from $\mathrm{CF}(G, B, \mathcal{K})$ onto $\mathrm{CF}(H, B', \mathcal{K})$. In the above $R^{\langle 1\rangle}$ is called an *isotypy* between B and B' and $(R^Q)_{\{Q(cyclic)\leq D\}}$ is called the *local system* of $R^{\langle 1\rangle}$.

2. Isotypies obtained from naturally Morita equivalences

In this section we state that naturally Morita equivalent blocks of finite groups in normal subgroups are isotypic. At first we recall the definition of naturally Morita equivalences of blocks following [Kül] and [K-H].

Definition. ([Kül] and [H-K]) Let $O = \mathcal{R}$ or $\mathcal{F}$. Let H be a subgroup of G and let A and B be blocks of OG and OH respectively. We say A and B are naturally Morita equivalent of degree n if there exists an O-subalgebra S of A such that $S \cong M_n(O)$ as O-algebras, $1_A \in S$ and a map $\phi : B \otimes_O S \to A$ given by $\phi(b \otimes s) = bs$ is an O-algebra isomorphism, where 1_A is the identity element of A.

If A and B are naturally Morita equivalent of degree n, then A and B are Morita equivalent. Moreover A covers B when H is normal in G and, A and B have a common defect group by [Kül], Theorem 7 and [H-K], Proposition 2.6. Moreover we have

Proposition 2.1. ([H-K], Proposition 2.4) Suppose that A and B are blocks of $\mathcal{R}G$ and $\mathcal{R}H$, respectively and put $\bar{A} = A/\mathrm{J}(\mathcal{R})A$ and $\bar{B} = B/\mathrm{J}(\mathcal{R})B$. Then the following are equivalent.

(i) A and B are naturally Morita equivalent of degree n.

(ii) $\bar{A}$ and $\bar{B}$ are naturally Morita equivalent of degree n.

We have the following.

Theorem 2.2. Let H be a normal subgroup of G, and A and B be blocks of OG and OH respectively. If A and B are naturally Morita equivalent of degree n and these blocks have a common defect group D, then the following hold.

(i) Let Q be a subgroup of D. Any block of $OC_H(Q)$ associated with B is covered by a unique block of $OC_G(Q)$ which is associated with A, and those blocks are naturally Morita equivalent of degree n.

(ii) The converse of (i) is true, that is, any block of $OC_G(Q)$ associated with A is naturally Morita equivalent of degree n to a unique block of $OC_H(Q)$ associated with B.

In the above the Morita equivalence between A and B gives a bijection between $\mathrm{Irr}(A)$ and $\mathrm{Irr}(B)$ as follows by [H-K], Proposition 2.6 : Let $\chi \in \mathrm{Irr}(A)$. Then the restriction χ_H of χ to H has a unique irreducible constituent ζ_χ, $\zeta_\chi \in \mathrm{Irr}(B)$ and $\chi_H = n\zeta_\chi$. Moreover the map $\mathrm{Irr}(A) \to \mathrm{Irr}(B), \chi \longmapsto \zeta_\chi$ gives a perfect isometry from $\mathcal{R}_\mathcal{K}(G, A)$ onto $\mathcal{R}_\mathcal{K}(H, B)$. Next let (D, A_D) be a maximal A-subpairs of G and (D, B_D) be a maximal B-subpairs of H such that A_D and B_D are naturally Morita equivalent of degree n. Such maximal subpairs exist by Theorem 2.2. For a subgroup Q of D, we denote by (Q, A_Q) (respectively, (Q, B_Q)) be the A-subpair (respectively, B-subpair) of G (respectively, H) contained in (D, A_D) (respectively, (D, B_D)). Then we can show A_Q and B_Q are naturally Morita equivalent of degree n. So let R^Q be the perfect isometry from $\mathcal{R}_\mathcal{K}(C_G(Q), A_Q)$ onto $\mathcal{R}_\mathcal{K}(C_H(Q), B_Q)$ for $Q \leq D$ and let $R = R^{\langle 1 \rangle}$.

Theorem 2.3. With the above notations, R is an isotypy with local system $(\pm R^Q)_{\{Q(\text{cyclic}) \leq D\}}$. (In fact R is an isotypy in the sense of Broué's good definition.)

Remark. Let H be a normal subgroup of G such that $G = HC_G(P)$ for a Sylow p-subgroup P of H and let B be the principal block of H. Moreover let A be a block of G such that $AB \neq \{0\}$ and P is a defect group of A. M. E. Harris proved that A and B are isotypic in this situation. As Harris showed, then A and B are naturally Morita equivalent. So Theorem 2.3 is a generalization of his result.

3. Isotypies obtained from Isaacs character correspondences

Let S act on G via automorphism such that $(|S|, |G|) = 1$ and $C = C_G(S)$. It is well known that in this situation there is a natural bijection $\pi(G, S)$ from $\mathrm{Irr}_S(G)$, the set of S-invariant irreducible characters of G, onto $\mathrm{Irr}(C)$. When S is solvable, this is obtained by G. Glauberman and when $|G|$ is odd this is obtained by I.M. Isaacs. In [Wa1] we showed that the Glauberman character correspondences give isotypies between blocks of G and C. And in [H] it is shown that the Isaacs character correspondences give perfect isometries between blocks of G and C. We will state that those are isotypies. Here we recall the definition of Isaacs correspondences.

Lemma 3.1. ([I], Corollary 10.7) With the above notations and with $|G|$ odd, let $H = [G, S]'C$. Then there exists a bijection $\sigma(G, H, S) : \mathrm{Irr}_S(G) \to \mathrm{Irr}_S(H)$ such that for $\chi \in \mathrm{Irr}_S(G)$, $\sigma(G, H, S)(\chi)$ is the unique S-invariant irreducible character α of H with (χ_H, α) odd.

Definition. ([I], Theorem 10.8) Let S act on G via automorphism such that $(|S|,|G|)=1$ and $C=C_G(S)$. Assume $|G|$ is odd. If $C<G$, then let

$$G=G_0>G_1>G_2>\cdots>G_n=C$$

by $G_{i+1}=[G_i,S]'C$, for $i\geq 0$. The Isaacs character correspondence $\pi(G,S):\mathrm{Irr}_S(G)\to\mathrm{Irr}(C)$ is defined as follows : If $C<G$, then $\pi(G,S)=\sigma(G_{n-1},C,S)\sigma(G_{n-2},G_{n-1},S)\cdots\sigma(G_2,G_1,S)\sigma(G,G_1,S)$, otherwise $\pi(G,S)$ is the identity map.

Hypothesis 3.2. Let S and G be finite groups such that S acts on G, $(|S|,|G|)=1$ and that $|G|$ is odd. Put $C=C_G(S)$.

Theorem 3.3. ([H], Theorem 1) Under the above hypothesis, let B be an S-invariant block of G such that a defect group D of B is centralized by S. Then there exists a block b of C such that $\mathrm{Irr}(b)=\{\pi(G,S)(\chi)|\chi\in\mathrm{Irr}(B)\}$ and $\pi(G,S)$ gives a perfect isometry R between B and b. Moreover D is a defect group of b.

In the above theorem the assumption for B implies that $\chi\in\mathrm{Irr}(B)$ is S-invariant by [Wal], Proposition 1. We call b in the theorem the *Isaacs correspondent of B*.

Proposition 3.4. With the same notations in the above theorem, let (Q,B_Q) be an S-invariant B-subpair of G such that $Q\subseteq D$ and that a defect group of B_Q is centralized by S. Then the Isaacs correspondent b_Q of B_Q is associated with b in the sense of Brauer.

With the same notation in Theorem 3.3, let (D,B_D) be an S-invariant maximal B-subpair and let (Q,B_Q) be a B-subpair contained in (D,B_D) for $Q\leq D$. Then B_Q is uniquely determined, and B_Q is S-invariant, because B_D is S-invariant. In fact let $(Q,B_Q)\trianglelefteq(R,B_R)$ be B-subpairs contained in (D,B_D). If B_R is S-invariant, then B_Q is S-invariant. Moreover we can show that a defect group of B_Q is centralized by S for any $Q\leq D$. In fact we show that a defect group of $(B_Q)^T$ is centralized by S where T is the inertial group of B_Q in $N_G(Q)$. Let U be a defect group of $(B_Q)^T$. Since $(B_Q)^T$ is associated with B, $Q^v\leq U^v\leq D$ for some $v\in G$. So we have $C_\Gamma(Q)\geq S^{v^{-1}}$ and $C_\Gamma(Q)\geq S$. Since $C_\Gamma(Q)=SC_G(Q)$, by the Schur-Zassenhaus theorem there exists an element $u\in C_G(Q)$ such that $S^{v^{-1}}=S^u$. Then $v^{-1}u^{-1}\in C$. Hence we have $U^{u^{-1}}\leq D^{v^{-1}u^{-1}}\subseteq C$. Thus $U^{u^{-1}}$ is a defect group of $(B_Q)^T$ centralized by S. Now let b_Q be the Isaacs correspondent of B_Q and let R^Q be the perfect isometry from $\mathcal{R}_\mathcal{K}(C_G(Q),B_Q)$ onto $\mathcal{R}_\mathcal{K}(C_C(Q),b_Q)$ for $Q\leq D$. We have the followings noticing that (Q,b_Q) is a b-subpair of C by Proposition 3.4.

Proposition 3.5. With the above notations we have the following.
(i) (D,b_D) is a maximal b-subpair of C and $(Q,b_Q)\subseteq(D,b_D)$ for any $Q\leq D$.

(ii) The Brauer categories $\mathbf{Br}_{B,D}(G)$ and $\mathbf{Br}_{b,D}(C)$ are equivalent.

Theorem 3.6. Assume Hypothesis 3.2 and let B be an S-invariant block of G such that a defect group D of B is centralized by S and b be the Isaacs correspondent of B. Then $R^{\langle 1\rangle}$ is an isotypy between B and b with local system $(\pm R^Q)_{\{Q(\mathrm{cyclic})\leq D\}}$, where R^Q is as in the above.

We use results in [I] and [Wo1, 2] in order to prove the above propositions and theorem. Details of this section will be found in [Wa2].

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