On $S_R((12))$-blocks for a symmetric group

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1　Main result

Let p be a prime number and let (K, R, k) be a p-modular system, that is, R is a noetherian, complete discrete valuation ring, K the quotient field of R of characteristic zero and k the residue class field of R of characteristic p. For a finite group G, we let $bl(G)$ denote the set of the blocks of RG. If B is a block of RG, then δ_B denotes the identity of B, which is called the block idempotent of B. In this article, we announce the following result:

Theorem A. Let $G = S_n$ be the symmetric group on n letters, $H = \{1, (12)\}$ and $I(H) = RG\sigma_H RG$, where $\sigma_H = 1 + (12)$. If $p > 2$, then $I(H) = \bigoplus_{B \in bl(G)} I(H)\delta_B$ is an indecomposable decomposition of $I(H)$ as a (G, G)-bimodule.

2　Necessary steps

Theorem A is closely related with the notion of the $S_R(H)$-block. Namely for any finite group G and any $H \leq G$, we know that $\operatorname{End}_{G \times G}(I(H))$ is isomorphic to the center of $\operatorname{End}_G(\sigma_H RG)$, where $\sigma_H = \sum_{x \in H} x$ (Robinson [4]). Hence Theorem A is equivalent to:

Theorem A′. With the assumptions of Theorem A, we have that $\{\delta_B e_H; B \in bl(G)\}$ is just the set of the block idempotents of the Hecke algebra $e_H RG e_H$, where $e_H = \sigma_H/2$.

We remark that Therem A′ (and hence Theorem A) remains true if R is replaced by k. To prove Theorem A, we need a further reduction. For $\chi \in \operatorname{Irr}(G)$, let $\operatorname{IBr}(\chi) = \{\psi \in \operatorname{IBr}(G); d_{\chi\psi} \neq 0\}$. Now Theorem A′ is equivalent to :

Theorem A″. With the assumptions of Theorem A, let B_0 be the principal

block of RG. Then for any $\chi, \chi' \in \mathrm{Irr}(B_0)\setminus\{1_G\}$, there exists a sequence:

$$\chi = \chi_0, \chi_1, \cdots, \chi_s = \chi'$$

of ordinary irreducible characters of G such that $\mathrm{IBr}(\chi_i) \cap \mathrm{IBr}(\chi_{i+1})$ contains an element other than the trivial character for each i $(0 \le i \le s-1)$.

Looking at some of the known decomposition matrices of the symmetric groups,one will be convinced that Theoem A'' must be true without taking much time.However to prove it is quite another. Now,let S^λ and $\overline{S}^\lambda$ be the Specht modules of G over K and k respectively associated with the partition $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_r)$ of n, i.e., $\lambda_1 \ge \lambda_2 \ge \ldots \ge \lambda_r$ are positive integers such that $\sum_{i=1}^{r} \lambda_i = n$. Let us write $r = d(\lambda)$. We call the partition λ to be p-regular if there is no i such that $\lambda_i = \lambda_{i+1} = \ldots = \lambda_{i+p-1}$. Let Λ and Λ_0 be the sets of the partitions and the p-regular ones of n respectively.As is well-known,we have that $\mathrm{Irr}(G) = \{S^\lambda ; \lambda \in \Lambda\}$. On the other hand, if $\lambda \in \Lambda_0$, $D^\lambda = hd(\overline{S}^\lambda)$ is simple and $\mathrm{IBr}(G) = \{D^\lambda ; \lambda \in \Lambda_0\}$. In particular,we have that $\mathrm{IBr}(S^\lambda\} \ni D^\lambda$ if λ is p-regular, where $\mathrm{IBr}(S^\lambda)$ is the set of irreducible constituents of the kG-module $\overline{S}^\lambda$.For the analysis of the sets $\mathrm{IBr}(S^\lambda)$ we need the theorems of Carter and Payne and of Schaper.To explain the theorem of Carter and Payne,let $[\lambda]$ be the Young diagram associated with the partition λ. We write $\lambda \to \mu$ if $[\mu]$ can be obtained from $[\lambda]$ by raising q spaces from a row of $[\lambda]$ to a higher row and each space is moved through a multiple of p^e spaces,where $p^e > q$.Then

Theorem B (Carter and Payne [1]). Let λ, $\mu \in \Lambda$. If $\lambda \to \mu$, then

$$\mathrm{Hom}_G(\overline{S}^\mu, \overline{S}^\lambda) \ne 0.$$

For the theorem of Schaper,see James [3].Using the theorem,we get the following result,which is crucial for the proof of Theorem A''.

Lemma C. Let λ be a p-regular partition of n different from (n).If $\mathrm{IBr}(S^\lambda) = \{D^\lambda, 1_G\}$,we have $d(\lambda) = 2$,unless $\lambda = (p-1, p-1, 1)$.

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References

[1] R.W.Carter and M.T.J.Payne: On homomorphisms between Weyl modules and Specht modules: Math.Proc.Camb.Phil.Soc.87, 419-425(1980).

[2] Y.Hieda and Y.Tsushima: On $S_R(H)$-blocks for finite groups, J.Alg.202 (1998), 583-588.

[3] G.James: The representation theory of the symmetric groups, Proc.Symposia in Pure Math.47(1987),111-126.

[4] G.R.Robinson: Some remarks on Hecke algebras, J.Alg.163(1994), 806-812.