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<th>DADE'S CONJECTURE FOR FINITE SPECIAL LINEAR GROUPS (Representation Theory of Finite Groups and Related Topics)</th>
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Kyoto University
DADE'S CONJECTURE FOR
FINITE SPECIAL LINEAR GROUPS

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1. DADE'S CONJECTURE

Let $p$ be a prime number, and let $G$ a finite group. A $p$-chain $C$ of $G$ is any strictly increasing chain

\[(1-1) \quad C : U_0 < U_1 < \cdots < U_m \]

of $p$-subgroups $U_i$ of $G$. We denote the length $m$ of $C$ by $|C|$. If $K$ is any group acting (exponentially) as automorphisms of $G$, then any $g \in K$ sends the $p$-chain $C$ to the $p$-chain

\[(1-2) \quad C^g : U_0^g < U_1^g < \cdots < U_m^g \]

of $G$. The normalizer $N_K(C)$ of $C$ in $K$ is the subgroup of all $g \in K$ such that $C = C^g$, i.e.,

\[N_K(C) = \bigcap_{i=0}^{m} N_K(U_i).\]

We say that the $p$-chain $C$ in (1-1) is radical (with respect with $G$) if $U_0$ is the largest normal $p$-subgroup $O_p(G)$ of $G$ and

\[U_i = O_p(\bigcap_{j=0}^{i} N_G(U_j)) \quad \text{for} \quad i = 1, 2, \cdots, m.\]

We denote by $\mathfrak{R}(G)$ the set of all radical $p$-chains of $G$. The set $\mathfrak{R}(G)$ is closed under the conjugation action (1-2) of $G$ on its $p$-chains. We denote by $\mathfrak{R}(G)/G$ any complete representatives for the $G$-conjugacy classes in $\mathfrak{R}(G)$.

For a $p$-block $B$ of $G$ and a non negative integer $d$, we denote by $\text{Irr}(H, B, d)$ the set of complex irreducible characters $\psi$ of $H$ such that

(i) the $p$-part of $|H|/\psi(1)$ is $p^d$, and
(ii) $\psi$ lies in a $p$-block $b$ of $H$ such that $b^G = B$.

In [D1], E. C. Dade gives the following conjecture.
Conjecture 1 (Dade’s ordinary conjecture). If $O_p(G) = 1$ and the defect of $B$ is positive, then

$$\sum_{C \in \mathfrak{R}(G)/G} (-1)^{|C|} |\text{Irr}(N_G(C), B, d)| = 0.$$ 

We mention a stronger conjecture.

Let $E$ be a finite group such that $G \triangleleft E$. By the conjugation action of $E$ on $G$, we define an action (1-2) of $E$ on the $p$-chains $C$ of $G$. So any such $C$ has a normalizer $N_E(C)$ in $E$, and we have $N_G(C) \triangleleft N_E(C)$. Thus $N_E(C)$ acts by conjugation on $\text{Irr}(N_G(C))$. For $\phi \in \text{Irr}(N_G(C))$, we write

$$T_{N_E(C)}(\phi) = \{g \in N_E(C) | \phi^g = \phi\}.$$ 

For $\overline{F} \triangleleft E/G$, we denote by $\text{Irr}(N_G(C), B, d, \overline{F})$ the set of $\phi \in \text{Irr}(N_G(C), B, d)$ such that

(iii) $G \cdot T_{N_E(C)}(\phi)/G = \overline{F}$.

The following conjecture is given in [D2].

Conjecture 2 (Dade’s invariant conjecture). If $O_p(G) = 1$ and the defect of $B$ is positive, then

$$\sum_{C \in \mathfrak{R}(G)/G} (-1)^{|C|} |\text{Irr}(N_G(C), B, d, \overline{F})| = 0.$$ 

Here, we treat a verification of Dade's invariant conjecture for $G = SL(n, q)$ and $E = GL(n, q)$ with $p | q$. This implies Dade’s invariant conjecture for $G = PSL(n, q)$ and $E = PGL(n, q)$.

2. ON RADICAL $p$-CHAINS OF A CHEVALLEY GROUP

In this section, let $G$ be a Chevalley group and let the defining field of $G$ characteristic $p$. Then $\mathfrak{R}(G)$ is the set of $p$-chains consisting of unipotent radicals of parabolic subgroups of $G$ [BT] [BW]. Now we fix a Borel subgroup $U$. Then we may take $\mathfrak{R}(G)/G$ to be the set of $p$-chains consisting of unipotent radicals of parabolic subgroups of $G$ containing $U$. Thus, for any $C \in \mathfrak{R}(G)/G$, $N_G(C)$ is some parabolic subgroup of $G$ containing $U$.

It is well known that the set of all parabolic subgroups of $G$ containing $U$ is parametrized by the set of subsets of a fundamental root system $I$ of $G$. Thus we denote by $P_J$ the parabolic subgroup corresponding to $J \subseteq I$.

By the above argument and [W] [KR], Conjecture 2 is equivalent to the following.

Conjecture 3. If $O_p(G) = 1$ and the defect of $B$ is positive, then

$$\sum_{J \subseteq I} (-1)^{|I \setminus J|} |\text{Irr}(P_J, B, d, \overline{F})| = 0.$$
3. The Case for $G = SL(n, q)$ and $E = GL(n, q)$ ($p|q$)

We consider the case for $G = SL(n, q)$ and $E = GL(n, q)$ with $p|q$.

We take $I = \{1, 2, \cdots, n-1\}$ as a fundamental root system and take the subgroup $U$ of lower triangular matrices in $GL(n, q)$ as a Borel subgroup of $GL(n, q)$. Then, if $J \subseteq I$ satisfying $\Gamma \setminus J = \{a_1, \cdots, a_k\}$, the parabolic subgroup $P_J$ of $GL(n, q)$ is

$$\{(p_{ij}) \in GL(n, q) | \text{If some } k \text{ satisfies } i \leq a_k \text{ and } j > a_k \text{, then } p_{ij} = 0\}.$$

Moreover $U \cap SL(n, q)$ is a Borel subgroup of $SL(n, q)$ and $P_J \cap SL(n, q)$ is a parabolic subgroup of $SL(n, q)$ containing $U \cap SL(n, q)$.

Here we restate Dade conjecture for $SL(n, q)$ to a statement on $GL(n, q)$. For a positive integer $s$, we denote by $\text{Irr}(J, B, d, s)$ the set of irreducible characters $\psi$ in $\text{Irr}(P_J \cap SL(n, q), B, d)$ such that the $GL(n, q)$-conjugacy class containing $\psi$ has $s$ elements. Because $GL(n, q)/SL(n, q)$ is cyclic and its order is relatively prime to $p$, Conjecture 3 for $G = SL(n, q)$ and $E = GL(n, q)$ is equivalent to the following: For any $p$-block $B$ of $SL(n, q)$ whose defect is positive, any non negative integer $d$ and any positive integer $s$,

$$\sum_{J \subseteq I}(-1)^{|I \setminus J|} \left| \text{Irr}(J, B, d, s) \right| = 0.$$

For a positive integer $s$ and a $p$-block $\tilde{B}$ of $GL(n, q)$, we denote by $\tilde{\text{Irr}}(J, \tilde{B}, d, s)$ the set of irreducible characters $\phi$ in $\text{Irr}(P_J \cap SL(n, q), \tilde{B}, d)$ such that the restriction of $\phi$ to $P_J \cap SL(n, q)$ has $s$ irreducible constituents. Then, we have the following theorem on $GL(n, q)$, slightly stronger than the above statement.

**Theorem** [S]. For any $p$-block $\tilde{B}$ of $GL(n, q)$ whose defect is positive, any non negative integer $d$ and positive integer $s$, the following holds:

$$\sum_{J \subseteq I}(-1)^{|I \setminus J|} \left| \tilde{\text{Irr}}(J, \tilde{B}, d, s) \right| = 0.$$

The proof of this theorem is an extension of the proof of Dade's ordinary conjecture for $GL(n, q)$ [OU].

Thus, we have

**Corollary.** If $p|q$, Conjecture 3 for $G = SL(n, q)$ and $E = GL(n, q)$ is true. Moreover conjecture 3 for $G = PSL(n, q)$ and $E = PGL(n, q)$ is true.

**References**


