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A remark on p-blocks of finite groups with abelian defect groups

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ABSTRACT. In modular representation theory of finite groups there is a well-known conjecture due to P.Donovan. The Donovan conjecture is on blocks of group algebras of finite groups over an algebraically closed field k of prime characteristic p, which says that, for any given finite p-group P, up to Morita equivalence, there are only finitely many block algebras with defect group P. We prove that the Donovan conjecture holds for principal block algebras in the case where P is elementary abelian 3-group of order 9.

In modular representation theory of finite groups, there are several important conjectures many people are interested in. One of them is the following, which is due to P.Donovan.

Donovan conjecture ([2, Conjecture M]). For any given prime p and for any given finite p-group P, up to Morita equivalence, there are only finitely many block algebras of finite groups with defect group P.

There are only a few cases where the Donovan conjecture has been checked. First of all, for the case that P is cyclic, which was due to works done by E.C.Dade, H.Kupisch and G.J.Janusz (see [8, Chap.VII]), and for the case that $p = 2$ and D is dihedral, semi-dihedral or quaternion, which was due to K. Erdmann [7]. The conjecture also holds when we consider only p-blocks of p-solvable groups which was done by B.Külshammer [14], and when we consider only p-blocks of symmetric groups which was done by J.Scopes [24].

In this note, we show that the Donovan conjecture is true also when we restrict ourselves to principal 3-blocks with elementary abelian Sylow 3-subgroups of order 9. We also show that, if $B_0(kG)$ is the principal block (algebra) of the group algebra kG of any finite group G with elementary abelian Sylow 3-subgrup of order 9 over an algebraically closed field k of characteristic 3, then the Loewy length (radical length) of $B_0(kG)$ is exactly 5 or 7. We should confess that these two results depend on the classification of finite simple groups.

Theorem 1. *Let P be the elementary abelian group of order* 9, *and let k be an algebraically closed field of characteristic* 3. *Then, there are only finitely many non-Morita equivalent principal block albebras of group algebras kG of finite groups G with such a Sylow 3-subgroup P.*

Theorem 2. *Let k be an arbitrary filed of characteristic 3, and let G be an arbitrary finite group with elementary abelian Sylow 3-subgroup P of order 9.*

(i) *The principal block algebra $B_0(kG)$ of kG has Loewy length 5 or 7.*

(ii) *The Loewy length of the projective cover $P(k_G)$ of the trivial module k_G over kG is also 5 or 7 for any finite group G with elementary abelian Sylow 3-subgroup P of order 9.*

Throughout this paper we use the following notation and teminology. In this paper G is always a finite group, and a *module* is always a finitely generated right module, unless stated otherwise. We write $(\mathcal{O}, \mathcal{K}, k)$ for a splitting p-modular system for all subgroups of G, that is, $\mathcal{O}$ is a complete discrete valuation ring of rank one with quotient field $\mathcal{K}$ and with residue field k such that $\mathcal{K}$ is a field of characteristic zero and k is a field of characteristic $p > 0$ and that $\mathcal{K}$ and k are both splitting fileds for all subgroups of G (note that only in the satement of Theorem 2 k is an arbitrary field of characteristic $p > 0$). We denote by $B_0(kG)$ the principal block algebra of the group algebra kG. We write k_G for the trivial kG-module of k-dimension one. For a block algebra A of kG, $\mathrm{Irr}(A)$ is the set of all irreducible ordinary characters of G in A. Let R be a ring. We write $J(R)$ for the Jacobson radical of R. For an R-module M we denote by $j(M)$ and $P(M)$ the Loewy length of M and the projective cover of M (of course, if they exist), that is, $j(M)$ is the least positive integer j such that $M{\cdot}J(R)^j = 0$. Let n be a positive integer. We then write C_n and Σ_n for the cyclic group of order n and the symmetric group on n letters, respectively. We denote by $\mathrm{GU}_n(q^2)$ the general unitary group of degree n over the Galoi field $\mathbb{F}_{q^2}$ of q^2 elements. For other notation and terminlogy we follow the book of Nagao-Tsushima [18].

The following proposition was informed by S.Yoshiara. The author is grateful to him.

Proposition 3 (S.Yoshiara). *Let G be a finite group with elementary abelian Sylow 3-subgroup of order 9 such that $O_{3'}(G) = 1$ and $O^{3'}(G) = G$. Then, G is one of the following (i)–(ii).*

(i) *$G = X \times Y$ for finite simple groups X and Y such that both of them have cyclic Sylow 3-subgroups of order 3.*

(ii) *G is a non-abelian finite simple group with elementary abelian Sylow 3-subgroup of order 9.*

By making use of the classification of finite simple groups and Proposition 3, we get the following list of finite non-abelian simple groups G with elementary abelian Sylow 3-subgroup of order 9.

Proposition 4. *If G is a non-abelian finite simple group with elementary abelian Sylow 3-subgroup of order 9, then G is one of the following nine types:*
(i) $A_6, A_7, A_8, M_{11}, M_{22}, M_{23}, HS$.
(ii) $\mathrm{PSL}_3(q)$ for a power q of a prime with $q \equiv 4$ or 7 (mod 9).
(iii) $\mathrm{PSU}_3(q^2)$ for a power q of a prime with $2 < q \equiv 2$ or 5 (mod 9).
(iv) $\mathrm{PSp}_4(q)$ for a power q of a prime with $q \equiv 4$ or 7 (mod 9).
(v) $\mathrm{PSp}_4(q)$ for a power q of a prime with $2 < q \equiv 2$ or 5 (mod 9).
(vi) $\mathrm{PSL}_4(q)$ for a power q of a prime with $2 < q \equiv 2$ or 5 (mod 9).
(vii) $\mathrm{PSU}_4(q^2)$ for a power q of a prime with $q \equiv 4$ or 7 (mod 9).
(viii) $\mathrm{PSL}_5(q)$ for a power q of a prime with $q \equiv 2$ or 5 (mod 9).
(ix) $\mathrm{PSU}_5(q^2)$ for a pwer q of a prime with $q \equiv 4$ or 7 (mod 9).

Proposition 5 (S.Koshitani and H.Miyachi). *Let G be* $\mathrm{GU}_4(q^2)$ *or* $\mathrm{GU}_5(q^2)$ *for a power q of a prime with $q \equiv 4$ or 7* (mod 9)*. Then, $B_0(\mathcal{O}G)$ and $B_0(\mathcal{O}H)$ are Puig equivalent, where H is the normalizer of a Sylow 3-subgroup of G.*

Proof. This follows from the fact that all simple kG-modules in $B_0(kG)$ are trivial source (p-permutation) modules and resulst of Okuyama [19, Lemma 2.2], Linckelmann [17, Theorem 2.1(iii)] and Rickard [23, Theorem 5.2].

Corollary 6 (S.Koshitani and H.Miyachi). *Let $G = \mathrm{PSU}_4(q^2)$ or $\mathrm{PSU}_5(q^2)$ for a power q of a prime with $q \equiv 4$ or 7* (mod 9)*. Then, $B_0(\mathcal{O}G)$ and $B_0(\mathcal{O}H)$ are Puig equivalent, where H is the normalizer of a Sylow 3-subgroup of G. (Hence, Broué conjecture* ([3, 6.2.Question], [4, 4.9.Conjecture]) *holds for $p = 3$ and for G here).*

Proof. This follows from Proposition 5 and a theorem of Alperin-Dade ([1] and [6]).

Proof of Theorem 1. First of all, a theorem of Külshammer [15, Proposition, p.305] implies that we may assume $O^{3'}(G) = G$. Then, by [16], [12], [22], [21], [13] and Corollary 6, we get the assertion.

Proof of Theorem 2. This is obtained by Proposition 3, Proposition 4, results of Waki ([25], [26], [27]), [12], [22], [13], Corollary 6 and [20].

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