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Kyoto University
A note on
the $D$-affinity of the flag variety in positive characteristic

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Let $G$ be a simply connected simple algebraic group over an algebraically closed field $\mathfrak{k}$ and let $B$ be a Borel subgroup of $G$. Let $\mathcal{X} = G/B$, $\mathcal{D}_\mathcal{X}$ the sheaf of $\mathfrak{k}$-algebras of differential operators on $\mathcal{X}$, $\mathcal{D}_\mathcal{X} \text{qc}$ the category of left $\mathcal{D}_\mathcal{X}$-modules that are quasi-coherent over the structure sheaf $\mathcal{O}_\mathcal{X}$ of $\mathcal{X}$, $\mathcal{D}(\mathcal{X}) = \Gamma(\mathcal{X}, \mathcal{D}_\mathcal{X})$ the $\mathfrak{k}$-algebra of differential operators on $\mathcal{X}$, and $\mathcal{D}(\mathcal{X}) \text{Mod}$ the category of left $\mathcal{D}(\mathcal{X})$-modules. We say $\mathcal{X}$ is $D$-affine iff for each $\mathcal{M} \in \mathcal{D}_\mathcal{X} \text{qc}$ (i) the natural morphism $\mathcal{D}_\mathcal{X} \otimes_{\mathcal{D}(\mathcal{X})} \Gamma(\mathcal{X}, \mathcal{M}) \to \mathcal{M}$ is epic, and (ii) $H^i(\mathcal{X}, \mathcal{M}) = 0$ $\forall i > 0$; equivalently, the functor $\Gamma(\mathcal{X}, ?) : \mathcal{D}_\mathcal{X} \text{qc} \to \mathcal{D}(\mathcal{X}) \text{Mod}$ gives an equivalence of categories with quasi-inverse $\mathcal{D}_\mathcal{X} \otimes_{\mathcal{D}(\mathcal{X})} ?$ (cf. [K98a, 1.6]).

In characteristic 0 a celebrated theorem of Beilinson and Bernstein [BB] affirms that $\mathcal{X}$ is $D$-affine. In positive characteristic B. Haastert [H87, 4.4.1] shows that in (i) even the natural morphism

$$O_\mathcal{X} \otimes_{\mathfrak{k}} \Gamma(\mathcal{X}, \mathcal{M}) \to \mathcal{M}$$

is epic.

Then by Grothendieck’s vanishing theorem (ii) will hold if $H^i(\mathcal{X}, \mathcal{D}_\mathcal{X}) = 0$ $\forall i > 0$. If $(\text{Diff}_m)_{m \in \mathbb{N}}$ is the standard filtration of $\mathcal{D}_\mathcal{X}$, however, [H87, 4.2.7] shows that if $p = \text{ch}\mathfrak{k} > h$ the Coxeter number of $G$ and if $G$ is not of type $A_1$, then

$$H^i(\mathcal{X}, \text{Diff}_p) \neq 0 \text{ for some } i \neq 0.$$

And yet there is another filtration, called the $p$-filtration, on $\mathcal{D}_\mathcal{X}$. If $O^{(p)}_\mathcal{X}$ is the sheaf of $\mathfrak{k}$-algebras such that $\Gamma(\mathcal{U}, O^{(p)}_\mathcal{X}) = \{ a^{p^r} \mid a \in \Gamma(\mathcal{U}, O_\mathcal{X}) \}$ for each open $\mathcal{U}$ of $\mathcal{X}$ and if $\mathcal{D}_r = \text{Mod}_{O^{(p)}_\mathcal{X}}(O_\mathcal{X}, O_\mathcal{X})$, then $\mathcal{D}_\mathcal{X} = \bigcup_{r \in \mathbb{N}} \mathcal{D}_r$. As $\mathcal{X}$ is noetherian,

$$H^i(\mathcal{X}, \mathcal{D}_\mathcal{X}) \simeq \lim_r H^i(\mathcal{X}, \mathcal{D}_r).$$

Let $G_r = \ker F^r$ with $F^r : G \to G^{(r)}$ the $r$-th Frobenius morphism [J, I.9], $\hat{\mathcal{Z}}_r$ the induction functor from the category $B\text{Mod}$ of $B$-modules to the category $G, B\text{Mod}$ of $G, B$-modules [J, I.3], and let $\mathcal{L}$ be the functor from $B\text{Mod}$ to the category of $G$-equivariant $O_\mathcal{X}$-modules $[J, I.5]$. Then by [H87, 4.3.3]

$$\mathcal{D}_r \simeq \mathcal{L}(\hat{\mathcal{Z}}_r(\mathfrak{k}^*) \simeq \mathcal{L}(\hat{\mathcal{Z}}_r(2(p^r - 1)\rho)),$$
where $\rho = \frac{1}{2} \sum_{\alpha \in R^+} \alpha$ with $R^+$ the positive system of roots of $G$ such that the roots of $B$ are $-R^+$. If $G = SL_2$ or $SL_3$, then the composition factors of $\hat{Z}_s(2(p^r-1)\rho)$ in $G, BMod$ have all dominant highest weights [H87, 4.5.4], hence $H^i(\mathfrak{X}, D_r) = 0 \forall i > 0$ by Kempf’s vanishing theorem, showing $\mathfrak{X}$ is $D$-affine in those cases. The argument unfortunately does not generalize.

There is another criterion for $\mathfrak{X}$ to be $D$-affine [Ka, Th. 1.4.1]: $\mathfrak{X}$ is $D$-affine iff there is a dominant weight $\lambda$ such that for all $r >> 0$ the natural morphism
\[(1) \quad D_\mathfrak{X} \otimes_{\mathcal{O}_\mathfrak{X}} \mathcal{L}(-r\lambda) \otimes H^0(\lambda) \to D_\mathfrak{X}\]
splits as a morphism of sheaves of abelian groups, where $H^0(\cdot) = H^0(\mathfrak{X}, \mathcal{L}(\cdot)) = \Gamma(\mathfrak{X}, \mathcal{L}(\cdot))$. If $\text{Dist}(G)$ (resp. $\text{Dist}(B)$) is the algebra of distributions on $G$ (resp. $B$), the natural morphism (5) can be described by the commutative diagram
\[(2) \quad \begin{array}{ccc}
D_\mathfrak{X} \otimes_{\mathcal{O}_\mathfrak{X}} \mathcal{L}(-r\lambda) \otimes H^0(\lambda) & \cong & D_\mathfrak{X} \\
\mathcal{L}(\text{Dist}(G) \otimes \text{Dist}(B) (-r\lambda) \otimes H^0(\lambda)) & \cong & \mathcal{L}(\text{Dist}(G)),
\end{array}\]
where $ev_{r\lambda} : H^0(\lambda) \to r\lambda$ is the evaluation at the identity element of $G$. In characteristic $0$ the map $\text{Dist}(G) \otimes \text{Dist}(B) (-r\lambda) \otimes ev_{r\lambda}$ has been proved to split in $BMod$ so that $\mathcal{L}(\text{Dist}(G) \otimes \text{Dist}(B) (-r\lambda) \otimes ev_{r\lambda})$ splits as a morphism of $G$-equivariant $\mathcal{O}_\mathfrak{X}$-modules to show the $D$-affinity of $\mathfrak{X}$ [BB].

Assume in the following that $\text{cht} = p > 0$. If $\mathfrak{X}$ is $D$-affine, in view of $1 \in D(\mathfrak{X})$ we must have for a given $r$ the morphism
\[(3) \quad D_s \otimes_{\mathcal{O}_\mathfrak{X}} \mathcal{L}(-r\lambda) \otimes H^0(\lambda) \to D_s\]
split as a morphism of sheaves of abelian groups for $s >> 0$. By (4) the morphism (7) reads as
\[\mathcal{L}(ev(\otimes ev)) : \mathcal{L}(\hat{Z}_s(2(p^s-1)\rho-r\lambda) \otimes H^0(\lambda)) \to \mathcal{L}(\hat{Z}_s(2(p^s-1)\rho)),\]
where $ev(\otimes ev) \in G_s BMod(\hat{Z}_s(2(p^r-1)\rho-r\lambda) \otimes H^0(\lambda), \hat{Z}_s(2(p^r-1)\rho))$ is induced by the Frobenius reciprocity from $ev \otimes ev \in BMod(\hat{Z}_s(2(p^r-1)\rho-r\lambda) \otimes H^0(\lambda), 2(p^r-1)\rho)$ the tensor product of evaluations $ev_{2(p^r-1)\rho-r\lambda} : \hat{Z}_s(2(p^r-1)\rho-r\lambda) \to 2(p^r-1)\rho - r\lambda$ and $ev_{r\lambda} : H^0(\lambda) \to r\lambda$.

Now $1 \in D_s$ belongs to $\mathcal{O}_\mathfrak{X}$ and $\mathcal{O}_\mathfrak{X}$ is a direct summand of $D_s$ as an $\mathcal{O}_\mathfrak{X}$-module, in fact, as a $G$-equivariant $\mathcal{O}_\mathfrak{X}$-module, corresponding to the splitting of the quotient $\pi : \hat{Z}_s(2(p^s-1)\rho) \to \text{hd}_G B\hat{Z}_s(2(p^s-1)\rho) = \mathfrak{e}$ in $BMod$. Then we should have at least the composite
\[\begin{array}{ccc}
H^0(\hat{Z}_s(2(p^s-1)\rho-r\lambda) \otimes H^0(\lambda)) & \to & \mathfrak{e} \\
H^0(\text{ev}(\otimes ev)) \ar@{->}[d] & & \ar@{->}[d] \\
H^0(\hat{Z}_s(2(p^s-1)\rho)) & &
\end{array}\]
to be surjective, that we will verify in what follows.
We will suppress $\mathfrak{f}$ in $\otimes_{\mathfrak{f}}$. By the tensor identity we have a commutative diagram

$$
\begin{array}{c}
\hat{Z}_s((2(p^s-1)\rho-r\lambda) \otimes H^0(r\lambda)) \\
\downarrow_{\sim} \\
\hat{Z}_s(2(p^s-1)\rho-r\lambda) \otimes H^0(r\lambda)
\end{array}
\xrightarrow{ev \otimes ev} 
\begin{array}{c}
\hat{Z}_s((2(p^s-1)\rho-r\lambda) \otimes H^0(r\lambda)) \\
\downarrow_{\sim} \\
\hat{Z}_s(2(p^s-1)\rho-r\lambda) \otimes H^0(r\lambda)
\end{array}
\xrightarrow{ev \otimes ev}
\hat{Z}_s(2(p^s-1)\rho-r\lambda) \otimes H^0(r\lambda)
$$

As $ev : H^0(r\lambda) \to r\lambda$ is surjective and as $\hat{Z}_s$ is exact, $ev \otimes ev$ is surjective, hence $\pi \circ ev \otimes ev$ is surjective. On the other hand,

$$
G_sB\text{Mod}(\hat{Z}_s((2(p^s-1)\rho-r\lambda) \otimes H^0(r\lambda)), $t) \\
\simeq G_sB\text{Mod}(\hat{Z}_s(r\lambda)^* \otimes H^0(r\lambda), $t)
$$

is $G_sB\text{Mod}(H^0(r\lambda), \hat{Z}_s(r\lambda))$, by the Frobenius reciprocity

\begin{equation}
\simeq t.
\end{equation}

If $Tr : \text{Mod}_t(\hat{Z}_s(r\lambda), \hat{Z}_s(r\lambda)) \to t$ is the trace map, the composite

$$
\hat{Z}_s(\lambda)^* \otimes H^0(r\lambda) \xrightarrow{ev} t
$$

also belongs to $G_sB\text{Mod}(\hat{Z}_s(r\lambda)^* \otimes H^0(r\lambda), $t)$, where $\text{res}_{r\lambda}$ is the restriction from $G$ to $G_sB$. Take $s$ so large that $(r\lambda, \alpha^\vee) < p^s$ for all simple root $\alpha$. Then $\text{res}_{r\lambda} : H^0(r\lambda) \to \hat{Z}_s(r\lambda)$ is injective, hence $Tr \circ (\hat{Z}_s(r\lambda)^* \otimes \text{res}_{r\lambda}) \neq 0$. It follows that

$$
\pi \circ ev \otimes ev = Tr \circ (\hat{Z}_s(r\lambda)^* \otimes \text{res}_{r\lambda}) \quad \text{up to } t^\chi.
$$

**Proposition.** Assume $p \geq 2(h-1)$. If $0 \leq (\nu + \rho, \alpha^\vee) < p^s$ for each simple root $\alpha$, then $H^0(\pi \circ ev \otimes ev) : H^0(\hat{Z}_s(2(p^s-1)\rho-\nu) \otimes H^0(\nu)) \to t$ is surjective.

**Proof.** By the argument above it is enough to show $H^0(\text{Tr} \circ (\hat{Z}_s(\nu)^* \otimes \text{res}_\nu)) : H^0(\hat{Z}_s(\nu)^* \otimes H^0(\nu)) \to t$ is surjective. By the hypothesis on $\nu$ we have from [J, II.11.13]

\begin{equation}
\text{hd}_G H^0(2(p^s-1)\rho) \simeq t \simeq \text{hd}_G H^0(2(p^s-1)\rho)
\end{equation}

and that the restriction

$$
\text{res}_{2(p^s-1)\rho-\nu} : H^0(2(p^s-1)\rho-\nu) \to \hat{Z}_s(2(p^s-1)\rho-\nu)
$$

is surjective.

On the other hand, $\text{res}_\nu : H^0(\nu) \to \hat{Z}_s(\nu)$ is injective. As $G_sB\text{Mod}(\hat{Z}_s(\nu)^* \otimes H^0(\nu), $t) \simeq t$, there is a commutative diagram up to $t^\chi$

$$
\begin{array}{c}
H^0(\nu)^* \otimes H^0(\nu) \\
\downarrow_{\text{res}_\nu \otimes H^0(\nu)} \\
\hat{Z}_s(\nu)^* \otimes H^0(\nu) \\
\downarrow_{\text{res}_\nu} \\
\hat{Z}_s(\nu)^* \otimes \hat{Z}_s(\nu)
\end{array}
\xrightarrow{\text{Tr}} 
\begin{array}{c}
G_sB\text{Mod}(\hat{Z}_s(\nu)^* \otimes H^0(\nu), $t)
\end{array}
\xrightarrow{\text{Tr}} 
\begin{array}{c}
t
\end{array}
$$

Hence we have only to show that $H^0(\text{Tr} \circ (\hat{Z}_s(\nu)^* \otimes H^0(\nu)))$ is surjective.
As \(G_sB\text{Mod}(\hat{Z}_s(\nu)^* \otimes \mathbb{H}^0(\nu), \not\in) \simeq \mathfrak{t}\) again, we have a commutative diagram in \(G_sB\text{Mod}\)

\[
\begin{array}{ccc}
\hat{Z}_s(\nu)^* \otimes \mathbb{H}^0(\nu) & \xrightarrow{\text{Tr}} & \mathfrak{t} \\
\text{res}^* \otimes \mathbb{H}^0(\nu) & \downarrow \sim & \\
\hat{Z}_s(2(p^s - 1)\rho - \nu) \otimes \mathbb{H}^0(\nu) & \xrightarrow{\pi} & \hat{Z}_s(2(p^s - 1)\rho)
\end{array}
\]

where the bottom horizontal map is the cup product surjective by Mathieu's theorem [M] (cf. also [K98b]). Moreover, if \(\pi_G: \mathbb{H}^0(2(p^s - 1)\rho) \to \text{hd}_{G_sB}(2(p^s - 1)\rho)\) is the quotient morphism, we have from (8) a commutative diagram

\[
\begin{array}{ccc}
\mathbb{H}^0(2(p^s - 1)\rho) & \xrightarrow{\text{res}_{2(p^s - 1)\rho}} & \hat{Z}_s(2(p^s - 1)\rho) \\
\pi_G & \downarrow \sim & \\
\text{hd}_{G_sB}(2(p^s - 1)\rho)
\end{array}
\]

Hence taking \(\mathbb{H}^0(\cdot)\) of (9) yields a commutative diagram

\[
\begin{array}{ccc}
\mathbb{H}^0(\hat{Z}_s(\nu)^* \otimes \mathbb{H}^0(\nu)) & \xrightarrow{\pi_G \circ (\text{res}_G^* \otimes \mathbb{H}^0(\nu))} & \mathfrak{t} \\
\mathbb{H}^0(\text{Tr}(\text{res}_G^* \otimes \mathbb{H}^0(\nu))) & \downarrow \pi_G & \\
\mathbb{H}^0(2(p^s - 1)\rho - \nu) \otimes \mathbb{H}^0(\nu) & \xrightarrow{\text{hd}_{\mathbb{H}^0(\mathbb{H}^0(\nu))}^0} & \mathbb{H}^0(2(p^s - 1)\rho)
\end{array}
\]

It follows that \(\mathbb{H}^0(\text{Tr} \circ (\text{res}_G^* \otimes \mathbb{H}^0(\nu))) \neq 0\), as desired.

References


