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A note on the D-affinity of the flag variety in positive characteristic

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Let G be a simply connected simple algebraic group over an algebraically closed field $\mathfrak{k}$ and let B be a Borel subgroup of G. Let $\mathfrak{X} = G/B$, $\mathcal{D}_{\mathfrak{X}}$ the sheaf of $\mathfrak{k}$-algebras of differential operators on $\mathfrak{X}$, $\mathcal{D}_{\mathfrak{X}}\mathbf{qc}$ the category of left $\mathcal{D}_{\mathfrak{X}}$-modules that are quasi-coherent over the structure sheaf $\mathcal{O}_{\mathfrak{X}}$ of $\mathfrak{X}$, $\mathcal{D}(\mathfrak{X}) = \Gamma(\mathfrak{X}, \mathcal{D}_{\mathfrak{X}})$ the $\mathfrak{k}$-algebra of differential operators on $\mathfrak{X}$, and $\mathcal{D}(\mathfrak{X})\mathbf{Mod}$ the category of left $\mathcal{D}(\mathfrak{X})$-modules. We say $\mathfrak{X}$ is D-affine iff for each $\mathcal{M} \in \mathcal{D}_{\mathfrak{X}}\mathbf{qc}$ (i) the natural morphism $\mathcal{D}_{\mathfrak{X}} \otimes_{\mathcal{D}(\mathfrak{X})} \Gamma(\mathfrak{X}, \mathcal{M}) \to \mathcal{M}$ is epic, and (ii) $\mathrm{H}^i(\mathfrak{X}, \mathcal{M}) = 0 \ \forall i > 0$; equivalently, the functor $\Gamma(\mathfrak{X}, ?) : \mathcal{D}_{\mathfrak{X}}\mathbf{qc} \to \mathcal{D}(\mathfrak{X})\mathbf{Mod}$ gives an equivalence of categories with quasi-inverse $\mathcal{D}_{\mathfrak{X}} \otimes_{\mathcal{D}(\mathfrak{X})} ?$ (cf. [K98a, 1.6]).

In characteristic 0 a celebrated theorem of Beilinson and Bernstein [BB] affirms that $\mathfrak{X}$ is D-affine. In positive characteristic B. Haastert [H87, 4.4.1] shows that in (i) even the natural morphism

(1) $$\mathcal{O}_{\mathfrak{X}} \otimes_{\mathfrak{k}} \Gamma(\mathfrak{X}, \mathcal{M}) \to \mathcal{M} \text{ is epic.}$$

Then by Grothendieck's vanishing theorem (ii) will hold if $\mathrm{H}^i(\mathfrak{X}, \mathcal{D}_{\mathfrak{X}}) = 0 \ \forall i > 0$. If $(\mathit{Diff}_m)_{m \in \mathbb{N}}$ is the standard filtration of $\mathcal{D}_{\mathfrak{X}}$, however, [H87, 4.2.7] shows that if $p = \mathrm{ch}\mathfrak{k} > h$ the Coxeter number of G and if G is not of type A_1, then

(2) $$\mathrm{H}^i(\mathfrak{X}, \mathit{Diff}_p) \neq 0 \quad \text{for some } i \neq 0.$$

And yet there is another filtration, called the p-filtration, on $\mathcal{D}_{\mathfrak{X}}$. If $\mathcal{O}_{\mathfrak{X}}^{(r)}$ is the sheaf of $\mathfrak{k}$-algebras such that $\Gamma(\mathfrak{U}, \mathcal{O}_{\mathfrak{X}}^{(r)}) = \{a^{p^r} | a \in \Gamma(\mathfrak{U}, \mathcal{O}_{\mathfrak{X}})\}$ for each open $\mathfrak{U}$ of $\mathfrak{X}$ and if $\mathcal{D}_r = \mathcal{M}od_{\mathcal{O}_{\mathfrak{X}}^{(r)}}(\mathcal{O}_{\mathfrak{X}}, \mathcal{O}_{\mathfrak{X}})$, then $\mathcal{D}_{\mathfrak{X}} = \cup_{r \in \mathbb{N}} \mathcal{D}_r$. As $\mathfrak{X}$ is noetherian,

(3) $$\mathrm{H}^{\cdot}(\mathfrak{X}, \mathcal{D}_{\mathfrak{X}}) \simeq \varinjlim_r \mathrm{H}^{\cdot}(\mathfrak{X}, \mathcal{D}_r).$$

Let $G_r = \ker F^r$ with $F^r : G \to G^{(r)}$ the r-th Frobenius morphism [J, I.9], $\hat{Z}_r$ the induction functor from the category $B\mathbf{Mod}$ of B-modules to the category $G_rB\mathbf{Mod}$ of G_rB-modules [J, I.3], and let $\mathcal{L}$ be the functor from $B\mathbf{Mod}$ to the category of G-equivariant $\mathcal{O}_{\mathfrak{X}}$-modules [J, I.5]. Then by [H87, 4.3.3]

(4) $$\mathcal{D}_r \simeq \mathcal{L}(\hat{Z}_r(\mathfrak{k})^*) \simeq \mathcal{L}(\hat{Z}_r(2(p^r - 1)\rho)),$$

where $\rho = \frac{1}{2}\sum_{\alpha \in R^+} \alpha$ with R^+ the positive system of roots of G such that the roots of B are $-R^+$. If $G = SL_2$ or SL_3, then the composition factors of $\hat{Z}_r(2(p^r-1)\rho)$ in $G_rB\mathbf{Mod}$ have all dominant highest weights [H87, 4.5.4], hence $\mathrm{H}^i(\mathfrak{X}, \mathcal{D}_r) = 0 \ \forall i > 0$ by Kempf's vanishing theorem, showing $\mathfrak{X}$ is D-affine in those cases. The argument unfortunately does not generalize.

There is another criterion for $\mathfrak{X}$ to be D-affine [Ka, Th. 1.4.1]: $\mathfrak{X}$ is D-affine iff there is a dominant weight λ such that for all $r >> 0$ the natural morphism

$$\mathcal{D}_{\mathfrak{X}} \otimes_{\mathcal{O}_{\mathfrak{X}}} \mathcal{L}(-r\lambda) \otimes_{\mathfrak{k}} \mathrm{H}^0(r\lambda) \to \mathcal{D}_{\mathfrak{X}} \tag{1}$$

splits as a morphism of sheaves of abelian groups, where $\mathrm{H}^0(?) = \mathrm{H}^0(\mathfrak{X}, \mathcal{L}(?)) = \Gamma(\mathfrak{X}, \mathcal{L}(?))$. If $\mathrm{Dist}(G)$ (resp. $\mathrm{Dist}(B)$) is the algebra of distributions on G (resp. B), the natural morphism (5) can be described by the commutative diagram

$$\begin{array}{ccc} \mathcal{D}_{\mathfrak{X}} \otimes_{\mathcal{O}_{\mathfrak{X}}} \mathcal{L}(-r\lambda) \otimes_{\mathfrak{k}} \mathrm{H}^0(r\lambda) & \longrightarrow & \mathcal{D}_{\mathfrak{X}} \\ \wr\big| & & \big|\wr \\ \mathcal{L}(\mathrm{Dist}(G) \otimes_{\mathrm{Dist}(B)} (-r\lambda) \otimes_{\mathfrak{k}} \mathrm{H}^0(r\lambda)) & \xrightarrow[\mathcal{L}(\mathrm{Dist}(G)\otimes_{\mathrm{Dist}(B)}(-r\lambda)\otimes_{\mathfrak{k}}\mathrm{ev}_{r\lambda})]{} & \mathcal{L}(\mathrm{Dist}(G)), \end{array} \tag{2}$$

where $\mathrm{ev}_{r\lambda} : \mathrm{H}^0(r\lambda) \to r\lambda$ is the evaluation at the identity element of G. In characteristic 0 the map $\mathrm{Dist}(G) \otimes_{\mathrm{Dist}(B)} (-r\lambda) \otimes_{\mathfrak{k}} \mathrm{ev}_{r\lambda}$ has been proved to split in $B\mathbf{Mod}$ so that $\mathcal{L}(\mathrm{Dist}(G) \otimes_{\mathrm{Dist}(B)} (-r\lambda) \otimes_{\mathfrak{k}} \mathrm{ev}_{r\lambda})$ splits as a morphism of G-equivariant $\mathcal{O}_{\mathfrak{X}}$-modules to show the D-affinity of $\mathfrak{X}$ [BB].

Assume in the following that $\mathrm{ch}\mathfrak{k} = p > 0$. If $\mathfrak{X}$ is D-affine, in view of $1 \in \mathcal{D}(\mathfrak{X})$ we must have for a given r the morphism

$$\mathcal{D}_s \otimes_{\mathcal{O}_{\mathfrak{X}}} \mathcal{L}(-r\lambda) \otimes_{\mathfrak{k}} \mathrm{H}^0(r\lambda) \to \mathcal{D}_s \tag{3}$$

split as a morphism of sheaves of abelian groups for $s >> 0$. By (4) the morphism (7) reads as

$$\mathcal{L}(\widehat{\mathrm{ev} \otimes_{\mathfrak{k}} \mathrm{ev}}) : \mathcal{L}(\hat{Z}_s(2(p^s-1)\rho - r\lambda) \otimes_{\mathfrak{k}} \mathrm{H}^0(r\lambda)) \to \mathcal{L}(\hat{Z}_s(2(p^s-1)\rho)),$$

where $\widehat{\mathrm{ev} \otimes_{\mathfrak{k}} \mathrm{ev}} \in G_sB\mathbf{Mod}(\hat{Z}_s(2(p^s-1)\rho - r\lambda) \otimes_{\mathfrak{k}} \mathrm{H}^0(r\lambda), \hat{Z}_s(2(p^s-1)\rho))$ is induced by the Frobenius reciprocity from $\mathrm{ev} \otimes_{\mathfrak{k}} \mathrm{ev} \in B\mathbf{Mod}(\hat{Z}_s(2(p^s-1)\rho - r\lambda) \otimes_{\mathfrak{k}} \mathrm{H}^0(r\lambda), 2(p^s-1)\rho)$ the tensor product of evaluations $\mathrm{ev}_{2(p^s-1)\rho - r\lambda} : \hat{Z}_s(2(p^s-1)\rho - r\lambda) \to 2(p^s-1)\rho - r\lambda$ and $\mathrm{ev}_{r\lambda} : \mathrm{H}^0(r\lambda) \to r\lambda$.

Now $1 \in \mathcal{D}_s$ belongs to $\mathcal{O}_{\mathfrak{X}}$ and $\mathcal{O}_{\mathfrak{X}}$ is a direct summand of $\mathcal{D}_s$ as an $\mathcal{O}_{\mathfrak{X}}$-module, in fact, as a G-equivariant $\mathcal{O}_{\mathfrak{X}}$-module, corresponding to the splitting of the quotient $\pi : \hat{Z}_s(2(p^s-1)\rho) \to \mathrm{hd}_{G_sB}\hat{Z}_s(2(p^s-1)\rho) = \mathfrak{k}$ in $B\mathbf{Mod}$. Then we should have at least the composite

$$\begin{array}{ccc} \mathrm{H}^0(\hat{Z}_s(2(p^s-1)\rho - r\lambda) \otimes \mathrm{H}^0(r\lambda)) & \dashrightarrow & \mathfrak{k} \\ & \searrow^{\mathrm{H}^0(\widehat{\mathrm{ev}\otimes_{\mathfrak{k}}\mathrm{ev}})} & \uparrow \mathrm{H}^0(\pi) \\ & & \mathrm{H}^0(\hat{Z}_s(2(p^s-1)\rho)) \end{array}$$

to be surjective, that we will verify in what follows.

We will suppress $\mathfrak{k}$ in $\otimes_{\mathfrak{k}}$. By the tensor identity we have a commutative diagram

$$\begin{array}{ccc}
\hat{Z}_s(2(p^s-1)\rho - r\lambda)\otimes_{\mathfrak{k}} \mathrm{H}^0(r\lambda) & \xrightarrow{\widehat{\mathrm{ev}\otimes\mathrm{ev}}} & \hat{Z}_s(2(p^s-1)\rho) \\
\wr\big| & \nearrow_{\hat{Z}_s((2(p^s-1)\rho - r\lambda)\otimes \mathrm{ev})} & \\
\hat{Z}_s((2(p^s-1)\rho - r\lambda)\otimes \mathrm{H}^0(r\lambda)) & &
\end{array}$$

As $\mathrm{ev}: \mathrm{H}^0(r\lambda)\to r\lambda$ is surjective and as $\hat{Z}_s$ is exact, $\widehat{\mathrm{ev}\otimes\mathrm{ev}}$ is surjective, hence $\pi\circ\widehat{\mathrm{ev}\otimes\mathrm{ev}}$ is surjective. On the other hand,

$$\begin{aligned}
G_sB\mathbf{Mod}(\hat{Z}_s(2(p^s-1)\rho - r\lambda)\otimes \mathrm{H}^0(r\lambda),\mathfrak{k}) &\simeq G_sB\mathbf{Mod}(\hat{Z}_s(r\lambda)^*\otimes \mathrm{H}^0(r\lambda),\mathfrak{k})\\
&\simeq G_sB\mathbf{Mod}(\mathrm{H}^0(r\lambda),\hat{Z}_s(r\lambda))\\
&\simeq B\mathbf{Mod}(\mathrm{H}^0(r\lambda), r\lambda) \quad \text{by the Frobenius reciprocity}\\
&\simeq \mathfrak{k}.
\end{aligned}$$

If $\mathrm{Tr}: \mathbf{Mod}_{\mathfrak{k}}(\hat{Z}_s(r\lambda),\hat{Z}_s(r\lambda))\to\mathfrak{k}$ is the trace map, the composite

$$\begin{array}{ccc}
\hat{Z}_s(\lambda)^*\otimes \mathrm{H}^0(r\lambda) & \dashrightarrow & \mathfrak{k}\\
{\scriptstyle \hat{Z}_s(\lambda)^*\otimes \mathrm{res}_{r\lambda}}\downarrow & & \uparrow{\scriptstyle \mathrm{Tr}}\\
\hat{Z}_s(r\lambda)^*\otimes \hat{Z}_s(r\lambda) & \underset{\sim}{\longrightarrow} & \mathbf{Mod}_{\mathfrak{k}}(\hat{Z}_s(r\lambda),\hat{Z}_s(r\lambda))
\end{array}$$

also belongs to $G_sB\mathbf{Mod}(\hat{Z}_s(r\lambda)^*\otimes \mathrm{H}^0(r\lambda),\mathfrak{k})$, where $\mathrm{res}_{r\lambda}$ is the restriction from G to G_sB. Take s so large that $\langle r\lambda,\alpha^\vee\rangle < p^s$ for all simple root α. Then $\mathrm{res}_{r\lambda}: \mathrm{H}^0(r\lambda)\to \hat{Z}_s(r\lambda)$ is injective, hence $\mathrm{Tr}\circ(\hat{Z}_s(r\lambda)^*\otimes \mathrm{res}_{r\lambda})\neq 0$. It follows that

$$\pi\circ\widehat{\mathrm{ev}\otimes\mathrm{ev}} = \mathrm{Tr}\circ(\hat{Z}_s(r\lambda)^*\otimes \mathrm{res}_{r\lambda}) \quad \text{up to } \mathfrak{k}^\times.$$

Proposition. *Assume $p\geq 2(h-1)$. If $0\leq\langle\nu+\rho,\alpha^\vee\rangle < p^s$ for each simple root α, then $\mathrm{H}^0(\pi\circ\widehat{\mathrm{ev}\otimes\mathrm{ev}}): \mathrm{H}^0(\hat{Z}_s(2(p^s-1)\rho-\nu)\otimes \mathrm{H}^0(\nu))\to\mathfrak{k}$ is surjective.*

Proof. By the argument above it is enough to show $\mathrm{H}^0(\mathrm{Tr}\circ(\hat{Z}_s(\nu)^*\otimes \mathrm{res}_\nu)): \mathrm{H}^0(\hat{Z}_s(\nu)^*\otimes \mathrm{H}^0(\nu))\to\mathfrak{k}$ is surjective. By the hypothesis on ν we have from [J, II.11.13]

(4) $$\mathrm{hd}_G\mathrm{H}^0(2(p^s-1)\rho)\simeq\mathfrak{k}\simeq \mathrm{hd}_{G_s}\mathrm{H}^0(2(p^s-1)\rho)$$

and that the restriction

$$\mathrm{res}_{2(p^s-1)\rho-\nu}: \mathrm{H}^0(2(p^s-1)\rho-\nu)\to\hat{Z}_s(2(p^s-1)\rho-\nu) \text{ is surjective.}$$

On the other hand, $\mathrm{res}_\nu: \mathrm{H}^0(\nu)\to\hat{Z}_s(\nu)$ is injective. As $G_sB\mathbf{Mod}(\hat{Z}_s(\nu)^*\otimes \mathrm{H}^0(\nu),\mathfrak{k})\simeq\mathfrak{k}$, there is a commutative diagram up to $\mathfrak{k}^\times$

$$\begin{array}{ccccc}
\mathrm{H}^0(\nu)^*\otimes \mathrm{H}^0(\nu) & \xleftarrow{\mathrm{res}_\nu^*\otimes \mathrm{H}^0(\nu)} & \hat{Z}_s(\nu)^*\otimes \mathrm{H}^0(\nu) & \xrightarrow{\hat{Z}_s(\nu)^*\otimes \mathrm{res}_\nu} & \hat{Z}_s(\nu)^*\otimes \hat{Z}_s(\nu).\\
& \searrow^{\mathrm{Tr}} & & {}^{\mathrm{Tr}}\swarrow & \\
& & \mathfrak{k} & &
\end{array}$$

Hence we have only to show that $\mathrm{H}^0(\mathrm{Tr}\circ(\mathrm{res}_\nu^*\otimes \mathrm{H}^0(\nu)))$ is surjective.

As $G_sB\mathbf{Mod}(Z_s(\nu)^*\otimes \mathrm{H}^0(\nu), \mathfrak{k}) \simeq \mathfrak{k}$ again, we have a commutative diagram in $G_sB\mathbf{Mod}$

$$
(5)\qquad
\begin{array}{ccc}
\mathrm{H}^0(\nu)^* \otimes \mathrm{H}^0(\nu) & \xrightarrow{\mathrm{Tr}} & \mathfrak{k} \\
\uparrow{\scriptstyle \mathrm{res}^*\otimes \mathrm{H}^0(\nu)} & & \Big|{\scriptstyle \sim} \\
\hat{Z}_s(\nu)^* \otimes \mathrm{H}^0(\nu) & & \mathrm{hd}_{G_sB}\hat{Z}_s(2(p^s-1)\rho) \\
{\scriptstyle \sim}\Big| & & \uparrow{\scriptstyle \pi} \\
\hat{Z}_s(2(p^s-1)\rho-\nu)\otimes \mathrm{H}^0(\nu) & \xrightarrow{\widehat{\mathrm{ev}\otimes\mathrm{ev}}} & \hat{Z}_s(2(p^s-1)\rho) \\
\uparrow{\scriptstyle \mathrm{res}_{2(p^s-1)\rho-\nu}\otimes \mathrm{H}^0(\nu)} & & \uparrow{\scriptstyle \mathrm{res}_{2(p^s-1)\rho}} \\
\mathrm{H}^0(2(p^s-1)\rho-\nu)\otimes \mathrm{H}^0(\nu) & \twoheadrightarrow & \mathrm{H}^0(2(p^s-1)\rho),
\end{array}
$$

where the bottom horizontal map is the cup product surjective by Mathieu's theorem [M] (cf. also [K98b]). Moreover, if $\pi_G : \mathrm{H}^0(2(p^s-1)\rho) \to \mathrm{hd}_G\mathrm{H}^0(2(p^s-1)\rho)$ is the quotient morphism, we have from (8) a commutative diagram

$$
\begin{array}{ccc}
\mathrm{H}^0(2(p^s-1)\rho) & \xrightarrow{\pi\circ\mathrm{res}_{2(p^s-1)\rho}} & \mathrm{hd}_{G_sB}\hat{Z}_s(2(p^s-1)\rho) \\
& \searrow{\scriptstyle \pi_G} & \Big|{\scriptstyle \sim} \\
& & \mathrm{hd}_G\mathrm{H}^0(2(p^s-1)\rho).
\end{array}
$$

Hence taking $\mathrm{H}^0(?)$ of (9) yields a commutative diagram

$$
\begin{array}{ccc}
\mathrm{H}^0(\hat{Z}_s(\nu)^*\otimes \mathrm{H}^0(\nu)) & \xrightarrow{\mathrm{H}^0(\mathrm{Tr}\circ(\mathrm{res}_\nu^*\otimes \mathrm{H}^0(\nu)))} & \mathfrak{k} \\
\uparrow{\scriptstyle \mathrm{H}^0(\mathrm{res}_\nu^*\otimes \mathrm{H}^0(\nu))} & & \uparrow{\scriptstyle \pi_G} \\
\mathrm{H}^0(2(p^s-1)\rho-\nu)\otimes \mathrm{H}^0(\nu). & \twoheadrightarrow & \mathrm{H}^0(2(p^s-1)\rho)
\end{array}
$$

It follows that $\mathrm{H}^0(\mathrm{Tr}\circ(\mathrm{res}_\nu^*\otimes \mathrm{H}^0(\nu))) \neq 0$, as desired.

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