数理解析研究所講究録
1149 巻 2000 年 1-15

Tensor products of representations of the symmetric groups and related groups[1]

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1 Introduction

An important problem in the representation theory of a finite group G over a field K is the computation of tensor products, i.e. given two KG-modules V and W, we consider the tensor product $V \otimes_K W$ with the group G acting diagonally. In general, it is very hard to determine such tensor products and only little information is known. For K a field of characteristic 0, it suffices to study the pointwise product $\chi_V \cdot \chi_W$ of the two corresponding characters, sometimes also called *Kronecker product.*

In the representation theory of finite groups and in applications the representations of the symmetric groups S_n and related groups have always played a special rôle. In particular, in many contexts the decomposition of tensor products of irreducible representations of such groups is of great interest. The description of the decomposition of such products is a central open problem even at characteristic 0. In the past 15 years a number of results have been obtained for computing the Kronecker product for special characters, for determining the multiplicity of special constituents, or for restricting the set of possible constituents. Such work was motivated from different sources, e.g. by the study of polynomial identities by Regev and his school or by the investigation of multiply transitive subgroups of S_n in the work of Saxl; algebraic combinatorialists have been interested in this problem because of its connection with symmetric functions. It is also

[1]This article is an extended version of the talk given at the conference.

of relevance to applications in chemistry and physics, as is evident e.g. from the number of papers on Kronecker products appearing in physics journals.

For K a field of characteristic p, i.e. for p-modular representations, the problem is much harder. In this case, even computing the tensor product with the sign representation is difficult; in fact, a combinatorial description conjectured by Mullineux in 1979 was only proved in recent years by the work of Kleshchev [K] and Ford-Kleshchev [FK], see also [BO].

In the following sections some recent work is described which started with the question of classifying irreducible Kronecker products for S_n. Going beyond the original question, in joint work with A. Kleshchev Kronecker products of complex S_n-characters with few different irreducible constituents were classified and as a consequence, homogeneous Kronecker products of A_n-characters, i.e. those with only one irreducible constituent (up to multiplicity), were characterized [BK]. Next, the Kronecker product problem is considered for the double cover $\widetilde{S}_n$ of the symmetric group S_n. Homogeneous Kronecker products of spin characters are characterized and families of homogeneous and almost homogeneous mixed products are described. Finally, we mention some recent progress on modular tensor products for S_n.
For detailed proofs of these results the reader is referred to [BK] resp. forthcoming papers.

2 Kronecker products for S_n and A_n at characteristic 0

2.1 Setup and some known results

First we recall the classification of the irreducible characters of S_n which was already achieved by Frobenius.

A *partition* $\lambda = (\lambda_1, \ldots, \lambda_l)$ of a natural number n is a weakly decreasing sequence $\lambda_1 \geq \ldots \geq \lambda_l > 0$ of integers with $\sum_{i=1}^{l} \lambda_i = n$, for short we write: $\lambda \vdash n$. The integer

$l = l(\lambda)$ is the *length* of λ, the numbers λ_i are the *parts* of λ. The partition is also written exponentially as $\lambda = (l_1^{a_1}, \ldots, l_m^{a_m})$, $l_1 > \ldots > l_m > 0$. We let $p(n)$ denote the number of partitions of n.

The irreducible complex characters of S_n are *naturally* labelled by partitions of n [JK]. We denote the complex irreducible character labelled by the partition λ by $[\lambda]$, and the set of irreducible characters is denoted by $\mathrm{Irr}(S_n) = \{[\lambda] \mid \lambda \vdash n\}$.

The character values can be computed by a combinatorial recursion formula, the well-known Murnaghan-Nakayama formula, which shows in particular that the character values are all integers.

We can now formulate our central problem on Kronecker products of complex characters of S_n:

Problem. Let μ and ν be partitions of n. Determine the coefficients $c^{\lambda}_{\mu,\nu} \in \mathbb{N}_0$ in the expansion

$$[\mu] \cdot [\nu] = \sum_{\lambda \vdash n} c^{\lambda}_{\mu,\nu} [\nu] \, .$$

Of course, one may compute the coefficients by "brute force", i.e. using the character inner product. But above, "determine" means to give an (effective) combinatorial algorithm for computing the coefficients.

Let us consider the easiest two cases. For the trivial character $[n]$ and a partition μ of n we have just

$$[\mu] \cdot [n] = [\mu] \, .$$

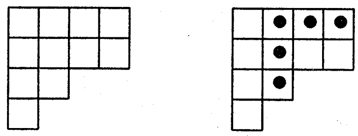
The first non-trivial case is the tensor product with the sign representation sgn $= [1^n]$. Here we have for an arbitrary $\mu \vdash n$:

$$[\mu] \cdot \mathrm{sgn} \ = [\mu] \cdot [1^n] = [\mu']$$

where the *conjugate partition* μ' is obtained from μ by reflecting its *Young diagram* in the main diagonal.

Recall that for $\lambda = (\lambda_1, \ldots, \lambda_l) \vdash n$, its *Young diagram* $Y(\lambda)$ has λ_i boxes in row i, for $i = 1, \ldots, n$. We will also need the notion of *hooks* in λ. The (i,j)-*hook* $H_{i,j}$ in λ consists of the box at position (i,j) (using matrix notation) together with all boxes in $Y(\lambda)$ to the right and below. The *hooklength* $h_{i,j} = h^\lambda_{i,j}$ counts the number of boxes in $H_{i,j}$. We illustrate these notions by an example.

Example. For $\lambda = (4^2, 2, 1)$, its Young diagram $Y(\lambda)$ is shown to the left, then the Young diagram with the (1,2)-hook $H_{1,2}$ indicated; here, $h_{1,2} = 5$.



As has already been mentioned in the introduction, there are many partial results known on the Kronecker product problem. In 1938, Murnaghan introduced the so-called reduced notation (or: n-independent notation) and proved a number of formulae based on this; further progress on this has been obtained in recent years (see [STW]). Littlewood [L] provided in 1956 a reduction argument, using the Littlewood-Richardson rule for computing outer tensor products and Frobenius reciprocity. In 1981, new impetus was provided by the book of James and Kerber [JK], which contained tables for the decomposition of Kronecker products for S_n up to $n = 8$. In recent times, several software packages have been developed that also allow the computation of Kronecker products. Apart from comparably big systems that have been developed for computations in group and representation theory like GAP and MAGMA which also provide character tables for S_n, there are the specialized MAPLE packages ACE (Algebraic Combinatorics Environment) by S. Veigneau et al. and SF (Symmetric functions) by J. Stembridge which are useful for computing Kronecker products.

In several papers the coefficients $c^\lambda_{\mu,\nu}$ are computed for special partitions μ, ν, in particular for hook partitions, 2-part partitions and special rectangular partitions, resp. for special $\lambda \vdash n$, notably those of small depth $n - \lambda_1$, see [CM], [G], [GR], [R], [RW], [V].

In some investigations, the focus was on computing tensor squares; for the constituents of squares slightly more information is available (see [S], [Z1], [Z2], [Z3], [MM]).

2.2 Irreducible Kronecker products for S_n and bounds

The starting point of the joint work with A. Kleshchev described in the next section was the following statement in [JK]:

"The inner tensor product of two ordinary irreducible representations of S_n is an ordinary representation of S_n which is in general reducible (...)."

Our original aim was to classify the irreducible products; in fact our methods gave much more. We learned only later that in fact, the problem of classifying the irreducible products had already been solved by Zisser [Z2].

Theorem 2.1 *[Z2] Let μ and ν be partitions of n. Then the Kronecker product $[\mu]\cdot[\nu]$ is irreducible if and only if one of μ, ν is (n) or (1^n).*

In other words, the tensor product of two irreducible S_n-representations at characteristic 0 is irreducible only in the two trivial cases mentioned before, namely when a representation is tensored by a 1-dimensional representation.

Zisser's proof only uses the following two facts. First, the complex characters are real-valued so that the scalar product of the characters can be rewritten as

$$([\mu]\cdot[\nu],[\mu]\cdot[\nu]) = ([\mu]^2,[\nu]^2)\ .$$

We may assume that $n \geq 4$, since for $n \leq 3$ the assumption is easily checked. Now the squares of all non-linear characters have both $[n]$ and $[n-2,2]$ as constituents; in fact, the multiplicity of $[n-2,2]$ in squares is known explicitly [S]. Hence the scalar product above is at least 2 and thus the product $[\mu]\cdot[\nu]$ is reducible.

The approach taken in [BK] started out with the following results by Dvir [D] resp. Clausen and Meier [CM] describing the rectangular hull of the partition labels of the constituents in $[\mu]\cdot[\nu]$ and the 'high' constituents. Below, $\mu\cap\nu$ denotes the partition obtained by intersecting the corresponding Young diagrams.

Theorem 2.2 *[D], [CM]. Let μ, ν be partitions of n. Then*

$$\begin{aligned}\max\{\lambda_1 \mid c^\lambda_{\mu,\nu}\neq 0 \text{ for some } \lambda=(\lambda_1,\lambda_2,\ldots)\} &= |\mu\cap\nu| \\ \max\{m \mid c^\lambda_{\mu,\nu}\neq 0 \text{ for some } \lambda=(\lambda_1\geq\ldots\geq\lambda_m>0)\} &= |\mu\cap\nu'|\end{aligned}$$

For partitions $\alpha\subseteq\beta$ (the inclusion meaning the inclusion of the corresponding Young diagrams), we denote by $[\beta/\alpha]$ the skew character corresponding to the skew diagram $\beta\setminus\alpha$ (see [JK]).

Theorem 2.3 *[D], [CM]. Let μ, ν and $\lambda=(\lambda_1,\lambda_2,\ldots)$ be partitions of n, and set $\widehat{\lambda}=(\lambda_2,\lambda_3,\ldots)$, $\gamma=\mu\cap\nu$. If $\lambda_1=|\mu\cap\nu|$, then*

$$c^\lambda_{\mu,\nu}=([\mu/\gamma]\cdot[\nu/\gamma],[\widehat{\lambda}])\,.$$

Note that this gives a recursion rule as the skew characters $[\mu/\gamma]$ and $[\nu/\gamma]$ can be computed by the Littlewood-Richardson rule.
There is also a dual result to Theorem 2.3 describing the multiplicity of constituents with partition labels of maximal length.

Now the following crucial result holds:

Theorem 2.4 *[BK] Let μ, ν be partitions of n, both different from (n) and (1^n). If $[\lambda]$ is a constituent of $[\mu]\cdot[\nu]$, then for the maximal hook length in λ we have*

$$h^\lambda_{11}<|\mu\cap\nu|+|\mu\cap\nu'|-1\,.$$

In particular, this means that for a product of two non-linear irreducible characters the rectangular hull of the labels of the constituents is not spanned by one single character label.

2.3 Kronecker products with few homogeneous components

Instead of considering only irreducible products, the previous result allows to classify also homogeneous products, i.e. those which are multiples of an irreducible character. More generally, we were interested in the situation where the product has few homogeneous components, i.e. few different irreducible constituents. The motivation for this was to obtain a classification of homogeneous products also in the case of A_n.

The results are collected in the following theorem; part (iii) was stated as a conjecture in [BK] but has been proved in the meantime.

Theorem 2.5 *[BK] Let μ and ν be partitions of n, and let r be the number of homogeneous components of the Kronecker product $[\mu]\cdot[\nu]$. Then*

(i) $r = 1$ if and only if one of the partitions μ, ν is (n) or (1^n) (and in this case the product is irreducible).

(ii) $r = 2$ if and only if one of the partitions μ, ν is a rectangle (a^b) with $a, b > 1$, and the other one is $(n-1,1)$ or $(2,1^{n-2})$. In these cases we have :

$$\begin{aligned} [n-1,1]\cdot[a^b] &= [a+1,a^{b-2},a-1]+[a^{b-1},a-1,1]\ , \\ [2,1^{n-2}]\cdot[a^b] &= [b+1,b^{a-2},b-1]+[b^{a-1},b-1,1]\ . \end{aligned}$$

(iii) $r = 3$ if and only if $n = 3$ and $\mu = \nu = (2,1)$ or $n = 4$ and $\mu = \nu = (2^2)$.

Remarks. Part (i) follows immediately from Theorem 2.4. For part (ii), a much more detailed analysis of the product is required (see [BK]). Note that a weaker version of (ii) was also obtained by Zisser [Z2]. For part (iii), the methods in [BK] have been refined and carried further.

The computer experiments also led to the following conjecture stated in [BK], where the 'if'-part is proved, describing the products explicitly:

Conjecture (notation as above) $r = 4$ if and only if one of the following holds:

(a) $n \geq 4$ and $\mu, \nu \in \{(n-1,1),(2,1^{n-2})\}$;

(b) $n = 2k+1$ for some $k \geq 2$, and one of μ, ν is in $\{(2k,1),(2,1^{2k-1})\}$ while the other one is in $\{(k+1,k),(2^k,1)\}$;

(c) $n = 6$ and $\mu, \nu \in \{(2^3),(3^2)\}$.

While it seems hopeful to find a proof of this conjecture, from the computer calculations it seems that there might not be a good characterization of products with r components, for general r. Also, for $r \leq 4$, all the products above have been found to be multiplicity-free; for $r = 5$ we have $[3,2] \cdot [3,1^2]$ as an example with 5 components and $[3,1^2]$ occurring with multiplicity 2.

Apart from the numerical investigations, further evidence for the conjecture above is given by the following result:

Theorem 2.6 *[BK] Let μ and ν be symmetric partitions of n. Then $[\mu] \cdot [\nu]$ never has exactly 4 homogeneous components.*

2.4 Homogeneous Kronecker products of A_n-characters

We first recall the classification of the complex irreducible A_n-characters (see [JK]). If μ is a non-symmetric partition of n, i.e. $\mu \neq \mu'$, then the restriction $[\mu]_{A_n}$ is again irreducible and it coincides with $[\mu']_{A_n}$. We denote the corresponding irreducible A_n-character by $\{\mu\} = \{\mu'\}$. If μ is symmetric, i.e. $\mu = \mu'$, then $[\mu]_{A_n}$ is a sum of two different irreducible A_n-characters, which we denote by $\{\mu\}_+$ and $\{\mu\}_-$; these characters are conjugate via a transposition in S_n. Then the set of irreducible characters of A_n is

$$\mathrm{Irr}(A_n) = \{\{\mu\}_+, \{\mu\}_- \mid \mu \vdash n, \mu = \mu'\} \cup \{\{\mu\} \mid \mu \vdash n, \mu \neq \mu'\}$$

We can now state the classification of the homogeneous Kronecker products of irreducible A_n-characters. Of course, we obtain irreducible products when one of the characters is of degree 1. For $n > 4$ the only 1-dimensional character is the trivial one. For $n = 3$ and 4 we also have the 1-dimensional characters $\{2,1\}_\pm$ and $\{2^2\}_\pm$.

The theorem below gives a family of non-trivial irreducible Kronecker products (see also [Z2] for an earlier weaker result).

Theorem 2.7 *[BK] Let ϕ, ψ be irreducible A_n-characters of degree greater than* 1. *Then $\phi \cdot \psi$ is homogeneous if and only if $n = a^2$ for some $a > 2$ and one of the characters is $\{n-1, 1\}$, while the other is $\{a^a\}_+$ or $\{a^a\}_-$. In the exceptional case we have:*

$$\{n-1, 1\} \cdot \{a^a\}_\pm = \{a+1, a^{a-2}, a-1\} .$$

3 Kronecker products of characters of $\widetilde{S}_n$

Let $n \geq 4$, and let $\widetilde{S}_n$ be one of the two double covers of S_n (except for $n = 6$ they are non-isomorphic); so $\widetilde{S}_n$ is a non-split extension of S_n by a central subgroup $\langle z \rangle$ of order 2. As the representation theory of the two double covers is 'the same' for all representation theoretical purposes, the choice does not matter below.

Now $\widetilde{S}_n$ has as irreducible complex characters the (non-faithful) irreducible characters lifted from S_n and the faithful characters, which are called *spin characters.* For the classification of the irreducible spin characters of the double cover $\widetilde{S}_n$ we have to introduce some notation.

The set of partitions of n into odd parts only is denoted by $O(n)$, and the set of partitions of n into distinct parts is denoted by $D(n)$. We write $D^+(n)$ resp. $D^-(n)$ for the sets of partitions λ in $D(n)$ with $n - l(\lambda)$ even resp. odd; the partition λ is then also called even resp. odd. The conjugacy classes of S_n which split in $\widetilde{S}_n$ (i.e. when g and gz are not conjugate) are labelled by the set $O(n) \cup D^-(n)$.

The associate classes of spin characters of $\widetilde{S}_n$ are labelled canonically by the partitions in $D(n)$. For each $\lambda \in D^+(n)$ there is a self-associate spin character $\langle \lambda \rangle = \operatorname{sgn} \langle \lambda \rangle$, and to each $\lambda \in D^-(n)$ there is a pair of associate spin characters $\langle \lambda \rangle, \langle \lambda \rangle' = \operatorname{sgn} \langle \lambda \rangle$. We write $\langle \lambda \rangle^o$ for a choice of associate, and

$$\widehat{\langle \lambda \rangle} = \begin{cases} \langle \lambda \rangle & \text{if } \lambda \in D^+(n) \\ \langle \lambda \rangle + \langle \lambda \rangle' & \text{if } \lambda \in D^-(n) \end{cases} .$$

The values of the spin characters on classes of type $O(n)$ can be computed by a spin analogue of the Murnaghan-Nakayama formula which is due to Morris. The values on the D^--classes are given explicitly in terms of the parts of the labelling partition λ.

3.1 Classification of homogeneous spin products

The theorem of Dvir resp. Clausen and Meier can be stated in short form by saying that the rectangular hull of (the partition labels of the constituents of) $[\mu]\cdot[\nu]$ is the rectangular partition $(|\mu\cap\nu|^{|\mu\cap\nu'|})$. The spin analogue of this result is slightly more complicated.

Theorem 3.1 *Let $n \geq 4$, and let μ, $\nu \in D(n)$.*

(a) Let $\mu = \nu \in D^-(n)$.
If $n - l(\mu) \equiv 1 \bmod 4$, then the rectangular hull of $\langle\mu\rangle \cdot \langle\mu\rangle$ is $((n-1)^n)$, unless $n = \binom{k+1}{2}$ is a triangular number and $\mu = (k, k-1, \ldots, 2, 1)$, when the rectangular hull is $((n-2)^n)$.
If $n - l(\mu) \equiv 3 \bmod 4$, then the rectangular hull of $\langle\mu\rangle\cdot\langle\mu\rangle$ is (n^{n-1}), unless $n = \binom{k+1}{2}$ is a triangular number and $\mu = (k, k-1, \ldots, 2, 1)$, when the rectangular hull is (n^{n-2}).

(b) If we are not in the situation described in (a), then the rectangular hull of $\langle\mu\rangle \cdot \langle\nu\rangle$ (and all associate products) is $(|\mu\cap\nu|^{|\mu\cap\nu|})$.

Theorem 3.2 *Let $n \geq 4$, μ, $\nu \in D(n)$. Then $\langle\mu\rangle \cdot \langle\nu\rangle$ is homogeneous if and only if n is a triangular number, say $n = \binom{k+1}{2}$, one of μ, ν is (n) and the other one is $(k, k-1, \ldots, 2, 1)$. In this case, we have*

$$\langle n\rangle \cdot \langle k, k-1, \ldots, 2, 1\rangle = 2^{a(k)}[k, k-1, \ldots, 2, 1]\,,$$

where

$$a(k) = \begin{cases} \frac{k-2}{2} & \text{if } k \text{ is even} \\ \frac{k-1}{2} & \text{if } k \equiv 1 \bmod 4 \\ \frac{k-3}{2} & \text{if } k \equiv 3 \bmod 4 \end{cases}$$

In particular, the only irreducible products occur for $n = 6$, namely:

$$\langle 6\rangle \cdot \langle 3,2,1\rangle = \langle 6\rangle' \cdot \langle 3,2,1\rangle = \langle 6\rangle \cdot \langle 3,2,1\rangle' = \langle 6\rangle' \cdot \langle 3,2,1\rangle' = [3,2,1]\,.$$

3.2 Mixed Kronecker products of characters of $\tilde{S}_n$

In this section we describe some families of characters with homogeneous and almost homogeneous mixed products.

In [St], Stembridge provided an explicit combinatorial description of the inner tensor products $\langle n\rangle[\mu]$. The coefficient $g_{\lambda\mu}$ appearing below is the number of "shifted tableaux" S of unshifted shape μ and content λ such that the tableau word $w = w(S)$ satisfies a suitable lattice property and the leftmost i of $|w|$ is unmarked in w for $1 \le i \le l(\lambda)$ (see [St] for details). Furthermore, we set

$$\varepsilon_\lambda = \begin{cases} 1 & \text{if } \lambda \in D^+(n) \\ \sqrt{2} & \text{if } \lambda \in D^-(n) \end{cases}$$

Theorem 3.3 *([St], 9.3) Let μ be a partition of n, $\lambda \in D(n)$. We have*

$$(\langle n\rangle[\mu], \langle\lambda\rangle^o) = \frac{1}{\varepsilon_\lambda \varepsilon_{(n)}} 2^{(l(\lambda)-1)/2} g_{\lambda\mu}\,,$$

unless $\lambda = (n)$, n is even, and μ is a hook partition. In that case, the multiplicity of $\langle\lambda\rangle^o$ is 0 or 1 according to choice of associates.

Theorem 3.4 *Let $\mu \vdash n$, $\mu \neq (n), (1^n)$. Then the product $\langle n\rangle \cdot [\mu]$ is almost homogeneous, i.e. of the form $c\langle\lambda\rangle$ or $c\widehat{\langle\lambda\rangle}$ for some $\lambda \in D(n)$ and $c \in \mathbb{N}$, if and only if μ is a rectangle. In this case, if $\mu = (b^a)$ with $1 < a \le b$, then for a odd and b even we have*

$$\langle n\rangle \cdot [b^a] = \langle n\rangle' \cdot [b^a] = 2^{\frac{a-3}{2}} \widehat{\langle a+b-1, a+b-3, \cdots, b-a+1\rangle}$$

while in all other cases we have

$$\langle n\rangle \cdot [b^a] = \langle n\rangle' \cdot [b^a] = 2^{[\frac{a-1}{2}]} \langle a+b-1, a+b-3, \cdots, b-a+1\rangle\,.$$

We have found a further family of almost homogeneous mixed products which do not involve the basic spin character:

Theorem 3.5 *Let n be a triangular number, say $n = \binom{k+1}{2}$. Then*

$$\langle k, k-1, \ldots, 2, 1\rangle \cdot [n-1, 1] = \langle k+1, k-1, \widehat{k}-2, \ldots, 3, 2\rangle\,.$$

4 Tensor products for S_n at characteristic p

We now turn to *p-modular representation theory* of S_n. Let F be a field of characteristic $p > 0$. Studying tensor products of modular representations is motivated by applications in the investigation of maximal subgroups of finite groups of Lie type.
The classification of the p-modular irreducible S_n-representations is wellknown, see [JK]. A partition λ of n is called p-regular, if no part is repeated p or more times. For each p-regular partition λ there is a corresponding irreducible module, denoted by D^λ. The modules D^λ, where λ runs through the p-regular partitions of n, form a complete system of representatives for the (isomorphism classes of) irreducible FS_n-modules.

In section 2, we have discussed tensor products of complex irreducible S_n-representations; while there was no good answer for general such tensor products, at least tensoring with the sign representation was easy. At characteristic $p > 2$, even computing the tensor product with the sign repesentation was a hard problem. In 1979, Mullineux [Mu] defined a rather complicated p-analogue of conjugation for p-regular partitions and conjectured that this gave the combinatorial answer to the question on the tensor product with the sign representation for p-modular irreducible S_n-representations; so for a p-regular partition λ the Mullineux map describes the p-regular partition λ^M defined by

$$D^\lambda \otimes \mathrm{sgn} \cong D^{\lambda^M}\,.$$

Applying his branching results, Kleshchev [K] had reduced this conjecture to a purely combinatorial conjecture which was subsequently proved by him and Ford [FK]; a short proof of this combinatorial conjecture providing further insights was given in [BO].

Despite these difficulties even in the first non-trivial case, recently strong information has been obtained on modular tensor products. On the basis of their results at characteristic 2 and experimental evidence, Gow and Kleshchev conjectured the following characterization of irreducible tensor products [GK]:

Conjecture. Let D_1 and D_2 be two irreducible FS_n-module of dimensions greater than 1. Then $D_1 \otimes D_2$ is irreducible if and only if $p = 2$, $n = 2 + 4l$ for some positive integer l, one of the modules corresponds to the partition $(2l + 2, 2l)$ and the other corresponds to a partition of the form $(n - 2j - 1, 2j + 1)$, $0 \leq j < l$. Moreover, in the exeptional cases one has

$$D^{(2l+2,2l)} \otimes D^{(n-2j-1,2j+1)} \cong D^{(2l+1-j,2l-j,j+1,j)}.$$

In the meantime, a big step towards this conjecture has been taken; in particular, it has been shown that indeed irreducible tensor products can only occur at characteristic $p = 2$ and when n is even.

Acknowledgements. Some of the recent work described in this article was done during a research stay in Japan, with support by the Deutsche Forschungsgemeinschaft via the joint German-Japanese research project on 'Representation Theory of Finite Groups and Algebraic Groups' (grant JAP-115/169/0) which is gratefully acknowledged. It is a great pleasure to thank Prof. Koshitani for the invitation to the conference and to thank him and Prof. Uno for their wonderful hospitality at Chiba University and at Osaka University; my thanks also go to the coordinators of the programme, Prof. Michler and Prof. Kawanaka.

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