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INTEGRABILITY AND SUPERSYMMETRY

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ABSTRACT. Based on the $\mathcal{C}$-cohomology theory, some relations between integrability of nonlinear systems and their supersymmetric properties are established.

There exists a number of criteria for integrability of systems with infinite number of degree of freedom (though the concept of integrability itself for these systems still remains rather fuzzy). These criteria include existence of a Lax pair, zero-curvature representation, bi-Hamiltonian structure, etc. (see, for instance, [2]). Perhaps, the main characteristic property of integrable systems is that they possess infinite series of *higher symmetries* [1, 7] and, moreover, these series are generated by so-called *recursion operators* [6, 9].

For the technical details related to the subsequent exposition we also refer the reader to the papers [4, 5, 8].

1. Differential geometry of infinite prolongations

Let M be a smooth manifold and $\pi\colon E \to M$ be a smooth locally trivial fiber bundle. Denote by $\pi_k\colon J^k(\pi) \to M$ the bundle of the corresponding k-jets, including the case $k = \infty$, and by $\pi_{s,k}\colon J^s(\pi) \to J^k(\pi)$, $s \geq k$, the natural projections. For any (local) section $f \in \Gamma(\pi)$ its k-jet is denoted by $j_k(f) \in \Gamma(\pi_k)$ and any point $\theta \in J^k(\pi)$ can be represented in the form $[f]_x^k = j_k(f)|_x$, $x = \pi_k(\theta) \in M$. Let $M_f^k \subset J^k(\pi)$ be the graph of $j_k(f)$.

A *partial differential equation* of order k, $k < \infty$, is a smooth submanifold $\mathcal{E} \subset J^k(\pi)$. Let $\theta \in \mathcal{E}$ and consider the set

$$\mathcal{E}_\theta^l = \{\, \theta' = [f]_x^{k+l} \in J^{k+l}(\pi) \mid M_f^k \text{ is tangent to } \mathcal{E} \text{ with order } l \text{ at } \theta \,\}.$$

Then $\mathcal{E}^l = \bigcup_{\theta\in\mathcal{E}} \mathcal{E}_\theta^l \subset J^{k+l}(\pi)$ is called the *l-th prolongation* of $\mathcal{E}$. An equation $\mathcal{E}$ is said to be *formally integrable*, if all $\mathcal{E}^l$ are smooth manifolds and projections $\mathcal{E}^{l+1} \to \mathcal{E}^l$ are smooth fiber bundles. In the sequel, we deal with equations of such a type only.

Geometrical structure of $J^k(\pi)$ is determined by the *Cartan distribution* $\mathcal{C}^k$, where

$$\mathcal{C}_\theta^k = \{\, \text{linear span of all } T_\theta M_f^k \mid M_f^k \text{ passes through } \theta \,\},$$

$\theta \in J^k(\pi)$. This structure restricts to $\mathcal{E}$ providing the equation with a distribution $\mathcal{C}(\mathcal{E})$ whose maximal n-dimensional integral manifolds coincide with (generalized) solutions of $\mathcal{E}$.

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When passing to infinity, the distribution $\mathcal{C}^\infty = \mathcal{C}$ becomes integrable (i.e., satisfies the Frobenius integrability conditions). Moreover, the planes $\mathcal{C}_\theta$ are n-dimensional and project to M nondegenerately under $d\pi_\infty$. The same holds for its restriction to $\mathcal{E}^\infty$. Thus, any infinite prolongation $\pi_\infty \colon \mathcal{E}^\infty \to M$ is endowed with a flat connection which we call the *Cartan connection* and denote by the same symbol $\mathcal{C} \colon \mathrm{D}(M) \to \mathrm{D}(\mathcal{E}^\infty)$, where $\mathrm{D}(N)$ is the Lie algebra of vector fields on N.

2. $\mathcal{C}$-COHOMOLOGY

A general scheme of $\mathcal{C}$-cohomology construction is valid for any fiber bundle with a flat connection and is based on the *Frölicher–Nijenhuis bracket.*

2.1. The Frölicher–Nijenhuis bracket. Let $\Bbbk$ be a field of characteristic $\neq 2$ and A be a commutative associative $\Bbbk$-algebra with unit. With any A-module P we associate the module $\mathrm{D}_i(A)$ of skew-symmetric i-derivations $\underbrace{A \otimes_\Bbbk \ldots \otimes_\Bbbk A}_{i \text{ times}} \to P$ and introduce the module $\Lambda^i = \Lambda^i(A)$ of i-forms as the representative object for the functor $\mathrm{D}_i(\cdot)$: $\mathrm{D}_i(P) = \hom_A(\Lambda^i, P)$. Similar to the geometric case, one has the de Rham differential $d \colon \Lambda^i \to \Lambda^{i+1}$ with $d \circ d = 0$ and the inner product $\mathrm{i}_X \omega \in \Lambda^{i-1}$, $X \in \mathrm{D}_1(A)$, $\omega \in \Lambda^i$, with the Lie derivative defined by the infinitesimal Stokes formula $\mathrm{L}_X = [\mathrm{i}_X, d]$.[1]

One can also consider the modules $\mathrm{D}_1(\Lambda^i)$ and define the Frölicher–Nijenhuis bracket $[\![\Omega, \Theta]\!] \in \mathrm{D}_1(\Lambda^{i+j})$ of two elements $\Omega \in \mathrm{D}_1(\Lambda^i)$ and $\Theta \in \mathrm{D}_1(\Lambda^j)$ by setting

$$[\![\Omega, \Theta]\!](a) = \Omega(\Theta(a)) - (-1)^{ij}\Theta(\Omega(a)), \quad a \in A.$$

Proposition 1. *The above defined Frölicher–Nijenhuis bracket determines a Lie superalgebra structure in* $\mathrm{D}_1(\Lambda^*) = \bigoplus_{i \geq 0} \mathrm{D}_1(\Lambda^i)$, *i.e., the operation* $[\![\cdot, \cdot]\!]$ *is* $\Bbbk$-*bilinear and enjoys the identities*

$$[\![\Omega, \Theta]\!] + (-1)^{ij}[\![\Theta, \Omega]\!] = 0,$$

$$(-1)^{(i+l)j}[\![\Omega, [\![\Theta, \Xi]\!]]\!] + (-1)^{(j+i)l}[\![\Theta, [\![\Xi, \Omega]\!]]\!] + (-1)^{(l+j)i}[\![\Xi, [\![\Omega, \Theta]\!]]\!] = 0$$

for any $\Omega \in \mathrm{D}_1(\Lambda^i)$, $\Theta \in \mathrm{D}_1(\Lambda^j)$, $\Xi \in \mathrm{D}_1(\Lambda^l)$.

Note also that the inner product operations

$$\mathrm{i} \colon \mathrm{D}_1(\Lambda^i) \otimes_\Bbbk \Lambda^j \to \Lambda^{i+j-1}, \qquad \mathrm{i} \colon \mathrm{D}_1(\Lambda^i) \otimes_\Bbbk \mathrm{D}_1(\Lambda^j) \to \mathrm{D}_1(\Lambda^{i+j-1})$$

are also defined. Using these operations one can define the Lie derivative $\mathrm{L}_\Omega \omega \in \Lambda^{i+j}$ of a form $\omega \in \Lambda^j$ along the element $\Omega \in \mathrm{D}_1(\Lambda^i)$ by setting $\mathrm{L}_\Omega = [\mathrm{i}_\Omega, d]$ and then the Frölicher–Nijenhuis bracket may be redefined by $\mathrm{L}_{[\![\Omega,\Theta]\!]} = [\mathrm{L}_\Omega, \mathrm{L}_\Theta]$.

2.2. $\mathcal{C}$-cohomology. Consider now a smooth fiber bundle $\xi \colon N \to M$ with a connection $\nabla \colon \mathrm{D}_1(M) \to \mathrm{D}_1(N)$. The we can consider the connection form U_∇ as an element of $\mathrm{D}_1(\Lambda^1(N))$ defined by

$$\mathrm{i}_X(U_\nabla(f)) = X^v(f), \qquad f \in C^\infty(N), \quad X \in \mathrm{D}_1(N),$$

where the field X^v is the vertical part of X:

$$X^v_y = \{\, \text{the projection of } X_y \text{ to } T^v_y \text{ parallel to } \nabla_y \,\},$$

where $y \in M$, T^v_y is the tangent plane to the fiber of ξ passing through y, and ∇_y is the horizontal plane of the connection ∇ passing through the same point.

[1] Here and below all commutators are understood in the graded sense.

Proposition 2. *The connection form U_∇ satisfies the identity*

$$[\![U_\nabla, U_\nabla]\!] = 2R_\nabla,$$

where R_∇ is the curvature form of ∇.

From this proposition and from the graded Jacobi identity for the Frölicher–Nijenhuis bracket it follows that if the connection ∇ is flat, i.e., $R_\nabla = 0$, then the mapping

$$\partial_\nabla = [\![U_\nabla, \cdot]\!] \colon \mathrm{D}_1(\Lambda^i(N)) \to \mathrm{D}_1(\Lambda^{i+1}(N))$$

is a differential, i.e., $\partial_\nabla \circ \partial_\nabla = 0$. Below we apply this construction to the case $N = \mathcal{E}^\infty$, $\xi = \pi_\infty \colon \mathcal{E}^\infty \to M$ and $\nabla = \mathcal{C}$ being the Cartan connection.

2.3. **The case of $\mathcal{E}^\infty$.** First of all, we restrict the above constructions to the vertical part of the modules $\mathrm{D}_1(\Lambda^i)$:

$$\mathrm{D}_1^v(\Lambda^i(\mathcal{E}^\infty)) = \mathrm{D}_1^v(\Lambda^i) = \{\, \Omega \in \mathrm{D}_1^v(\Lambda^i(\mathcal{E}^\infty)) \mid \Omega(f) = 0, \forall f \in C^\infty(M)\,\}.$$

It is easily checked that

$$U_\mathcal{C} \in \mathrm{D}_1^v(\Lambda^1), \quad \partial_\mathcal{C}(\mathrm{D}_1^v(\Lambda^i)) \subset \mathrm{D}_1^v(\Lambda^{i+1}).$$

The corresponding cohomology groups are called the *$\mathcal{C}$-cohomology* of $\mathcal{E}$ and denoted by $H^i_\mathcal{C}(\mathcal{E})$.

Theorem 1. *Let $\mathcal{E}$ be a formally integrable equation. Then*

1. *The modules $H^i_\mathcal{C}(\mathcal{E})$ inherit both the Frölicher–Nijenhuis bracket and the inner product operation.*
2. *The modules $H^i_\mathcal{C}(\mathcal{E})$ act on the de Rham cohomology of the manifold $\mathcal{E}^\infty$ by Lie derivatives and inner product.*
3. *The module $H^0_\mathcal{C}(\mathcal{E})$ coincides with the Lie algebra* $\operatorname{sym}\mathcal{E}$ *of higher infinitesimal symmetries of the equation $\mathcal{E}$.*
4. *The module $H^1_\mathcal{C}(\mathcal{E})$ is identified with the equivalence classes of infinitesimal deformations of the equation structure $U_\mathcal{C}$.*
5. *The module $H^2_\mathcal{C}(\mathcal{E})$ contains obstructions for continuation of infinitesimal deformations up to formal ones.*

Let us introduce the mappings

$$d_\mathcal{C} = \mathrm{L}_{U_\mathcal{C}}, d_h = d - d_\mathcal{C} \colon \Lambda^i \to \Lambda^{i+1}$$

and call them the Cartan and the horizontal differentials respectively.

Proposition 3. *For any formally integrable equation one has:*

1. *The pair $(d_\mathcal{C}, d_h)$ forms a bicomplex structure on $\Lambda^*(\mathcal{E}^\infty) = \bigoplus_{i\geq 0}\Lambda^i(\mathcal{E}^\infty)$ with the total differential d.*
2. *The corresponding spectral sequence coincides with the Vinogradov $\mathcal{C}$-spectral sequence* [11].
3. *Due to the first statement, there is a direct sum decomposition*

$$\Lambda^i(\mathcal{E}^\infty) = \bigoplus_{p+q=i} \Lambda^q_h(\mathcal{E}^\infty) \otimes \mathcal{C}^p\Lambda(\mathcal{E}^\infty),$$

where $\Lambda_h^1(\mathcal{E}^\infty)$ *and* $\mathcal{C}^1\Lambda(\mathcal{E}^\infty)$ *are the submodules in* $\Lambda^1(\mathcal{E}^\infty)$ *spanned by the images of* d_h *and* $d_\mathcal{C}$ *respectively, while*

$$\Lambda_h^q(\mathcal{E}^\infty) = \underbrace{\Lambda_h^1(\mathcal{E}^\infty)\wedge\ldots\wedge\Lambda_h^1(\mathcal{E}^\infty)}_{q \text{ times}}, \quad \mathcal{C}\Lambda^p(\mathcal{E}^\infty) = \underbrace{\mathcal{C}\Lambda^1(\mathcal{E}^\infty)\wedge\ldots\wedge\mathcal{C}\Lambda^1(\mathcal{E}^\infty)}_{p \text{ times}}.$$

We conclude this section with a result describing $\mathcal{C}$-cohomology in two specific but important cases.

Theorem 2. *Let* π *be a vector bundle.*[2]

1. *For the "void" equation* $\mathcal{E}^\infty = J^\infty(\pi)$ *the module* $H_\mathcal{C}^*(\mathcal{E}^\infty)$ *is isomorphic to* $\mathcal{C}^*\Lambda(\mathcal{E}^\infty)\otimes\Gamma(\pi)$ (*see* [5]).
2. *Let* $\ell_\mathcal{E}$ *be the universal linearization operator corresponding to* $\mathcal{E}$ *and assume that the compatibility complex of* $\ell_\mathcal{E}$ *is of length* 2. *Then*

$$H_\mathcal{C}^*(\mathcal{E}^\infty) = \ker\ell_\mathcal{E}\oplus\operatorname{coker}\ell_\mathcal{E}.$$

(*see* [10]).

3. Passing to nonlocalities (recursion operators)

As a consequence of Theorem 1, we obtain the following

Proposition 4. *Under the assumptions of Theorem* 1 *the following statements are valid*:

1. *The module* $H_\mathcal{C}^1(\mathcal{E}^\infty)$ *is an associative algebra with respect to the inner product, the class of* $U_\mathcal{C}$ *being its unit.*
2. *The correspondence* $\mathcal{R}\colon H_\mathcal{C}^1(\mathcal{E}^\infty)\to\operatorname{End}H_\mathcal{C}^0(\mathcal{E}^\infty)$, $\mathcal{R}(\Omega)(X) = \mathrm{i}_X\Omega$, $\Omega\in H_\mathcal{C}^1(\mathcal{E}^\infty)$, $X\in H_\mathcal{C}^0(\mathcal{E}^\infty)$, *is a representation of this algebra in* $H_\mathcal{C}^0(\mathcal{E}^\infty) = \operatorname{sym}\mathcal{E}$.

Thus, having a symmetry $X\in\operatorname{sym}\mathcal{E}$ and a nontrivial element $\Omega\in H_\mathcal{C}^1(\mathcal{E}^\infty)$, we obtain a whole series of symmetries $X_i = \mathcal{R}(\Omega)^i(X)$. In other words, elements of $H_\mathcal{C}^1(\mathcal{E}^\infty)$ act as recursion operators on symmetries (cf. [9]) and may be computed by means of Theorem 2.

Remark. It can be shown that nontrivial recursion operators are identified with $\ker\ell_\mathcal{E}^{[1]}$, where $\ker\ell_\mathcal{E}^{[1]}$ acts on $\mathcal{C}^1\Lambda(\mathcal{E}^\infty)\otimes\Gamma(\pi)$, and this fact is independent of the length of the compatibility complex for $\ell_\mathcal{E}$.

The real problem that arises in solving the equation $\ell_\mathcal{E}^{[1]}\Omega = 0$ is that it usually has no nontrivial solutions, even though the initial equation $\mathcal{E}$ may possess recursion operators (e.g., the KdV equation, etc.). A way to enrich the situation is to introduce the so-called *nonlocal variables* using the theory of coverings [8].

Recall that a *covering* over an equation $\mathcal{E}$ is a smooth fiber bundle $\tau\colon W\to\mathcal{E}^\infty$ such that the total space W is endowed with an integrable $\dim M$-dimensional distribution $\tilde{\mathcal{C}}$ satisfying $d\tau(\tilde{\mathcal{C}}_\theta) = \mathcal{C}_{\tau(\theta)}$. The conceptual scheme of calculus over $\mathcal{E}^\infty$ can be almost literally implemented for coverings and, in particular, one can consider the equation $\tilde{\ell}_\mathcal{E}^{[1]}\Omega = 0$, where $\tilde{\ell}$ denotes the natural lifting of the linearization operator to the space of covering. Choosing an adequate covering is a rather well-structured procedure, though not completely formal one (see examples in [3]), which in practice leads to the desired operators.

[2]This assumption is not essential but simplifies the formulation.

Remark. In fact, solving the equation $\tilde{\ell}^{[1]}_{\mathcal{E}}\Omega = 0$, we obtain the so-called *shadows* of recursion operators (or of symmetries, when one solves $\tilde{\ell}_{\mathcal{E}}\Omega = 0$). Nevertheless, there exist regular means of reconstructing real recursion operators (resp., symmetries) from their shadows, see [4, 8].

4. The superization functor

Let us now put into correspondence to any equation $\mathcal{E}$ its "superization" $\mathcal{SE}$ — a superdifferential equation naturally associated to $\mathcal{E}$. A coordinate-free construction for $\mathcal{SE}$ is as follows. In the algebra $\Lambda^*(\mathcal{E}^\infty)$ of all differential forms on $\mathcal{E}^\infty$ consider the subalgebra $\mathcal{C}^*\Lambda(\mathcal{E}^\infty)$ consisting of forms vanishing on vector fields $\mathcal{C}_X$, $X \in \mathrm{D}(M)$, i.e., fields lifted from M to $\mathcal{E}^\infty$ by the Cartan connection (see above). Then $\mathcal{C}^*\Lambda(\mathcal{E}^\infty)$ may be considered as the function algebra on a supermanifold whose even component is $\mathcal{E}^\infty$. This supermanifold is the infinite prolongation of a *supersymmetric extension* [6] of $\mathcal{E}$ which serves the role of $\mathcal{SE}$. The equation $\mathcal{SE}$ is a *covering* over $\mathcal{E}$ and consequently elements of $\operatorname{sym}\mathcal{SE}$ are *nonlocal supersymmetries* of $\mathcal{E}$ [6, 8].

In a similar way, we can distinguish in $H^*_{\mathcal{C}}(\mathcal{E})$ the subalgebra consisting of cohomology classes whose representatives vanish on the fields $\mathcal{C}_X$, $X \in \mathrm{D}(M)$. Denote this subalgebra by $H^{*,0}_{\mathcal{C}}(\mathcal{E})$.

There exist two ways to embed the Lie superalgebra $H^{*,0}_{\mathcal{C}}(\mathcal{E})$ to $\operatorname{sym}\mathcal{E}$:

$$\mathrm{i}\colon H^{*,0}_{\mathcal{C}}(\mathcal{E}) \to \operatorname{sym}\mathcal{SE} \quad \text{and} \quad \mathrm{L}\colon H^{*,0}_{\mathcal{C}}(\mathcal{E}) \to \operatorname{sym}\mathcal{SE},$$

where, as before, i and L stand for inner product and Lie derivative respectively. Namely, for any cohomology class $[\Omega] \in H^{*,0}_{\mathcal{C}}(\mathcal{E})$ and a form $\rho \in \mathcal{C}^*\Lambda^*(\mathcal{E}^\infty)$ we set $\mathrm{i}_{[\Omega]}(\rho) = \mathrm{i}_\Omega(\rho)$ and $\mathrm{L}_{[\Omega]}(\rho) = \mathrm{L}_\Omega(\rho)$, and this definition is independent of the choice of a representative.

Theorem 3. *The Lie superalgebra* $\operatorname{sym}\mathcal{SE}$ *is a semi-direct product of* $\operatorname{im}\mathrm{i}$ *and* $\operatorname{im}\mathrm{L}$.

Thus we see that supersymmetry properties of $\mathcal{SE}$ are determined by the cohomology $H^{*,0}_{\mathcal{C}}(\mathcal{E})$. In particular, if $\mathcal{E}$ is integrable, i.e., possesses a recursion operator, then $\mathcal{SE}$ possesses the corresponding supersymmetry and vice versa.

Remark. Note that $\mathcal{SE}$ always has at least two symmetries: $\mathrm{i}_{[U_{\mathcal{C}}]}$ and $\mathrm{L}_{[U_{\mathcal{C}}]}$. They form a two-dimensional solvable Lie algebra.

Another result closely relates to supersymmetry properties of nonlinear differential equations.

Theorem 4. *Consider the graded differential algebra* $(\Lambda^*(\mathcal{E}^\infty), d_h)$, *where* d_h *is the horizontal de Rham differential. Then the Lie superalgebra of homotopy classes (with respect to* d_h*) of graded derivations* $\Delta\colon \Lambda^*(\mathcal{E}^\infty) \to \Lambda^*(\mathcal{E}^\infty)$ *is a semi-direct product of two copies of* $H^*_{\mathcal{C}}(\mathcal{E})$.

Remark. Note that this result is in a full agreement with A. M. Vinogradov's approach to Secondary Calculus in the category of differential equations, cf. [12].

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