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LIE SPHERE GEOMETRY AND LIE CONTACT STRUCTURES

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§1. Introduction

On a manifold with a Riemannian metric, we consider the Riemannian structure as an $O(n)$-reduction of the linear frame bundle. The conformal or projective structure, on the other hand, is a reduction of the second order frame bundles of the underlying manifolds, which was shown by T. Ochiai [O] according to the fundumental works on G-structures by S. Kobayashi [K] using jet bundles. Classically these structures were investigated by H. Weyl and a canonical connection on the structure is called the Weyl connection. These structures are understood as structures with group of which Lie algebra has the grading of the first order, or the G-structure with semi-simple flat homogeneous space as model space. N. Tanaka [T] generalizes the Cartan theory to these structures, and shows the existence of the normal Cartan connection on it. He further extended the theory to structures associated with a vast class of graded Lie algebras.

An almost forgotten classical geometry, the so-called Lie sphere geometry, is a geometry of the space of oriented hyperspheres on a sphere (Remember that the conformal geometry is a geometry on the space of hyperspheres in a sphere). On this space is acting the so-called Lie transformation group $PO(n+1,2)$, and this is the motivation of Sophus Lie to consider more general transformation groups later, which we now call the "Lie groups".

Corresponding to this old geometry, H. Sato [SY1] considers the Lie contact structure on manifolds with contact structures. This concept is not familiar, since the Lie sphere geometry itself is not well-known, and moreover since this structure lies on tangent bundles or on higher order frame bundles, and seems harder than usual G structures.

Recently, E. Ferapontov [F] found that the Lie contact transformation corresponds to the reciprocal transformation on the Hamiltonian system of the hydrodynamic type. This Hamiltonian system is closely related to the hypersurface geometry in the space form, especially Dupin hypersurfaces and isoparametric hypersurfaces. Important examples of integrable systems contain homogeneous hypersurfaces and their Lie images. It is reasonable to ask the role of Lie transformation here, because it preserves the integrability and some other nice properties.

Motivated by these recent movement, we would like to briefly introduce what is the Lie sphere geometry and the Lie contact structures, and then try to describe the relation between hypersurface geometry and Hamiltonian systems. In more details, see [M2].

§2. Lie sphere geoemetry

Consider the space $\mathcal{S}$ of oriented hyperspheres in the unit sphere $S^n \subset \mathbb{R}^{n+1}$, as

$$\mathcal{S} = \{(p, \cos\theta, \sin\theta) \in \mathbb{R}^{n+3}\}/\sim \subset bRP^{n+2}$$

where $\sim$ means the projectification, $p \in S^n$ is the center and θ is the oriented radius of the hypersphere, so that $(p, 0, 1)$ represents the totally geodesic hypersphere whose orientation is anticlockwise when looking down from p. Thus the hypersphere with another orientation is expressed as

$$(p, \cos\theta, -\sin\theta),$$

and a point of the sphere is identified with $(p, 1, 0)$. If we consider an inner product in $\mathbb{R}^{n+3}$ given by

$$\langle z, z\rangle = z_0^2 + \cdots + z_n^2 - z_{n+1}^2 - z_{n+2}^2, \quad z = (z_0, \ldots, z_{n+2}),$$

$\mathcal{S}$ is identified with

$$Q^{n+1} = \{[z] = [z_0, \ldots, z_{n+2}] \in \mathbb{R}P^{n+2} \mid \langle z, z\rangle = 0\} \subset \mathbb{R}P^{n+2}.$$

Two elements $[k_1]$ and $[k_2]$ in $\mathcal{S}$ are in oriented contact if

$$\langle k_1, k_2\rangle = 0,$$

which is an easy exercise using additive formula for trigonometric functions. In particular, a point $p \in S^n$ lies in $[k] \in \mathcal{S}$ when

$$\langle (p, 1, 0), k\rangle = 0.$$

U. Pinkall [P] shows that the Lie transformation group $PO(n+1, 2)$ acts on $\mathcal{S}$ preserving the oriented contactness of two elements. Intuitively, this group action consists of conformal transformations on S^n and the dilation by which we mean taking parallel hyperspheres in S^n.

For $[k_1]$ and $[k_2]$ in $\mathcal{S}$ satisfying

$$\langle k_1, k_2\rangle = 0,$$

the family of the linear combinations

$$l = \{[ak_1 + bk_2] \mid a, b \in \mathbb{R}\}$$

generates a line of Q^{n+1}, consisting of oriented hyperspheres in S^n which contact to each other at a point $p \in S^n$ orientedly. Consider the space of lines in Q^{n+1}. The group $PO(n+1, 2)$ acts on transitively, and is a homogeneous space $PO(n+1, 2)/H$, for some isotropy subgroup H.

Identifying $(p, n) \in T^1S^n$ with the line generated by

$$k_1 = (p, 1, 0), \quad k_2 = (n, 0, 1),$$

we know

Lemma 2.1. *$\cong T^1S^n$, where T^1S^n is the unit tangent bundle of S^n. Thus is a contact manifold of dimension $2n-1$.*

§3. Lie contact structures

The Lie contact structure is a G-structure on a contact manifold N^{2n-1} with the model space $\cong T^1S^n$. Consider the linear isotropy representation

$$\rho : H \to GL(2n-1, \mathbb{R}),$$

and denote $\tilde{H} = \rho(H)$. This is a subgroup of contact type, i.e. the structure group of the contact frame bundle. When the structure group has a reduction to $\tilde{H}$, we call the reduction a Lie contact structure [HY1]. A typical example was found by H. Sato on the tangent sphere bundle of a manifold with conformal structure. The Lie algebra of the structure group plays an important role when asking what is the cannonical connection of the structure. H. Sato and K. Yamaguchi pointed out that since the Lie algebra of $PO(n+1,2)$ is a graded Lie algebla of the second kind, we can apply the Cartan-Tanaka theory to it. The author constructed the connection and computed the curvature explicitly. When the curvature vanishes identically, we call the manifold Lie flat.

Theorem [M1,HY2]. *A conformal manifold M is Lie flat if and only if M is conformally flat.*

Through the proof of this theorem, we see that the Lie contact structrue is the structure naturally extended via flat extension from the lift of the underlying conformal structure on the tangent sphere bundle of M.

§4. Hypersurface geometry and Hamiltonian systems

When we are given an oriented immersed hypersurface $f : M \to S^n$, we can lift it to $T^1S^n \cong^{2n-1}$ by

$$\mathcal{L} : M \ni p \mapsto (p, n) \in T^1S^n$$

where n is the unit normal vector of M at p. This is called the Legendre map of the hypersurface, because the contact form vanishes along its image. Since $PO(n+1,2)$ acts on T^1S^n, we define

$$f_\varphi : M \to S^n, \quad f_\varphi = \pi \circ \varphi \circ \mathcal{L}, \quad \varphi \in PO(n+1,2),$$

where $\pi : T^1S^n \to S^n$ is the natural projection. We call this new hypersurface f_φ a Lie image of the original hypersurface. Intuitively again, a Lie image is obtained by conformally transforming M and then taking a parallel hypersurface, and repeating these procedures. The extension of group actions on hypersurfaces from the conformal group $PO(n+1,1)$ to the Lie transformation group $PO(n+1,2)$ plays an important role in the theory of Dupin hypersurfaces.

The Dupin hypersurfaces are by definition, hypersurfaces each of which principal curvature is constant along its leaf, in other words, such hypersurfaces foliated by spheres. When the radius of the leaf of each principal distribution is constant, we call them isoparametric hypersurfaces. A remarkable fact is that the Lie images of Dupin hypersurfaces are Dupin.

E. Ferapontov pointed out the relation between hypersurface geometry and Hamiltonian systems of hydrodynamic type, which is a Hamiltonian system given by

$$u_t^i = \frac{{}^2 h}{u^i u^j} u_x^j,$$

where $u = u(x,t)$ is a curve on an infinite dimensional symplectic manifold $\mathcal{P}$ consisting of functions on $\mathbb{R}$ with certain decay condition, and $h = h(u)$ is a functional on $\mathcal{P}$ independent of u_x, u_{xx} etc. He gave an explicit correspondence between the solution of this system and a hypersurface in the space form. In this correspondence, the Hessian of h corresponds to the Weingarten map of the hypersurface, and so the constantness of the eigenvalues of the Hessian along eigen-directions implies Dupin. He showed that if the system is integrable then the corresponding hypersurf aces must be Dupin. Moreover, there are parameter transformations $(x,t) \to (X,T)$ called reciprocal transformations, which correspond in hypersurface side to Lie contact transformations. The integrability of the system is invariant under reciprocal, hence Lie transformations, and we need to investigate the system in relation to the Lie sphere geometry.

The theory of integrable systems with multi spatial dimension is not well-developed. This is closely related to the motion of surfaces and hypersurfaces, and a geometrical approach to this field seems very important.

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