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ANOMALOUS QUADRATIC EXPONENTIALS IN THE STAR-PRODUCTS

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1. EXTENSIONS OF PRODUCT FORMULA

The Weyl algebra $W_\hbar$ is the associative algebra generated over $\mathbb{C}$ by u, v with the fundamental relation $u * v - v * u = -\hbar i$ where $\hbar$ is a positive constant.

For such a noncommutative algebra, the *ordering problem* may be the viewed as the problem of expression of elements of the algebra in a unique way. In the Weyl algebra, three kind of orderings; normal ordering, anti-normal ordering, and Weyl ordering, are mainly used. Through such an ordering, one can linearly identify the algebra with the space of all polynomials.

Another word, through such an ordering, one can view that the Weyl algebra is a non commutative associative product structure defined on the space $\mathbb{C}[u, v]$ of all polynomials. The product formulas are give respectively as follows:

- In the normal ordering: the product $*$ of the Weyl algebra is given by the ΨDO-*product formula* as follows:
$$f(u,v) * g(u,v) = f e^{\hbar i \overleftarrow{\partial_v}\cdot\overrightarrow{\partial_u}} g$$
- In the anti-normal ordering: the product $*$ of the Weyl algebra is given by the $\bar{\Psi}$DO-*product formula* as follows:
$$f(u,v) * g(u,v) = f e^{-\hbar i \overleftarrow{\partial_u}\cdot\overrightarrow{\partial_v}} g$$
- In the Weyl ordering: the product $*$ of the Weyl algebra is given by the *Moyal-product formula* as follows:
$$f(u,v) * g(u,v) = f e^{\frac{\hbar i}{2}\overleftarrow{\partial_v}\wedge\overrightarrow{\partial_u}} g.$$

Every product formula yields $u*v - v*u = -\hbar i$, and hence defines the Weyl algebra. Here, commutative products play only a supplementary role to express elements in the unique way.

Since every of three product formula is given by concrete forms, these extends to the following:

Let $\mathcal{H}(\mathbb{C}^2)$ be the space of all entire functions on $\mathbb{C}^2$ with the compact open topology.

- $f * g$ is defined if one of f, g is a polynomial.
- For every polynomial $p = p(u, v)$, the left-(resp. right-) multiplication $p*$ (resp. $*p$) is a continuous linear mapping of $\mathcal{H}(\mathbb{C}^2)$ into itself.

We call such a system a two-sided $(\mathbb{C}[u, v]; *)$-module.

Proposition 1. *In every product formula mentioned above,* $(\mathcal{H}_C(\mathbb{C}^2), \mathbb{C}[u, v], *)$ *is a two-sided* $(\mathbb{C}[u, v]; *)$*-module.*

By the polynomial approximation theorem, the associativity $f*(g*h) = (f*g)*h$ holds if two of f, g, h are polynomials.

Starting from a two-sided $(\mathbb{C}[u, v]; *)$-module, $*$-product extends to a wider class of functions. Let $\mathcal{E}^{(1)}(\mathbb{C}^2)$ be the commutative algebra with respect to the ordinary product generated by all polynomials $p(u, v)$ and exponential functions e^{au+bv}.

By each product formula, we can compute $e_*^{su} * e_*^{tv}$ as follows:

- $e_*^{su} * e_*^{tv} = e_\circ^{su+tv}$ in the ΨDO-product formula.
- $e_*^{su} * e_*^{tv} = e^{\hbar ist} e_\bullet^{su+tv}$ in the $\bar{\Psi}$DO-product formula.
- $e_*^{su} * e_*^{tv} = e^{\frac{\hbar ist}{2}} e_\cdot^{su+tv}$ in the Moyal product formula

where $\circ, \bullet, \cdot$ indicate the commutative product used in each product formula.

For every positive $p > 0$, set

$$\mathcal{E}_p(\mathbb{C}^2) = \{f \in \mathcal{E}(\mathbb{C}^2) \mid \|f\|_{p,s} = \sup |f|\, e^{(-s|\xi|^p)} < \infty, \forall s > 0\} \tag{1.1}$$

where $|\xi| = (|u|^2 + |v|^2)^{1/2}$. The family $\{\| \|_{p,s}\}_{s>0}$ induces a topology on $\mathcal{E}_p(\mathbb{C}^2)$ and $(\mathcal{E}_p(\mathbb{C}^2), \cdot)$ is an associative commutative Fréchet algebra, where the dott $\cdot$ is the ordinary multiplication for functions in $\mathcal{E}_p(\mathbb{C}^2)$. It is easily seen that for $0 < p < p'$, we have a continuous embedding

$$\mathcal{E}_p(\mathbb{C}^2) \subset \mathcal{E}_{p'}(\mathbb{C}^2) \tag{1.2}$$

as a commutative Fréchet algebra (cf.[GS]).

It is obvious that every polynomial is contained in $\mathcal{E}_p(\mathbb{C}^2)$ and $\mathcal{P}(\mathbb{C}^2)$ is dense in $\mathcal{E}_p(\mathbb{C}^2)$ for any $p > 0$.

We remark that every exponential function $e^{\alpha u+\beta v}$ is contained in $\mathcal{E}_p(\mathbb{C}^2)$ for any $p > 1$, but not in $\mathcal{E}_1(\mathbb{C}^2)$, and functions such as $e^{au^2+bv^2+2cuv}$ are contained in $\mathcal{E}_p(\mathbb{C}^2)$ for any $p > 2$, but not in $\mathcal{E}_2(\mathbb{C}^2)$.

Theorem 2. *The Moyal product formula* (2.1) *gives the following:*

(i): *For $0 < p \leq 2$, the space $(\mathcal{E}_p(\mathbb{C}^2), *_\hbar)$ forms a topological associative algebra.*

(ii): *For $p > 2$ and a fixed $\hbar \in \mathbb{R}$, the Moyal product formula gives a continuous bi-liner mapping of*

$$\begin{aligned} &\mathcal{E}_p(\mathbb{C}^2) \times \mathcal{E}_{p'}(\mathbb{C}^2) \to \mathcal{E}_p(\mathbb{C}^2), \\ &\mathcal{E}_{p'}(\mathbb{C}^2) \times \mathcal{E}_p(\mathbb{C}^2) \to \mathcal{E}_p(\mathbb{C}^2), \end{aligned} \tag{1.3}$$

for every p' such that $\frac{1}{p} + \frac{1}{p'} \geq 1$.

We remark here about the statement (ii). Since $p > 2$, p' must be $p' \leq 2$, hence the statement (i) gives that $(\mathcal{E}_{p'}(\mathbb{C}^2); *_\hbar$ is a Fréchet algebra. So the statement (ii) means that every $\mathcal{E}_p(\mathbb{C}^2)$, $p > 2$, is a topological 2-sided $\mathcal{E}_{p'}(\mathbb{C}^2)$-module.

We remark also that if $\hbar > 0$, then $e^{\pm\frac{2}{\hbar}(au^2+bv^2+2cuv)} \in \mathcal{E}_p(\mathbb{C}^2)$ for every $p > 2$.

Let $\mathcal{E}_{2+}(\mathbb{C}^2) = \bigcap_{p>2} \mathcal{E}_p(\mathbb{C}^2)$. $\mathcal{E}_{2+}(\mathbb{C}^2)$ is a Fréchet space, but this is not closed under the $*$-product, e.g. $e^{\frac{2i}{\hbar}uv} * e^{-\frac{2i}{\hbar}uv}$ diverges.

In the space $\mathcal{E}_{2+}(\mathbb{C}^2)$, the $*$-product behaves anoumalously, that we are going to talking about.

2. Quadratic forms

For every $(a, b, c) \in \mathbb{C}^3$, we consider quadratic forms $Q(u, v) = au^2 + bv^2 + 2cuv$. We define the product $*$ by the Moyal product formula:

$$f * g = f \exp \frac{\hbar i}{2}\{\overleftarrow{\partial_v} \wedge \overrightarrow{\partial_u}\} g. \tag{2.1}$$

It is easy to see that $X=\frac{1}{\hbar}u^2, Y=\frac{1}{\hbar}v^2, H=\frac{1}{\hbar}uv$ form a Lie algebra with respect to the commutator product $[\ ,\ ]_*$. Since

$$[\frac{i}{2\hbar}uv,\frac{1}{\hbar\sqrt{8}}u^2]=-\frac{1}{\hbar\sqrt{8}}u^2,\quad [\frac{i}{2\hbar}uv,\frac{1}{\hbar\sqrt{8}}v^2]=\frac{1}{\hbar\sqrt{8}}v^2,\quad [\frac{1}{\hbar\sqrt{8}}u^2,\frac{1}{\hbar\sqrt{8}}v^2]=-\frac{i}{2\hbar}uv,$$

this is the Lie algeba of $SL(2,\mathbb{C})$. X,Y,H generate an associative algebra in the space $\mathbb{C}[u,v]$ of all polynomials. This is an enveloping algebra of $\mathfrak{sl}(2,\mathbb{C})$.

The Casimir element $C=H^2+(X{*}Y+Y{*}X)$, that is

$$C=(\frac{i}{2\hbar}uv)^2_*+\frac{1}{\hbar\sqrt{8}}u^2{*}\frac{1}{\hbar\sqrt{8}}v^2+\frac{1}{\hbar\sqrt{8}}v^2{*}\frac{1}{\hbar\sqrt{8}}u^2$$

is given by

$$8\hbar^2C=u^2{*}v^2+v^2{*}u^2-2(u{*}v+\frac{\hbar i}{2})^2=u^2{*}v^2+v^2{*}u^2-2u{*}v{*}u{*}v-2\hbar iu{*}v+\frac{\hbar^2}{2}$$

Hence, $C=-\frac{3}{16}$. This means that our enveloping algebra is restricted in the space $C=-\frac{3}{16}$.

For every point $(a,b,c;s)$ in $\mathbb{C}^4$, consider a curve $s(t)e^{\frac{1}{\hbar}(a(t)u^2+b(t)v^2+2c(t)uv)}$ starting at the point $se^{\frac{1}{\hbar}(au^2+bv^2+2cuv)}$ then the tangent vector is given as

$$(\frac{t}{\hbar}((a'u^2+b'v^2+2c'uv)s+s')e^{\frac{1}{\hbar}(au^2+bv^2+2cuv)}$$

On the other hand, consider the quantity

$$\frac{d}{dt}\Big|_{t=0}e^{\frac{t}{\hbar}(a'u^2+b'v^2+2c'uv)}*se^{\frac{1}{\hbar}(au^2+bv^2+2cuv)}.$$

This is computed as follows:

$$\begin{aligned}&\frac{1}{\hbar}(a'u^2+b'v^2+2c'uv)*se^{\frac{1}{\hbar}(au^2+bv^2+2cuv)}\\ &\quad=\frac{1}{\hbar}(a'u^2+b'v^2+2c'uv)se^{\frac{1}{\hbar}(au^2+bv^2+2cuv)}\\ &\quad+\frac{2i}{\hbar}\{(b'v+c'u)(au+cv)-(a'u+c'v)(bv+cu)\}se^{\frac{1}{\hbar}(au^2+bv^2+2cuv)}\\ &\quad-\frac{1}{2\hbar}\{b'(\hbar a+2(au+cv)^2)-2c'(\hbar c+2(au+cv)(bv+cu))\\ &\qquad\qquad+a'(\hbar b+2(bv+cu)^2)\}se^{\frac{1}{\hbar}(au^2+bv^2+2cuv)}\end{aligned}$$

This may be written as

$$(2.2)\qquad \frac{1}{\hbar}(a',b',c')\begin{bmatrix}-(c+i)^2, & -b^2, & -b(c+i), & -\frac{b}{2}\\ -a^2, & -(c-i)^2, & -a(c-i), & -\frac{a}{2}\\ 2a(c+i), & 2b(c-i), & 1+ab+c^2, & c\end{bmatrix}\begin{bmatrix}u^2\\ v^2\\ 2uv\\ 1\end{bmatrix}se^{\frac{1}{\hbar}(au^2+bv^2+2cuv)}$$

We denote this matrix by $M(a,b,c;s)$, and by $M(a,b,c)$ the submatrix of first three columns.

3. $*$-EXPONENTIALS AND VACUUMS

In this section we define $e_*^{t(au^2+bv^2+2cuv)}$. Set $e_*^{t(au^2+bv^2+2cuv)} = F(t,u,v)$, and consider the differential equation

$$\frac{\partial}{\partial t}F(t,u,v) = (au^2+bv^2+2cuv) * F(t,u,v), \quad F(0,u,v) = 1 \tag{3.1}$$

If we assume that $e_*^{t(au^2+bv^2+2cuv)} = se^{a(t)u^2+b(t)v^2+2c(t)uv}$, then we have

$$\frac{d}{dt}(a(t),b(t),c(t)) = (a,b,c)M(a(t),b(t),c(t)), \quad (a(0),b(0),c(0)) = (0,0,0).$$

The right hand side of (3.1) is computed by the Moyal product formula as follows:

$$\begin{aligned}(au^2+bv^2+2cuv) * F(t,u,v) = (au^2+bv^2+2cuv)F+\hbar i\{(bv+cu)\partial_u F - (au+cv)\partial_v F\}\\ - \frac{\hbar^2}{4}\{b\partial_u^2 F - 2c\partial_v\partial_u F + a\partial_v^2 F\}\end{aligned}$$

If $ab-c^2>0$, then this is the heat equation and the existence of solutions is not ensured in general. However, the uniqueness holds in the category of real analytic functions in t. Hence we assume that $e_*^{t(au^2+bv^2+2cuv)}$ us a function of au^2+bv^2+2cuv; that is $e_*^{t(au^2+bv^2+2cuv)} = f_t(au^2+bv^2+2cuv)$. Then, we have

$$\begin{aligned}&(au^2+bv^2+2cuv) * f_t(au^2+bv^2+2cuv)\\ &=(au^2+bv^2+2cuv)\, f_t(au^2+bv^2+2cuv)\\ &\quad - \hbar^2(ab-c^2)\big(f_t'(au^2+v^2+2cuv) + f_t''(au^2+bv^2+2cuv)(au^2+bv^2+2cuv)\big).\end{aligned}$$

Setting $x = au^2+bv^2+2cuv$, we have

$$\frac{d}{dt}f_t(x) = xf_t(x) - \hbar^2(ab-c^2)\big(f_t'(x) + xf_t''(x)\big) \tag{3.2}$$

Lemma 3. *The solution of* (3.2) *with the initial function* 1 *is given by*

$$f_t(x) = \frac{1}{\cosh(\hbar\sqrt{ab-c^2}t)}\exp\{\frac{x}{\hbar\sqrt{ab-c^2}}\tanh(\hbar\sqrt{ab-c^2}\;t)\}$$

Proof. Assuming the shape $f_t(x) = g(t)e^{h(t)x}$, we see that

$$\{g'(t) + (ab-c^2)\hbar^2 g(t)h(t) + xg(t)\{h'(t) - 1 + (ab-c^2)\hbar^2 h(t)^2\}\}e^{h(t)x} = 0$$

and hence we have $h'(t) - 1 + (ab-c^2)\hbar^2 h(t)^2 = 0$. $h(t)$ is given as

$$h(t) = \frac{1}{\hbar\sqrt{ab-c^2}}\tanh(\hbar(\sqrt{ab-c^2})t).$$

Note that the ambiguity of $\sqrt{ab-c^2}$ does not suffer the result.

Next, we solve the equation

$$g'(t) + g(t)(ab-c^2)\hbar^2\frac{1}{\hbar\sqrt{ab-c^2}}\tanh(\hbar(\sqrt{ab-c^2})t) = 0$$

to obtain $g(t) = \frac{1}{\cosh(\hbar(\sqrt{ab-c^2})t)}$. This also does not depend on the sign of $\pm\sqrt{ab-c^2}$. In this argument t need not be restricted in the real number. $\square$

By Lemma 3, we have

$$
\begin{aligned}
e_*^{t(au^2+bv^2+2cuv)} &= \frac{1}{\cosh(\hbar\sqrt{ab-c^2}t)} e^{(au^2+bv^2+2cuv)(\frac{1}{\hbar\sqrt{ab-c^2}}\tanh(\hbar\sqrt{ab-c^2}\,t))} \\
&= \frac{1}{\cos(\hbar\sqrt{c^2-ab}t)} e^{(au^2+bv^2+2cuv)(\frac{1}{\hbar\sqrt{c^2-ab}}\tan(\hbar\sqrt{c^2-ab}\,t))}
\end{aligned}
\tag{3.3}
$$

It is equivalent with

$$
\sqrt{c^2-ab+1}\, e^{\frac{1}{\hbar}(au^2+bv^2+2cuv)} = e_*^{\frac{1}{\hbar\sqrt{c^2-ab}}(\arctan\sqrt{c^2-ab})(au^2+bv^2+2cuv)}. \tag{3.4}
$$

If $ab-c^2=0$, then $\frac{1}{\hbar\sqrt{ab-c^2}}\tanh(\hbar\sqrt{ab-c^2}\,t)=t$, and

$$
e_*^{t(au^2+bv^2+2cuv)} = e^{t(au^2+bv^2+2cuv)}, \quad ab-c^2=0.
$$

This means that if $au^2+bv^2+2cuv=(\sqrt{a}u+\sqrt{b}v)^2$, then the $*$-exponential coincides with the ordinary exponential function.

By the uniqueness of analytic solutions, the exponential law

$$
e_*^{isx} * e_*^{itx} = e_*^{i(s+t)x}
$$

holds where both sides are defined. If $\sqrt{ab-c^2}t\in\mathbb{R}$, then e_*^{itx} forms a one parameter group.

Lemma 4. *For $s,\sigma\in\mathbb{C}$ such that $1+s\sigma(ab-c^2)\neq 0$, we have*

$$
e^{\frac{s}{\hbar}(au^2+bv^2+2cuv)} * e^{\frac{\sigma}{\hbar}(au^2+bv^2+2cuv)} = \frac{1}{1+s\sigma(ab-c^2)} e^{\frac{s+\sigma}{\hbar(1+s\sigma(ab-c^2))}(au^2+bv^2+2cuv)}
$$

Thus, we have idempotent elements

$$
2e^{\pm\frac{1}{\hbar\sqrt{ab-c^2}}(au^2+bv^2+2cuv)} * 2e^{\pm\frac{1}{\hbar\sqrt{ab-c^2}}(au^2+bv^2+2cuv)} = 2e^{\pm\frac{1}{\hbar\sqrt{ab-c^2}}(au^2+bv^2+2cuv)}.
$$

We call $2e^{\frac{1}{\hbar\sqrt{ab-c^2}}(au^2+bv^2+2cuv)}$ a *vacuum.* By the Moyal product formula, we easily see that

$$
(\gamma u+\delta v) * e^{\frac{2i}{\hbar}(\alpha u+\beta v)(\gamma u+\delta v)} = 0, \quad \text{for} \quad \alpha\delta-\beta\gamma=1.
$$

Corollary 5.

$$
2e^{\frac{1}{\hbar\sqrt{ab-c^2}}(au^2+bv^2+2cuv)} = \lim_{t\to\infty} e^{it\sqrt{ab-c^2}} e_*^{\frac{t}{\hbar\sqrt{ab-c^2}}(au^2+bv^2+2cuv)}
$$

is a vacuum.

$e_*^{\frac{\pi}{\hbar}\frac{1}{\sqrt{c^2-ab}}(au^2+bv^2+2cuv)} = -1$, *and* $e_*^{\frac{\pi}{2\hbar}\frac{1}{\sqrt{c^2-ab}}(au^2+bv^2+2cuv)}$ *is singular.*

We show that $\{\exp_*(au^2+bv^2+2cuv);\, c^2-ab+1\neq 0\}$ form a group which is isomorphic to $SL(2,\mathbb{C})$.

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