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Construction of contact diffeomorphisms from Schwarzian derivatives

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I talk on my joint work with Tetsuya OZAWA ([O-S]).

1 Contact Schwarzian derivative

On the affine 3-space $\mathbb{K}^3$ ($\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$) with the usual coordinate $(x, y, z)$, we give the contact form $\alpha = dy - zdx$. Put

$$v_1 = \frac{\partial}{\partial x} + z \frac{\partial}{\partial y}, \quad v_2 = \frac{\partial}{\partial z}, \quad v_3 = \frac{\partial}{\partial y}, \quad v_4 = v_2v_1 + v_1v_2.$$ 

A local diffeomorphism $\phi$ is a contact diffeomorphism, if it satisfies $\phi^*(\alpha) = \rho \alpha$ for some nonvanishing function $\rho$. For a contact diffeomorphism $\phi : (x, y, z) \mapsto (X, Y, Z)$, we define the contact Schwarzian derivatives as follows: for $i, j, k = 1, 2$, set

$$s_{[ij,k]}(\phi) = v_i v_j(X) v_k(Z) - v_i v_j(Z) v_k(X),$$

and

$$S_{ijk}(\phi) = \frac{1}{3\Delta(\phi)} (s_{ij,k}(\phi) + s_{jk,i}(\phi) + s_{ki,j}(\phi),)$$

where $\Delta(\phi) = v_1(X)v_2(Z) - v_1(Z)v_2(X)$. We call the functions $S_{ijk}(\phi)$ the contact Schwarzian derivatives of the contact diffeomorphism $\phi$. We denote the quadruple of functions by

$$S(\phi) = (S_{[111]}(\phi), S_{[112]}(\phi), S_{[122]}(\phi), S_{[222]}(\phi)).$$

Proposition 1.1. The inverse $\phi^{-1}$ of a contact diffeomorphism $\phi : \mathbb{K}^3 \to \mathbb{K}^3$ maps the differential equation $Y''' = 0$ to

$$y''' = S_{[112]}(\phi, x) + 3S_{[111]}(\phi, x)y'' + 3S_{[222]}(\phi, x)(y'')^2 + S_{[122]}(\phi, x)(y'')^3.$$
By [S-Y], the condition that $y'''' = f(x, y, y', y'')$ is mapped to $y'''' = 0$ by a contact diffeomorphism is the vanishing of two curvatures $A$ and $b$. We obtain that $b = 0$ is equivalent to $\partial^4 f / \partial x^4 = 0$. Let us consider

$$y'''' = P + 3Qy' + 3R(y'')^2 + S(y'')^3,$$

where $P = P(x, y, y')$, $Q = Q(x, y, y')$, $R = R(x, y, y')$, $S = S(x, y, y')$. Then $b = 0$ and the condition $A = 0$ is equal to

$$
\begin{align*}
v_3(P) &= 2(v_1 - 2Q)(M_{11}) + 4PM_4 \\
3v_3(Q) &= 2(v_2 - 4R)(M_{11}) + 4(v_1 + Q)(M_4) + 4PM_{22} \\
3v_3(R) &= 2(v_1 + 4Q)(M_{22}) + 4(v_2 - R)(M_4) - 4SM_{11} \\
v_3(S) &= 2(v_2 + 2R)(M_{22}) - 4SM_4.
\end{align*}
$$

where we put

$$
\begin{align*}
M_{11} &= -\frac{1}{4}(v_1(Q) - v_2(P) - 2Q^2 + 2PR) \\
M_4 &= -\frac{1}{4}(v_1(R) - v_2(Q) - QR + PS) \\
M_{22} &= -\frac{1}{4}(v_1(S) - v_2(R) - 2R^2 + 2QS).
\end{align*}
$$

**Theorem 1.1.** Four function $P, Q, R, S$ on $\mathbb{K}^3$ is the Schwarzian derivarives of a contact diffeomorphism $\phi : \mathbb{K}^3 \rightarrow \mathbb{K}^3$;

$$(P, Q, R, S) = S(\phi),$$

if and only if the system of the nonlinear differential equations (IC) is satisfied.

We seek a system of linear differential equations whose integrability equation is equal to (IC) and its solutions give the contact diffeomorphism. We call the linear system the linearization of (IC)

### 2 Fundamental system

Here is the linear differential system:

$$
\begin{align*}
v_1^2(\vartheta) &= Qv_1(\vartheta) - P v_2(\vartheta) + M_{11} \vartheta \\
v_4(\vartheta) &= 2(Rv_1(\vartheta) - Qv_2(\vartheta) + M_{4} \vartheta) \\
v_2^2(\vartheta) &= Sv_1(\vartheta) - Rv_2(\vartheta) + M_{22} \vartheta
\end{align*}
$$

(Sp)
Theorem 2.1. The necessary and sufficient condition for the linear PDE system \((Sp)\) to have 4-dimensional solution space is equal to the nonlinear PDE system \((IC)\).

Proposition 2.1. For any two solutions \(\alpha\) and \(\beta\) of the PDE system \((Sp)\), the function \(I(\alpha, \beta)\) defined by

\[
I(\alpha, \beta) = \frac{1}{2} \alpha v_3(\beta) - \frac{1}{2} v_3(\alpha)\beta + v_1(\alpha) v_2(\beta) - v_2(\alpha) v_1(\beta) \tag{1}
\]

is constant on \((x, y, z)\). Moreover this skew product \(I(\alpha, \beta)\) is non-degenerate, and thus it defines a symplectic structure on the solution space \(S(P, Q, R, S)\) of \((Sp)\), provided the dimension of \(S(P, Q, R, S)\) is equal to 4.

Theorem 2.2. If a map \(\phi: (x, y, z) \mapsto (X, Y, Z)\) is contact, then there exists a symplectic basis \(\{\partial, \xi, \zeta, \eta\}\) of the solution space \(S(S(\phi))\) of the PDE system \((Sp)\) such that \(\phi\) is given by

\[
(x, y, z) \mapsto \left( \frac{\xi}{\partial}, \frac{\eta}{\partial} + \frac{\xi \zeta}{\partial^2}, \frac{\zeta}{\partial} \right). \tag{2}
\]

Conversely, given a symplectic basis \(\{\partial, \xi, \zeta, \eta\}\) of the solution space \(S(P, Q, R, S)\) of \((Sp)\), the map \(\phi\) defined by (2) is a contact diffeomorphism whose contact Schwarzian derivatives are equal to

\[
S(\phi) = (P, Q, R, S).
\]

References.


