<table>
<thead>
<tr>
<th>Title</th>
<th>Berezin Transforms and Laplace-Beltrami Operators on Homogeneous Siegel Domains: commutativity, symmetry of the domain and a Cayley transform (Lie Groups, Geometric Structures and Differential Equations: One Hundred Years after Sophus Lie)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Author(s)</td>
<td>Nomura, Takaaki</td>
</tr>
<tr>
<td>Citation</td>
<td>数理解析研究所講究録 (2000), 1150: 72-80</td>
</tr>
<tr>
<td>Issue Date</td>
<td>2000-04</td>
</tr>
<tr>
<td>URL</td>
<td><a href="http://hdl.handle.net/2433/64057">http://hdl.handle.net/2433/64057</a></td>
</tr>
<tr>
<td>Type</td>
<td>Departmental Bulletin Paper</td>
</tr>
<tr>
<td>Textversion</td>
<td>publisher</td>
</tr>
</tbody>
</table>

京都大学
Berezin Transforms and Laplace-Beltrami Operators
on Homogeneous Siegel Domains
— commutativity, symmetry of the domain and a Cayley transform —

TAKAAKI NOMURA¹
(Kyoto University)

1. Preliminaries

Homogeneous Siegel domains are described in terms of normal \( j \)-algebras (cf. [15]), of which we are going to give the definition. Let \( g \) be a split solvable Lie algebra, \( J \) a linear operator on \( g \) with \( J^2 = -I \) and \( \omega \) a linear form on \( g \). Then the triple \( (g, J, \omega) \) is called a normal \( j \)-algebra if

\[
\begin{align*}
(1.1) \quad [Jx, Jy] &= [x, y] + J[Jx, y] + J[x, Jy] \quad \text{(for all } x, y \in g), \\
(1.2) \quad \langle x \mid y \rangle_\omega := \langle [Jx, y], \omega \rangle \text{ defines a } J \text{-invariant inner product on } g.
\end{align*}
\]

We describe here some basic facts about normal \( j \)-algebras following [15] and [17] (see also [16]). Let \( (g, J, \omega) \) be a normal \( j \)-algebra. Let \( n := [g, g] \) be the derived algebra of \( g \), and \( a \) the orthogonal complement of \( n \) in \( g \) relative to the inner product \( \langle \cdot \mid \cdot \rangle_\omega \). Evidently we have \( g = a + n \). Moreover, \( a \) is a commutative subalgebra of \( g \) such that \( \text{ad}(a) \) consists of semisimple operators on \( g \). For every \( \alpha \in a^* \) we set

\[
n_\alpha := \{ x \in n ; [h, x] = \langle h, \alpha \rangle x \quad \text{for all } h \in a \}.
\]

Take all \( \alpha \in a^* \) such that \( n_\alpha \neq \{0\} \) and \( Jn_\alpha \subset a \), and number them as \( \alpha_1, \ldots, \alpha_r \). We have \( \dim a = r \) and \( \dim n_{\alpha_k} = 1 \) for every \( k \). The number \( r \) is called the rank of the normal \( j \)-algebra \( g \). We can reorder \( \alpha_1, \ldots, \alpha_r \), if necessary, so that all the \( \alpha \) such that \( n_\alpha \neq \{0\} \) (such an \( \alpha \) is called a root of the normal \( j \)-algebra) are of the following form (some roots might be missing):

\[
\begin{align*}
\frac{1}{2}(\alpha_m + \alpha_k) \quad (1 \leq k < m \leq r), \\
\frac{1}{2}(\alpha_m - \alpha_k) \quad (1 \leq k < m \leq r), \\
\frac{1}{2}\alpha_k \quad (1 \leq k \leq r), \\
\alpha_k \quad (1 \leq k \leq r).
\end{align*}
\]

¹E-mail: nomura@kusm.kyoto-u.ac.jp
We note that if $\alpha, \beta$ are distinct roots, then $n_\alpha$ is orthogonal to $n_\beta$. Put
\[
g(0) := a \oplus \sum_{m>k} n_{(\alpha_m - \alpha_k)/2}, \quad g(1/2) := \sum_{i=1}^{r} n_{\alpha_i/2},
\]
\[
g(1) := \sum_{i=1}^{r} n_{\alpha_i} \oplus \sum_{m>k} n_{(\alpha_m + \alpha_k)/2}.
\]
Understanding $g(i) = 0$ for $i > 1$, we have $[g(i), g(j)] \subset g(i + j)$. Moreover
\[
Jn_{(\alpha_m - \alpha_k)/2} = n_{(\alpha_m + \alpha_k)/2} \quad (m > k), \quad Jn_{\alpha_i/2} = n_{\alpha_i/2} \quad (1 \leqq i \leqq r),
\]
so that $Jg(0) = g(1)$ and $Jg(1/2) = g(1/2)$. Taking $E_i \in n_{\alpha_i}$ ($i = 1, \ldots, r$) such that $\alpha_k(JE_i) = \delta_{ki}$, we put $H_i := JE_i \in a$ and
\[
(1.3) \quad H := H_1 + \cdots + H_r, \quad E := E_1 + \cdots + E_r.
\]
We write down here the constants used frequently in this note:
\[
n_{mk} := \dim_{\mathbb{R}} n_{(\alpha_m - \alpha_k)/2} = \dim_{\mathbb{R}} n_{(\alpha_m + \alpha_k)/2} \quad (1 \leqq k < m \leqq r),
\]
\[
b_i := \frac{1}{2} \dim_{\mathbb{R}} n_{\alpha_i/2} \quad (1 \leqq i \leqq r),
\]
\[
d_j := 1 + \frac{1}{2} \left( \sum_{k>j} n_{kj} + \sum_{i<j} n_{ji} \right) \quad (1 \leqq j \leqq r).
\]

Let $G = \exp g$ be the connected and simply connected Lie group corresponding to $g$. Note that $g(0)$ is a Lie subalgebra of $g$. We denote by $G(0)$ the corresponding subgroup $\exp g(0)$ of $G$. The group $G(0)$ acts on $V := g(1)$ by adjoint action. Let $\Omega$ be the $G(0)$-orbit through $E$. By [17, Theorem 4.15] $\Omega$ is a regular open convex cone in $V$, and $G(0)$ acts on $\Omega$ simply transitively. Being invariant under $J$, the subspace $g(1/2)$ is considered as a complex vector space by means of $-J$. We shall write this complex vector space by $U$. We put $W := V_\mathbb{C}$, the complexification of $V$. The conjugation of $W$ relative to the real form $V$ is written as $w \mapsto w^\ast$. The real bilinear map $Q$ defined by
\[
Q(u, u') := \frac{1}{2} ([Ju, u'] - i[u, u']) \quad (u, u' \in g(1/2))
\]
turns out to be a complex sesqui-linear (complex linear in the first variable and antilinear in the second) Hermitian map $U \times U \rightarrow W$ which is $\Omega$-positive. This means that
\[
Q(u', u) = Q(u, u')^\ast \quad (u, u' \in U), \quad Q(u, u) \in \overline{\Omega} \setminus \{0\} \quad \text{for all} \ u \in U \setminus \{0\}.
\]
With these data we define the Siegel domain $D$ corresponding to the normal $j$-algebra $(g, J, \omega)$ to be

$$D := \{(u, w) \in U \times W ; w + w^* - Q(u, u) \in \Omega\}.$$ 

Note that we take a generalized right half plane rather than a more familiar upper half plane.

Consider the Lie subalgebra $n_D := g(1) + g(1/2)$. It is at most 2-step nilpotent. Let $N_D = \exp n_D$ be the corresponding connected and simply connected nilpotent Lie group contained in $G$. We write the elements of $N_D$ by $n(a, b)(a \in g(1), b \in g(1/2))$. The group $N_D$ acts on $D$ by

$$(1.5) \quad n(a, b) \cdot (u, w) = (u + b, w + ia + \frac{1}{2}Q(b, b) + Q(u, b)) \quad ((u, w) \in D).$$

On the other hand, the adjoint action of $G(0)$ on $g(1/2)$ commutes with $J$. This implies that $G(0)$ acts on $U$ complex-linearly. Moreover the adjoint action of $G(0)$ on $V = g(1)$ extends complex-linearly to $W$, so that $G(0)$ acts on $D$ complex-linearly. Hence $G = N_D \times G(0)$ acts on $D$ simply transitively. To see this more explicitly, put $e := (0, E) \in D$. Then given $z = (u, w) \in D$, we can find a unique $h \in G(0)$ satisfying $hE = \text{Re} w - Q(u, u)/2$. Taking $n = n(\text{Im} w, u) \in N_D$, we see by (1.5) that $z = nh \cdot e$.

For every $s = (s_1, \ldots, s_r) \in \mathbb{C}^r$ let $\chi_s$ be the one-dimensional representation of $A := \exp a$ defined by

$$\chi_s \left( \exp \sum_k t_k H_k \right) = \exp \left( \sum_k s_k t_k \right) \quad (t_1, \ldots, t_r \in \mathbb{R}).$$

Let $N := \exp n$. It is clear that $G = N \rtimes A$. We extend $\chi_s$ to a one-dimensional representation of $G$ by defining $\chi_s(n) = 1$ for $n \in N$. Let us define functions $\Delta_s (s \in \mathbb{C}^r)$ on $\Omega$ by $\Delta_s(hE) = \chi_s(h)$ ($h \in G(0)$). Evidently it holds that

$$(1.6) \quad \Delta_s(hx) = \chi_s(h) \Delta_s(x) \quad (h \in G(0), x \in \Omega).$$

We know that $\Delta_s$ extends to a holomorphic function on the tube domain $\Omega + iV$ (cf. for example [7, Corollary 2.5]).

For $h \in G(0)$, let $\text{Ad}_{g(1)}(h) := (\text{Ad} h)|_{g(1)}$. Moreover let $\text{Ad}_U(h)$ stand for the complex linear operator on $U$ defined by the adjoint action of $h \in G(0)$ on $g(1/2)$, and $\det \text{Ad}_U(h)$ its determinant as a complex linear operator. Then, with $d := (d_1, \ldots, d_r)$ and $b := (b_1, \ldots, b_r)$, we have for $h \in G(0)$

$$(1.7) \quad \det \text{Ad}_{g(1)}(h) = \chi_d(h), \quad |\det \text{Ad}_U(h)|^2 = \chi_b(h).$$
By [6, §5] or [18, §II.6], it is known that $D$ has a Bergman kernel $\kappa$. If $\text{Hol}(D)$ denotes the Lie group of the holomorphic automorphisms of $D$, then $\kappa$ satisfies

\begin{equation}
\kappa(z_1, z_2) = \kappa(g \cdot z_1, g \cdot z_2) \det g'(z_1) \det g'(z_2) \quad (g \in \text{Hol}(D), \ z_1, z_2 \in D),
\end{equation}

where $g'(z)$ is the complex Jacobian map of $g$ at $z \in D$. The description of the simple transitive action of $G$ on $D$ together with the property (1.7) and (1.8) shows

\begin{equation}
\kappa(z_1, z_2) = C \cdot \Delta_{-2d-b}(w_1 + w_2^* - Q(u_1, u_2)) \quad (z_j = (u_j, w_j) \in D)
\end{equation}

with $C = \kappa(e, e) \Delta_{2d+b}(2E) > 0$. We put $\eta := \Delta_{-2d-b}$ in what follows for simplicity.

2. Cayley transform

Let $D_v$ be the directional derivative in the direction $v \in V$ given by

\begin{equation}
D_v f(x) = \frac{d}{dt} f(x + tv)|_{t=0}.
\end{equation}

For every $x \in \Omega$ we define $\mathcal{I}(x) \in V^*$ to be $-\nabla \log \eta(x)$, that is,

\begin{equation}
\langle v, \mathcal{I}(x) \rangle = -D_v \log \eta(x) \quad (v \in V).
\end{equation}

$\mathcal{I}$ is called the pseudoinverse map. By [3, §2], $\mathcal{I}$ gives a diffeomorphism of $\Omega$ onto the dual cone $\Omega^*$ in $V^*$, where

\begin{equation}
\Omega^* := \{ \xi \in V^* ; \langle x, \xi \rangle > 0 \text{ for all } x \in \overline{\Omega} \setminus \{0\} \}.
\end{equation}

The group $G(0)$ acts also on $V^*$ by the coadjoint action: $h \cdot \xi = \xi \circ h^{-1}$, where $h \in G(0)$ and $\xi \in V^*$. It is easy to show by using (1.6) that $\mathcal{I}$ is $G(0)$-equivariant:

\begin{equation}
\mathcal{I}(hx) = h \cdot \mathcal{I}(x) \quad (h \in G(0), \ x \in \Omega).
\end{equation}

In particular, $\mathcal{I}(\lambda x) = \lambda^{-1} \mathcal{I}(x)$ for all $\lambda > 0$, and $G(0)$ acts on $\Omega^*$ simply transitively. Moreover, $\mathcal{I}$ can be extended to a rational map $W \to W^*$ [4, Satz I.2.3].

In order to find an inverse map of $\mathcal{I}$, we need to dualize the above matters concerning $\mathcal{I}$. First we define $E^{*}_1, \ldots, E^{*}_r \in V^*$ by

\begin{equation}
\left\langle \sum_{j=1}^{r} x_j E_j + \sum_{m>k} X_{mk}, E^*_i \right\rangle = x_i \quad (x_j \in \mathbb{R}, \ X_{mk} \in \mathfrak{n}_{(a_m+a_k)/2}),
\end{equation}

and for every $s = (s_1, \ldots, s_r) \in \mathbb{R}^r$,

\begin{equation}
E^*_s := s_1 E^*_1 + \cdots + s_r E^*_r \in V^*.
\end{equation}

We can show that $\mathcal{I}(E) = E^*_{2d+b}$. Next we put $s^* := (s_r, \ldots, s_1)$ and set

\begin{equation}
\chi^*_s := \chi_{-s^*}, \quad \Delta^*_s(h \cdot E^*_{2d+b}) := \chi^*_s(h) \quad (h \in G(0)).
\end{equation}
\( \Delta^*_s \) is a function on \( \Omega^* \) such that \( \Delta^*_s(h \cdot \xi) = \chi^*_s(h) \Delta^*_s(\xi) \) for \( h \in G(0) \) and \( \xi \in V^* \).

We define \( \eta^* := \Delta^{*-2d-\cdot} \), and

\[
\langle I^*(\xi), f \rangle := -D_f \log \eta^*(\xi) \quad (\xi \in \Omega^*, f \in V^*).
\]

Thus \( I^*(\xi) \in V \) and \( I^* \) gives a diffeomorphism of \( \Omega^* \) onto \( \Omega \).

Moreover, \( I^* \) is \( G(O) \)-equivariant, that is, \( I^*(h \cdot \xi) = h(I^*(\xi)) \) for any \( h \in G(O) \).

We can prove that \( I^* \) is extended to a rational map \( W^* \rightharpoonup W \).

**Proposition 2.1.** \( I^* = I^{-1} \).

**Theorem 2.2** ([11]). (1) \( I \) is holomorphic on \( \Omega + iV \), and \( I^* \) is holomorphic on \( \Omega^* + iV^* \).

(2) \( I(\Omega + iV) \) is contained in the holomorphic domain of \( I^* \), and \( I^*(\Omega^* + iV^*) \) is contained in the holomorphic domain of \( I \).

**Remark 2.3.** In general we cannot have \( I(\Omega + iV) \subset \Omega^* + iV^* \) if \( \Omega \) is no longer selfdual. This failure is given by an example where \( \Omega \) is the Vinberg cone. See [11] for details.

Now considering \( E^*_{2d+b} \) naturally as an element of \( W^* \), we define

\[
C(w) := E^*_{2d+b} - 2I(w + E) \in W^* \quad (w \in W).
\]

It is evident that \( C \) is a rational mapping \( W \rightharpoonup W^* \) which is holomorphic on \( \Omega + iV \).

Let \( U^\dagger \) denote the space of all antilinear forms on \( U \). We set for \( z = (u, w) \in U \times W \)

\[
C(z) := (2I(w + E) \circ Q(u, \cdot), C(w)) \in U^\dagger \times W^*.
\]

Clearly \( C \) is a rational map \( U \times W \rightharpoonup U^\dagger \times W^* \). It should be noted that if \( z = (u, w) \in D \), then we have \( w \in \Omega + iV \), so that \( C(z) \) is holomorphic on \( D \). We call \( C \) a Cayley transform. This is a slight modification of Penney's [14]. By a verbal translation of Penney's proof [14] we have

**Proposition 2.4.** The image \( C(D) \) of \( D \) is bounded.
Next we put

\[(2.2) \quad (u_1 | u_2)_\eta := \langle Q(u_1, u_2) | E \rangle_\eta \quad (u_1, u_2 \in U).\]

It is easy to see that this is a Hermitian inner product on \( U \). Now define linear maps \( F \mapsto \bar{F} \) from \( U^\dagger \) to \( U \) and \( u \mapsto \hat{u} \) from \( U \) to \( U^\dagger \) by

\[
(\bar{F} | u')_\eta = \langle u', F \rangle, \quad (u', \hat{u}) = (u | u')_\eta \quad (u' \in U).
\]

Obviously they are inverse to one another. Moreover, for every \( w \in W \), let \( \varphi(w) \) be the complex linear operator on \( U \) determined through

\[(2.3) \quad (\varphi(w)u_1 | u_2)_\eta = \langle Q(u_1, u_2) | w \rangle_\eta \quad (u_1, u_2 \in U).\]

Clearly \( \varphi(E) \) is the identity operator, and it is easy to see that \( \varphi(w^*) = \varphi(w)^* \). Let us set

\[
B(f) := 2I^*(E_{2d+b}^* - f) - E \in W \quad (f \in W^*),
\]

\[
B(F, f) := (\varphi(E - \bar{f})^{-1}\bar{F}, B(f)) \in U \times W \quad ((F, f) \in U^\dagger \times W^*).
\]

It is evident that both \( B \) and \( B \) are rational mappings.

**Theorem 2.5** ([11]). \( C : D \rightarrow C(D) \) is biholomorphic and birational with \( C^{-1} = B \).

**Remark 2.6.** Suppose that \( D \) is *quasisymmetric* in this remark. This means that \( \Omega \) is selfdual with respect to the inner product \( \langle \cdot | \cdot \rangle_\eta \) defined by (2.1). We identify \( V^* \) with \( V \) and \( W \) with \( W^* \) by \( \langle \cdot | \cdot \rangle_\eta \). Then by [1, Proposition 3] the product \( \circ \) defined by

\[
\langle v_1 \circ v_2 | v_3 \rangle_\eta := -\frac{1}{2} D_{v_1}D_{v_2}D_{v_3} \log \eta(E) \quad (v_1, v_2, v_3 \in V)
\]

is a Jordan algebra product, so that \( V \) is a Euclidean Jordan algebra in the sense of [5]. The identity element is \( E \), and by the above identification we have \( I(x) = x^{-1} \), the Jordan algebra inverse of \( x \). Identifying further \( U^\dagger \) with \( U \) by means of \( \langle \cdot | \cdot \rangle_\eta \) in (2.2), we get

\[
C(u, w) = (2\varphi(w + E)^{-1}u, (w - E)(w + E)^{-1})
\]

Thus our \( C \) coincides with Dorfmeister's in [2, (2.8)] for quasisymmetric \( D \). We note that the map \( w \mapsto \varphi(w) \) with \( \varphi(w) \) as in (2.3) is a representation of the complex Jordan algebra \( W = V_C \) in the present case (cf. [2, Theorem 2.1]).
3. A characterization of symmetric Siegel domains

By definition, the spaces $g(1/2)$ and $V = g(1)$ have the real inner product $\langle \cdot | \cdot \rangle_\omega$ of (1.2). We first export this inner product to $V^*$ canonically by identifying $V^*$ with $V$ by $\langle \cdot | \cdot \rangle_\omega$. Note that this identification is not quite the same as in Remark 2.6 in general. The real inner product on $V^*$ obtained this way is again denoted by $\langle \cdot | \cdot \rangle_\omega$, which is extended naturally to a Hermitian inner product $(\cdot | \cdot)_\omega$ on $W^*$. On the other hand the complex vector space $U$ has a Hermitian inner product $(\cdot | \cdot)_\omega$ defined by

\[(u_1 | u_2)_\omega := 2 \langle Q(u_1, u_2), \omega \rangle = \langle [Ju_1, u_2], \omega \rangle - i \langle [u_1, u_2], \omega \rangle.\]

We note that $\text{Re}(u_1 | u_2)_\omega = \langle u_1 | u_2 \rangle_\omega$ for $u_1, u_2 \in U$. By a procedure similar to the above we introduce a Hermitian inner product $(\cdot | \cdot)_\omega$ on $U^\uparrow$ by importing the Hermitian inner product (3.1) from $U$.

Let $\beta \in g^*$ be the Koszul form given by

\[\langle x, \beta \rangle := \text{tr}(\text{ad}(Jx) - J \circ \text{ad} x) \quad (x \in g).\]

It is known by [10] (see also [9, §5]) that $\langle [Jx, y], \beta \rangle$ is (the real part of) the inner product on $g$ induced by the Bergman metric of the corresponding Siegel domain $D$ up to a positive multiple. Indeed we can show that $\beta|_n$ is equal to $E_{2d+b}^*$ extended to $n$ by zero-extension.

**Theorem 3.1** ([12]). One has $||C(g \cdot e)||_\omega = ||C(g^{-1} \cdot e)||_\omega$ for all $g \in G$ if and only if the following two conditions are satisfied:

1. $D$ is symmetric,
2. $\omega|_n$ is equal to a positive number multiple of $\beta|_n$.

**Remark 3.2.** Since $C : D \to C(D)$ is biholomorphic with $C(e) = 0$, we have

\[||C(g \cdot e)||_\omega = ||C(g^{-1} \cdot e)||_\omega \quad \text{for all } g \in G\]

\[\iff ||h \cdot 0||_\omega = ||h^{-1} \cdot 0||_\omega \quad \text{for all } h \in G := C \circ G \circ C^{-1}.\]

4. Berezin transforms

For simplicity we set

\[\lambda_0 := \max_{1 \leq j \leq r} \frac{b_j + d_j + p_j/2}{b_j + 2d_j},\]

where $p_j := \sum_{k>j} n_{kj}$. Let $\lambda > \lambda_0$. This is the condition for the non-triviality of certain Hilbert spaces $H^2_\lambda(D)$ of holomorphic functions on $D$ (cf. [17] or [7]). Let $\kappa$
be the Bergman kernel of $D$ (see (1.9)). The Berezin kernel $A_\lambda$ on $D$ is given by

$$A_\lambda(z_1, z_2) := \left( \frac{\kappa(z_1, z_2)^2}{\kappa(z_1, z_1) \kappa(z_2, z_2)} \right)^\lambda \quad (z_1, z_2 \in D).$$

We put $a_\lambda(g) := A_\lambda(g \cdot e, e) \ (g \in G)$. Then it is easy to see that $a_\lambda(g) = a_\lambda(g^{-1})$. We know that $a_\lambda$ is integrable on $G$ with respect to the left Haar measure. Consider the space $L^2(G)$ on $G$ for the left Haar measure. The Berezin transform $B_\lambda$, when transferred to $L^2(G)$, is given by the convolution operator

$$B_\lambda f(x) := \int_G f(y) a_\lambda(y^{-1}x) \, dy = f * a_\lambda(x) \quad (f \in L^2(G)).$$

On the other hand, the inner product $\langle \cdot | \cdot \rangle_\omega$ on $g$ defines a left invariant Riemannian metric on $G$, relative to which we have the Laplace-Beltrami operator $\mathcal{L}_\omega$ on $G$. In order to express $\mathcal{L}_\omega$ in terms of the elements of the enveloping algebra $U(g)$, we set for $X \in g$

$$Xf(x) := \frac{d}{dt} f((\exp-tX)x) \big|_{t=0}, \quad \overline{X}f(x) := \frac{d}{dt} f(x \exp tX) \big|_{t=0}.\]$$

These are extended to $U(g)$ by homomorphisms. Though the following lemma holds for any connected Lie group, we write it down here in our situation. See [19, Theorem 1] for a proof.

**Lemma 4.1.** Take $\Psi \in g$ for which one has $\langle X | \Psi \rangle_\omega = \text{tr} \, \text{ad} \, (X)$ for all $x \in g$. Then $\mathcal{L}_\omega = -\overline{\Lambda} + \overline{\Psi}$, where $\Lambda := X_1^2 + \cdots + X_{2N}^2$ with an orthonormal basis $\{X_j\}_{j=1}^{2N}$ of $g$ relative to $\langle \cdot | \cdot \rangle_\omega$.

We note that $\Psi \in a$ in our case.

**Theorem 4.2** ([13]). Let $\lambda > \lambda_0$ be fixed. Then, $B_\lambda$ commutes with $\mathcal{L}_\omega$ if and only if $D$ is symmetric and $\omega|_\mathfrak{n}$ is equal to a positive number multiple of $\beta|_\mathfrak{n}$.

We indicate here how Theorem 4.2 is derived from Theorem 3.1.

(1) $B_\lambda$ commutes with $\mathcal{L}_\omega$ $\iff$ $(-\overline{\Lambda} + \overline{\Psi}) a_\lambda = (-\Lambda + \Psi) a_\lambda$.

(2) Since $a_\lambda(g) = a_\lambda(g^{-1})$, we have $\overline{X} a_\lambda(g) = X a_\lambda(g^{-1})$ for all $X \in U(g)$ and $g \in G$.

(3) $(-\Lambda - \Psi) a_\lambda(g) = \lambda a_\lambda(g) \left( \lambda \lVert C(g \cdot e) \lVert^2_\omega - \langle \Psi, \alpha \rangle \right)$ for some $\alpha \in a^*$.

**References**


Department of Mathematics
Faculty of Science
Kyoto University
Sakyo-ku 606-8502
Kyoto
Japan