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Berezin Transforms and Laplace-Beltrami Operators

on Homogeneous Siegel Domains

— commutativity, symmetry of the domain and a Cayley transform —

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1. Preliminaries

Homogeneous Siegel domains are described in terms of normal $j$-algebras (cf. [15]), of which we are going to give the definition. Let $g$ be a split solvable Lie algebra, $J$ a linear operator on $g$ with $J^2 = -I$ and $\omega$ a linear form on $g$. Then the triple $(g, J, \omega)$ is called a normal $j$-algebra if

\begin{align}
(1.1) \quad [Jx, Jy] &= [x, y] + J[Jx, y] + J[x, Jy] \quad (\text{for all } x, y \in g), \\
(1.2) \quad \langle x \mid y \rangle_\omega &= \langle [Jx, y], \omega \rangle \text{ defines a } J\text{-invariant inner product on } g.
\end{align}

We describe here some basic facts about normal $j$-algebras following [15] and [17] (see also [16]). Let $(g, J, \omega)$ be a normal $j$-algebra. Let $n := [g, g]$ be the derived algebra of $g$, and $a$ the orthogonal complement of $n$ in $g$ relative to the inner product $\langle \cdot \mid \cdot \rangle_\omega$. Evidently we have $g = a + n$. Moreover, $a$ is a commutative subalgebra of $g$ such that $\text{ad}(a)$ consists of semisimple operators on $g$. For every $\alpha \in a^*$ we set

$$n_\alpha := \{x \in n; [h, x] = \langle h, \alpha \rangle x \quad \text{for all } h \in a\}.$$ 

Take all $\alpha \in a^*$ such that $n_\alpha \neq \{0\}$ and $Jn_\alpha \subset a$, and number them as $\alpha_1, \ldots, \alpha_r$. We have $\dim a = r$ and $\dim n_{\alpha_k} = 1$ for every $k$. The number $r$ is called the rank of the normal $j$-algebra $g$. We can reorder $\alpha_1, \ldots, \alpha_r$, if necessary, so that all the $\alpha$ such that $n_\alpha \neq \{0\}$ (such an $\alpha$ is called a root of the normal $j$-algebra) are of the following form (some roots might be missing):

$$\frac{1}{2}(\alpha_m + \alpha_k) \quad (1 \leq k < m \leq r), \quad \frac{1}{2}(\alpha_m - \alpha_k) \quad (1 \leq k < m \leq r), \quad \frac{1}{2}\alpha_k \quad (1 \leq k \leq r), \quad \alpha_k \quad (1 \leq k \leq r).$$

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We note that if $\alpha, \beta$ are distinct roots, then $n_{\alpha}$ is orthogonal to $n_{\beta}$. Put

$$g(0) := a \oplus \sum_{m>k} n_{(\alpha_{m}-\alpha_{k})/2}, \quad g(1/2) := \sum_{i=1}^{r} n_{\alpha_{i}/2},$$

$$g(1) := \sum_{i=1}^{r} n_{\alpha_{i}} \oplus \sum_{m>k} n_{(\alpha_{m}+\alpha_{k})/2}.$$

Understanding $g(i) = 0$ for $i > 1$, we have $[g(i), g(j)] \subset g(i+j)$. Moreover

$$Jn_{(\alpha_{m}-\alpha_{k})/2} = n_{(\alpha_{m}+\alpha_{k})/2} \quad (m > k), \quad Jn_{\alpha_{i}/2} = n_{\alpha_{i}/2} \quad (1 \leqq i \leqq r),$$

so that $Jg(0) = g(1)$ and $Jg(1/2) = g(1/2)$. Taking $E_{i} \in n_{\alpha_{i}} \ (i = 1, \ldots, r)$ such that $\alpha_{k}(JE_{i}) = \delta_{ki}$, we put $H_{i} := JE_{i} \in a \oplus n_{\alpha_{i}/2}$ and

$$(1.3) \quad H := H_{1} + \cdots + H_{r}, \quad E := E_{1} + \cdots + E_{r}.$$We write down here the constants used frequently in this note:

$$n_{mk} := \dim_{\mathbb{R}} n_{(\alpha_{m}-\alpha_{k})/2} = \dim_{\mathbb{R}} n_{(\alpha_{m}+\alpha_{k})/2} \quad (1 \leqq k < m \leqq r),$$

$$b_{i} := \frac{1}{2} \dim_{\mathbb{R}} n_{\alpha_{i}/2} \quad (1 \leqq i \leqq r),$$

$$d_{j} := 1 + \frac{1}{2} \left( \sum_{k>j} n_{kj} + \sum_{i<j} n_{ji} \right) \quad (1 \leqq j \leqq r).$$

Let $G = \exp g$ be the connected and simply connected Lie group corresponding to $g$. Note that $g(0)$ is a Lie subalgebra of $g$. We denote by $G(0)$ the corresponding subgroup $\exp g(0)$ of $G$. The group $G(0)$ acts on $V := g(1)$ by adjoint action. Let $\Omega$ be the $G(0)$-orbit through $E$. By [17, Theorem 4.15] $\Omega$ is a regular open convex cone in $V$, and $G(0)$ acts on $\Omega$ simply transitively. Being invariant under $J$, the subspace $g(1/2)$ is considered as a complex vector space by means of $-J$. We shall write this complex vector space by $U$. We put $W := V_{\mathbb{C}}$, the complexification of $V$. The conjugation of $W$ relative to the real form $V$ is written as $w \mapsto w^{*}$. The real bilinear map $Q$ defined by

$$Q(u, u') := \frac{1}{2} ([Ju, u'] - i[u, u']) \quad (u, u' \in g(1/2))$$

turns out to be a complex sesqui-linear (complex linear in the first variable and antilinear in the second) Hermitian map $U \times U \to W$ which is $\Omega$-positive. This means that

$$Q(u', u) = Q(u, u')^{*} \quad (u, u' \in U), \quad Q(u, u) \in \overline{\Omega} \setminus \{0\} \text{ for all } u \in U \setminus \{0\}.$$
With these data we define the Siegel domain \( D \) corresponding to the normal \( j \)-algebra \((g, J, \omega)\) to be

\[
D := \{(u, w) \in U \times W ; w + w^* - Q(u, u) \in \Omega\}.
\]

Note that we take a generalized right half plane rather than a more familiar upper half plane.

Consider the Lie subalgebra \( \mathfrak{n}_D := g(1) + g(1/2) \). It is at most 2-step nilpotent. Let \( N_D = \exp \mathfrak{n}_D \) be the corresponding connected and simply connected nilpotent Lie group contained in \( G \). We write the elements of \( N_D \) by \( n(a, b)(a \in g(1), b \in g(1/2)) \).

The group \( N_D \) acts on \( D \) by

\[
(1.5) \quad n(a, b) \cdot (u, w) = (u + b, w + ia + \frac{1}{2}Q(b, b) + Q(u, b)) \quad ((u, w) \in D).
\]

On the other hand, the adjoint action of \( G(0) \) on \( g(1/2) \) commutes with \( J \). This implies that \( G(0) \) acts on \( U \) complex-linearly. Moreover the adjoint action of \( G(0) \) on \( V = g(1) \) extends complex-linearly to \( W \), so that \( G(0) \) acts on \( D \) complex-linearly. Hence \( G = N_D \times G(0) \) acts on \( D \) simply transitively. To see this more explicitly, put \( e := (0, E) \). Then given \( z = (u, w) \in D \), we can find a unique \( h \in G(0) \) satisfying \( hE = \text{Re} w - Q(u, u)/2 \). Taking \( n = n(\text{Im} w, u) \in N_D \), we see by (1.5) that \( z = nh \cdot e \).

For every \( s = (s_1, \ldots, s_r) \in \mathbb{C}^r \) let \( \chi_s \) be the one-dimensional representation of \( A := \exp \mathfrak{a} \) defined by

\[
\chi_s \left( \exp \sum_k t_k H_k \right) = \exp \left( \sum_s s_k t_k \right) \quad (t_1, \ldots, t_r \in \mathbb{R}).
\]

Let \( N := \exp \mathfrak{n} \). It is clear that \( G = N \rtimes A \). We extend \( \chi_s \) to a one-dimensional representation of \( G \) by defining \( \chi_s(n) = 1 \) for \( n \in N \). Let us define functions \( \Delta_s (s \in \mathbb{C}^r) \) on \( \Omega \) by \( \Delta_s(hE) = \chi_s(h) \) \( (h \in G(0)) \). Evidently it holds that

\[
(1.6) \quad \Delta_s(hx) = \chi_s(h) \Delta_s(x) \quad (h \in G(0), \ x \in \Omega).
\]

We know that \( \Delta_s \) extends to a holomorphic function on the tube domain \( \Omega + iV \) (cf. for example [7, Corollary 2.5]).

For \( h \in G(0) \), let \( \text{Ad}_{g(1)}(h) := (\text{Ad} h)|_{g(1)} \). Moreover let \( \text{Ad}_U(h) \) stand for the complex linear operator on \( U \) defined by the adjoint action of \( h \in G(0) \) on \( g(1/2) \), and \( \det \text{Ad}_U(h) \) its determinant as a complex linear operator. Then, with \( d := (d_1, \ldots, d_r) \) and \( b := (b_1, \ldots, b_r) \), we have for \( h \in G(0) \)

\[
(1.7) \quad \det \text{Ad}_{g(1)}(h) = \chi_d(h), \quad |\det \text{Ad}_U(h)|^2 = \chi_b(h).
\]
By [6, §5] or [18, §II.6], it is known that $D$ has a Bergman kernel $\kappa$. If Hol $(D)$ denotes the Lie group of the holomorphic automorphisms of $D$, then $\kappa$ satisfies

\begin{equation}
\kappa(z_1, z_2) = \kappa(g \cdot z_1, g \cdot z_2) \det g'(z_1) \det g'(z_2) \quad (g \in \text{Hol } (D), \; z_1, z_2 \in D),
\end{equation}

where $g'(z)$ is the complex Jacobian map of $g$ at $z \in D$. The description of the simple transitive action of $G$ on $D$ together with the property (1.7) and (1.8) shows

\begin{equation}
\kappa(z_1, z_2) = C \cdot \Delta_{-2d-b}(w_1 + w_2^* - Q(u_1, u_2)) \quad (z_j = (u_j, w_j) \in D)
\end{equation}

with $C = \kappa(e, e) \Delta_{2d+b}(2E) > 0$. We put $\eta := \Delta_{-2d-b}$ in what follows for simplicity.

2. Cayley transform

Let $D_v$ be the directional derivative in the direction $v \in V$ given by

$$D_v f(x) = \frac{d}{dt} f(x + tv)|_{t=0}.$$ 

For every $x \in \Omega$ we define $\mathcal{I}(x) \in V^*$ to be $-\nabla \log \eta(x)$, that is,

$$\langle v, \mathcal{I}(x) \rangle = -D_v \log \eta(x) \quad (v \in V).$$

$\mathcal{I}$ is called the pseudoinverse map. By [3, §2], $\mathcal{I}$ gives a diffeomorphism of $\Omega$ onto the dual cone $\Omega^*$ in $V^*$, where

$$\Omega^* := \{ \xi \in V^* ; \; \langle x, \xi \rangle > 0 \; \text{for all } x \in \overline{\Omega} \setminus \{0\} \}.$$

The group $G(0)$ acts also on $V^*$ by the coadjoint action: $h \cdot \xi = \xi \circ h^{-1}$, where $h \in G(0)$ and $\xi \in V^*$. It is easy to show by using (1.6) that $\mathcal{I}$ is $G(0)$-equivariant:

$$\mathcal{I}(hx) = h \cdot \mathcal{I}(x) \quad (h \in G(0), \; x \in \Omega).$$

In particular, $\mathcal{I}(\lambda x) = \lambda^{-1} \mathcal{I}(x)$ for all $\lambda > 0$, and $G(0)$ acts on $\Omega^*$ simply transitively. Moreover, $\mathcal{I}$ can be extended to a rational map $W \to W^*$ [4, Satz I.2.3].

In order to find an inverse map of $\mathcal{I}$, we need to dualize the above matters concerning $\mathcal{I}$. First we define $E_1^*, \ldots, E_r^* \in V^*$ by

$$\left\langle \sum_{j=1}^r x_j E_j + \sum_{m>k} X_{mk}, E_i^* \right\rangle = x_i \quad (x_j \in \mathbb{R}, \; X_{mk} \in \mathfrak{n}_{(\alpha_m + \alpha_k)/2}),$$

and for every $s = (s_1, \ldots, s_r) \in \mathbb{R}^r$,

$$E_s^* := s_1 E_1^* + \cdots + s_r E_r^* \in V^*.$$ 

We can show that $\mathcal{I}(E) = E_{2d+b}^*$. Next we put $s^* := (s_r, \ldots, s_1)$ and set

$$x_s^* := x_{-s^*}, \quad \Delta_s^* (h \cdot E_{2d+b}^*) := x_s^*(h) \quad (h \in G(0)).$$
\( \Delta^*_s \) is a function on \( \Omega^* \) such that \( \Delta^*_s(h \cdot \xi) = \chi^*_s(h) \Delta^*_s(\xi) \) for \( h \in G(0) \) and \( \xi \in V^* \).

We define \( \eta^* := \Delta^* - 2d^* - b^* \) and
\[
\langle \mathcal{I}^*(\xi), f \rangle := -D_f \log \eta^*(\xi) \quad (\xi \in \Omega^*, f \in V^*).
\]
Thus \( \mathcal{I}^*(\xi) \in V \) and \( \mathcal{I}^* \) gives a diffeomorphism of \( \Omega^* \) onto \( \Omega \).

Moreover, \( \mathcal{I}^* \) is \( G(O) \)-equivariant, that is, \( \mathcal{I}^*(h \cdot \xi) = h(\mathcal{I}^*(\xi)) \) for any \( h \in G(O) \).

We can prove that \( \mathcal{I}^* \) is extended to a rational map \( W^* \rightarrow W \).

**Proposition 2.1.** \( \mathcal{I}^* = \mathcal{I}^{-1} \).

**Theorem 2.2** ([11]). (1) \( \mathcal{I} \) is holomorphic on \( \Omega + iV \), and \( \mathcal{I}^* \) is holomorphic on \( \Omega^* + iV^* \).

(2) \( \mathcal{I}(\Omega + iV) \) is contained in the holomorphic domain of \( \mathcal{I}^* \), and \( \mathcal{I}^*(\Omega^* + iV^*) \) is contained in the holomorphic domain of \( \mathcal{I} \).

**Remark 2.3.** In general we cannot have \( \mathcal{I}(\Omega + iV) \subset \Omega^* + iV^* \) if \( \Omega \) is no longer selfdual. This failure is given by an example where \( \Omega \) is the Vinberg cone. See [11] for details.

Now considering \( E_{2d+b}^* \) naturally as an element of \( W^* \), we define
\[
C(w) := E_{2d+b}^* - 2 \mathcal{I}(w + E) \in W^* \quad (w \in W).
\]
It is evident that \( C \) is a rational mapping \( W \rightarrow W^* \) which is holomorphic on \( \Omega + iV \).

Let \( U^\dagger \) denote the space of all antilinear forms on \( U \). We set for \( z = (u, w) \in U \times W \)
\[
C(z) := (2 \mathcal{I}(w + E) \circ Q(u, \cdot), C(w)) \in U^\dagger \times W^*.
\]
Clearly \( C \) is a rational map \( U \times W \rightarrow U^\dagger \times W^* \). It should be noted that if \( z = (u, w) \in D \), then we have \( w \in \Omega + iV \), so that \( C(z) \) is holomorphic on \( D \). We call \( C \) a Cayley transform. This is a slight modification of Penney’s [14]. By a verbal translation of Penney’s proof [14] we have

**Proposition 2.4.** The image \( C(D) \) of \( D \) is bounded.
Next we put

\[(2.2)\quad (u_1 | u_2)_\eta := \langle Q(u_1, u_2) | E \rangle_\eta \quad (u_1, u_2 \in U).\]

It is easy to see that this is a Hermitian inner product on \(U\). Now define linear maps \(F \mapsto \overline{F}\) from \(U^\dagger\) to \(U\) and \(u \mapsto \widehat{u}\) from \(U\) to \(U^\dagger\) by

\[(\overline{F} | u')_\eta = \langle u', F \rangle, \quad (u', \widehat{u}) = (u | u')_\eta \quad (u' \in U).\]

Obviously they are inverse to one another. Moreover, for every \(w \in W\), let \(\varphi(w)\) be the complex linear operator on \(U\) determined through

\[(2.3)\quad (\varphi(w)u_1 | u_2)_\eta = \langle Q(u_1, u_2) | w \rangle_\eta \quad (u_1, u_2 \in U).\]

Clearly \(\varphi(E)\) is the identity operator, and it is easy to see that \(\varphi(w^*) = \varphi(w)^*\). Let us set

\[B(f) := 2\mathcal{I}^*(E_{2d+b} - f) - E \in W \quad (f \in W^*),\]

\[B(F, f) := (\varphi(E - \overline{f})^{-1} \overline{F}, B(f)) \in U \times W \quad ((F, f) \in U^\dagger \times W^*).\]

It is evident that both \(B\) and \(\mathcal{B}\) are rational mappings.

**Theorem 2.5** ([11]). \(C : D \rightarrow C(D)\) is biholomorphic and birational with \(C^{-1} = B\).

**Remark 2.6.** Suppose that \(D\) is quasisymmetric in this remark. This means that \(\Omega\) is selfdual with respect to the inner product \(\langle \cdot | \cdot \rangle_\eta\) defined by \((2.1)\). We identify \(V^*\) with \(V\) and \(W\) with \(W^*\) by \(\langle \cdot | \cdot \rangle_\eta\). Then by [1, Proposition 3] the product \(\circ\) defined by

\[\langle v_1 \circ v_2 | v_3 \rangle_\eta := -\frac{1}{2} D_{v_1} D_{v_2} D_{v_3} \log \eta(E) \quad (v_1, v_2, v_3 \in V)\]

is a Jordan algebra product, so that \(V\) is a Euclidean Jordan algebra in the sense of [5]. The identity element is \(E\), and by the above identification we have \(\mathcal{I}(x) = x^{-1}\), the Jordan algebra inverse of \(x\). Identifying further \(U^\dagger\) with \(U\) by means of \(\langle \cdot | \cdot \rangle_\eta\) in \((2.2)\), we get

\[C(u, w) = (2 \varphi(w + E)^{-1} u, (w - E)(w + E)^{-1}).\]

Thus our \(C\) coincides with Dorfmeister’s in [2, (2.8)] for quasisymmetric \(D\). We note that the map \(w \mapsto \varphi(w)\) with \(\varphi(w)\) as in \((2.3)\) is a representation of the complex Jordan algebra \(W = V_C\) in the present case (cf. [2, Theorem 2.1]).
3. A characterization of symmetric Siegel domains

By definition, the spaces $g(1/2)$ and $V = g(1)$ have the real inner product $\langle \cdot | \cdot \rangle_\omega$ of (1.2). We first export this inner product to $V^*$ canonically by identifying $V^*$ with $V$ by $\langle \cdot | \cdot \rangle_\omega$. Note that this identification is not quite the same as in Remark 2.6 in general. The real inner product on $V^*$ obtained this way is again denoted by $\langle \cdot | \cdot \rangle_\omega$, which is extended naturally to a Hermitian inner product $\langle \cdot | \cdot \rangle_\omega$ on $W^*$. On the other hand the complex vector space $U$ has a Hermitian inner product $\langle \cdot | \cdot \rangle_\omega$ defined by

\[(u_1 | u_2)_\omega := 2 \langle Q(u_1, u_2), \omega \rangle = \langle [Ju_1, u_2], \omega \rangle - i \langle [u_1, u_2], \omega \rangle.
\]

We note that $\text{Re}(u_1 | u_2)_\omega = \langle u_1 | u_2 \rangle_\omega$ for $u_1, u_2 \in U$. By a procedure similar to the above we introduce a Hermitian inner product $\langle \cdot | \cdot \rangle_\omega$ on $U^\uparrow$ by importing the Hermitian inner product (3.1) from $U$.

Let $\beta \in g^*$ be the Koszul form given by

$$\langle x, \beta \rangle := \text{tr} (\text{ad} (Jx) - J \circ (\text{ad} x)) \quad (x \in g).$$

It is known by [10] (see also [9, §5]) that $\langle [Jx, y], \beta \rangle$ is (the real part of) the inner product on $g$ induced by the Bergman metric of the corresponding Siegel domain $D$ up to a positive multiple. Indeed we can show that $\beta|_n$ is equal to $E_{2d+b}^*$ extended to $n$ by zero-extension.

Theorem 3.1 ([12]). One has $\|C(g \cdot e)\|_\omega = \|C(g^{-1} \cdot e)\|_\omega$ for all $g \in G$ if and only if the following two conditions are satisfied:

1. $D$ is symmetric,
2. $\omega|_n$ is equal to a positive number multiple of $\beta|_n$.

Remark 3.2. Since $C : D \to C(D)$ is biholomorphic with $C(e) = 0$, we have

$$\|C(g \cdot e)\|_\omega = \|C(g^{-1} \cdot e)\|_\omega \quad \text{for all } g \in G$$

$$\iff \|h \cdot 0\|_\omega = \|h^{-1} \cdot 0\|_\omega \quad \text{for all } h \in G := C \circ G \circ C^{-1}.$$

4. Berezin transforms

For simplicity we set

$$\lambda_0 := \max_{1 \leq j \leq r} \frac{b_j + d_j + p_j/2}{b_j + 2d_j},$$

where $p_j := \sum_{k>j} n_{kj}$. Let $\lambda > \lambda_0$. This is the condition for the non-triviality of certain Hilbert spaces $H_\lambda^2(D)$ of holomorphic functions on $D$ (cf. [17] or [7]). Let $\kappa$
be the Bergman kernel of $D$ (see (1.9)). The Bergman kernel $A_\lambda$ on $D$ is given by
\[
A_\lambda(z_1, z_2) := \left( \frac{\kappa(z_1, z_2)^2}{\kappa(z_1, z_1) \kappa(z_2, z_2)} \right)^\lambda, \quad (z_1, z_2 \in D).
\]
We put $a_\lambda(g) := A_\lambda(g \cdot e, e) \quad (g \in G)$. Then it is easy to see that $a_\lambda(g) = a_\lambda(g^{-1})$.

We know that $a_\lambda$ is integrable on $G$ with respect to the left Haar measure. Consider the space $L^2(G)$ on $G$ for the left Haar measure. The Berezin transform $B_\lambda$, when transferred to $L^2(G)$, is given by the convolution operator
\[
B_\lambda f(x) := \int_G f(y) a_\lambda(y^{-1}x) \, dy = f * a_\lambda(x) \quad (f \in L^2(G)).
\]

On the other hand, the inner product $\langle \cdot | \cdot \rangle_\omega$ on $g$ defines a left invariant Riemannian metric on $G$, relative to which we have the Laplace-Beltrami operator $\mathcal{L}_\omega$ on $G$. In order to express $\mathcal{L}_\omega$ in terms of the elements of the enveloping algebra $U(g)$, we set for $X \in g$
\[
Xf(x) := \frac{d}{dt} f((\exp - tX)x)|_{t=0}, \quad \bar{X}f(x) := \frac{d}{dt} f(x \exp tX)|_{t=0}.
\]
These are extended to $U(g)$ by homomorphisms. Though the following lemma holds for any connected Lie group, we write it down here in our situation. See [19, Theorem 1] for a proof.

**Lemma 4.1.** Take $\Psi \in g$ for which one has $\langle X | \Psi \rangle_\omega = \text{tr} \text{ad} (X)$ for all $x \in g$.
Then $\mathcal{L}_\omega = -\bar{\Lambda} + \bar{\Psi}$, where $\Lambda := X_1^2 + \cdots + X_{2N}^2$ with an orthonormal basis $\{X_j\}_{j=1}^{2N}$ of $g$ relative to $\langle \cdot | \cdot \rangle_\omega$.

We note that $\Psi \in a$ in our case.

**Theorem 4.2 ([13]).** Let $\lambda > \lambda_0$ be fixed. Then, $B_\lambda$ commutes with $\mathcal{L}_\omega$ if and only if $D$ is symmetric and $\omega|_\mathfrak{n}$ is equal to a positive number multiple of $\beta|_\mathfrak{n}$.

We indicate here how Theorem 4.2 is derived from Theorem 3.1.

1. $B_\lambda$ commutes with $\mathcal{L}_\omega$ $\iff$ $(-\bar{\Lambda} + \bar{\Psi})a_\lambda = (-\Lambda + \Psi)a_\lambda$.
2. Since $a_\lambda(g) = a_\lambda(g^{-1})$, we have $\bar{X}a_\lambda(g) = Xa_\lambda(g^{-1})$ for all $X \in U(g)$ and $g \in G$.
3. $(\Lambda - \Psi)a_\lambda(g) = \lambda a_\lambda(g) (\lambda \|C(g \cdot e)\|^2_\omega - \langle \Psi, \alpha \rangle)$ for some $\alpha \in a^*$.

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