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COMPLETE INTEGRABILITY OF HAMILTONIAN SYSTEMS AND DIFFERENTIAL GALOIS GROUPS

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The subject of the lecture is a recent new approach of the old problem of integrability of systems of classical mechanics based upon differential Galois theory and algebraic groups. (The lecture is mainly based upon a joint work with Juan Morales from Barcelona.)

Roughly speaking an hamiltonian system with \( n \) degrees of freedom is integrable if there exists sufficiently many first integrals (integrals of motion) such that its integration can be reduced to quadratures. Here we are mainly interested in completely integrable systems in the classical Liouville sense (i.e. there must exists \( n \) first integrals generically independent and in involution).
The investigation of such systems was a central subject during the last century and stimulated the appearance of the theory of Lie groups. Single out integrable systems among all the Hamiltonian systems remain a big open question. H. Poincaré proved that the existence of global integrals of motion is exceptional, a fortiori completely integrable systems are even more exceptional and only a small number of such systems are known. Recently the situation was perfectly stated by Perelomov (1990): To find a general criterion for complete integrability seems at present a hopeless task. In this lecture we will present such a criterion. It will sound quite abstract, but it is (surprisingly even for me...) easy to apply to actual problems: it gives new elegant solutions of some classical problems (as Lagrange top) and allows us to solve very easily a lot of open problems.

1. Ordinary differential equations, first integrals, linearized (variational) equations.
2. Hamiltonian systems (the symplectic formulation). Completely integrable systems.
3. A first (weak) formulation of our theorem: if an hamiltonian system is completely integrable, then the Lie algebra of the Zariski closure in the complex linear symplectic group \( Sp(2n;C) \) of the monodromy group of a variational equation along a solution (in complexified time) is abelian.
4. Basics about differential Galois theory
5. Our main theorem: if an hamiltonian system is completely integrable,
then the Lie algebra of the differential Galois group of the variational equation along any solution is *abelian*.

6. Applications.

Recipe for applications. Examples of solutions of classical open problems:

- Collinear three body problems with homogeneous potentials, Bianchi IX Cosmological model, spring pendulum... (by J.Morales and J.P. Ramis),
- Collinear four bodies problems with potential $1/r^2$ (solution of the first open problem after Jacobi and Moser-Calogero results, by Emmanuelle Juillard-Tosel 1999),
- Non integrability of the planar Newton three bodies problem near the equilateral Lagrange solution by Delphine Boucher and Alexei Tsygvintsev 1999),
- Henon-Heiles systems...