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Chaotic maps on measure spaces and behavior of states

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Introduction. As well known, chaotic maps are considered as those $\varphi$'s which have the following property (cf.[1]).

(1) The set of all periodic points for $\varphi$ are dense.
(2) $\varphi$ is transitive.
(3) $\varphi$ depends on sensitive initial condition.

Those properties are concerned with the orbit of a given initial point. In this note, we consider how probability density functions changed by iteration of chaotic maps. More generally, we study behavior of states by $\ast$-endomorphisms of von Neumann algebras associated with chaotic maps. In particular, we show some theorems concerning the limits of iterated states, which are stated as follows.

(4) The sequence of iterated states by a chaotic map converges to a unique state in the norm topology.

In Section 1 and 2, we note some results related to $\ast$-endomorphisms of von Neumann algebras and iterated states by chaotic maps respectively, which are stated without proof. Section 3 consists of examples only which give us the meaning of theorems in Section 2 and provide fruitful discussion on our theory. Moreover we can find deep relationship between our study and wavelets theory (cf.[4]). This note is a continuation of [5].

§1. A $\ast$-endomorphism of von Neumann algebra associated with a family of isometries. Let $\mathcal{H}$ be a Hilbert space with inner product $\langle \cdot, \cdot \rangle$. In this note $\{V_i\}_{i=1}^n$ means a family of isometries on $\mathcal{H}$ satisfying the following property and is said to be a FIC on $\mathcal{H}$ for short.

(C.1) $\{V_iV_i^*\}_{i=1}^n$ is a set of mutually orthogonal projections and $\sum_{i=1}^n V_iV_i^* = I$.

Of course, this family $\{V_i\}_{i=1}^n$ on $\mathcal{H}$ is the generators of the image of a representation of Cuntz-algebra $O_n$ [3]. Moreover we can define a $\ast$-endomorphism $\alpha_V$ of the full operator algebra $B(\mathcal{H})$ as follows. 

(C.2) $\alpha_V(T) = \sum_{i=1}^n V_iTV_i^*, (T \in B(\mathcal{H}))$
If a von Neumann algebra $M$ on $\mathcal{H}$ is invariant for $\alpha_V$, then $\alpha_V$ becomes a $*$-endomorphism of $M$. For $n$ and a positive integer $k$, we denote by $I(n)$ the set $\{1, 2, \ldots, n\}$ and $I(n)^k$ the set of all $k$-tuples $\mu = (j_1, \ldots, j_k)$ with $j_i \in \{1, 2, \ldots, n\}$. For $\mu$ in $I(n)^k$ we denote by $V(\mu)$ the isometry $V_{j_1}V_{j_2} \cdots V_{j_k}$ on $(\mathcal{H})$. Then $\{V(\mu)\mu \in I(n)^k\}$ is a family of isometries whose final projections are mutually orthogonal. When $\alpha_V$ is a $*$-endomorphism of $M$, $\alpha_V^*\alpha_V$ is of the form:

$$\alpha_V^k(T) = \sum_{\mu\in I(n)^k} V(\mu)TV(\mu)^*, \quad (T \in M).$$

Proposition 1.1. Let $\{V_i\}_{i=1}^n$ be a FIC on $\mathcal{H}$ and $e$ a unit vector in $\mathcal{H}$ such that $V_1e = e$. We put

$$\text{ONS}(e, V) = \bigcup_{k=1}^\infty \{V(\mu)e|\mu \in I(n)^k\}.$$  

Then $\text{ONS}(e, V)$ is an orthonormal system.

Remark. An orthonormal system $\text{ONS}(e, V)$ in the proposition above is regarded as the sequence $\{e_k\}_{k=1}^\infty$ which is inductively defined as follows: $e_1 = e$ and

$$e_{i+n(i-1)} = V_ie_\ell \quad (i \in I(n), \ell \in \mathbb{N}).$$

(c.f. 2 of [2])

For a von Neumann algebra $M$ on $\mathcal{H}$, $M_*$ denotes the predual of $M$. We denote by $\alpha_V^*$ the transpose map of $\alpha_V$ with respect to the duality of $M$ and $M_*$. The vector state in $M_*$ associated with unit vector $\xi$ in $\mathcal{H}$ is denoted by $\omega_\xi$, that is, for $T$ in $M$, $\omega_\xi(T) = <T\xi, \xi>$ and

$$\omega_\xi(\alpha_V(T)) = <\alpha_V(T)\xi, \xi> = \alpha_V^*(\omega_\xi)(T).$$

Moreover we have

$$\omega_\xi(\alpha_V(\omega_\xi)) = \sum_{i=1}^n \omega_i\alpha_V^*(\omega_\xi).$$

When $e$ is a unit vector such that $V_1e = e$, namely, it is an eigenvector for eigenvalue 1 of $V_1$, we denote by $\mathcal{H}_e$ the subspace of $\mathcal{H}$ spanned by $\text{ONS}(e, V)$.

Proposition 1.2. Let $\{V_i\}_{i=1}^n$ be a FIC on $\mathcal{H}$. If there exists a unit vector $e$ such that $V_1e = e$, then for any unit vector $\xi$ in the subspace $\mathcal{H}_e$ it follows that

$$\lim_{n \to \infty}(\alpha_V^*)^n(\omega_\xi) = \omega_e \quad (\text{norm topology}).$$

Proposition 1.3. Let $\{V_i\}_{i=1}^n$ be a FIC on $\mathcal{H}$. If there exists a unit vector $e$ such that $V_1e = e$, then for any state $\omega$ of the form $\omega = \sum_{k=1}^\infty \omega_{\xi_k}$ where $\xi_k$'s are in $\mathcal{H}_e$, it follows
that
\[ \lim_{n \to \infty} (\alpha^*_V)^n(\omega) = \omega_e \quad (\text{norm topology}). \]

Proposition 1.4. Let \( \{V_i\}_{i=1}^n \) be a FIC on \( \mathcal{H} \) and \( e \) a unit vector such that \( V_ie = e \). If \( ONS(e, V) \) is complete, then for any state \( \omega \) in the predual of \( B(\mathcal{H}) \) it follows that
\[ \lim_{n \to \infty} (\alpha^*_V)^n(\omega) = \omega_e \quad (\text{norm topology}). \]

Proposition 1.5. Let \( M \) be a Neumann algebra on \( \mathcal{H} \) and \( \{V_i\}_{i=1}^n \) and \( \{W_i\}_{i=1}^n \) be a couple of families of isometries on \( \mathcal{H} \) satisfying (1.1). Suppose that \( M \) is invariant for \( \alpha_V \) and \( \alpha_W \). Then following conditions are equivalent.

1. \( \alpha_V(T) = \alpha_W(T) \) for all \( T \) in \( M \).

2. \( (W_1, \ldots, W_n) = (V_1, \ldots, V_n) \)

that is, \( W_i = \sum_{j=1}^n V_j h_{ji}, \ (1 \leq i \leq n) \), where each \( h_{ij} \) is a unitary element in the commutant \( M' \) of \( M \) on the Hilbert space \( \mathcal{H} \).

§2. Chaotic maps and behavior of states. Let \( X \) be a measure space with measure \( m \) and \( \varphi \) a measurable map on \( X \). Here we note some notations concerning \( X \) and \( \varphi \).

1. \( m \circ \varphi \) denotes the measure on \( X \) defined by \( m \circ \varphi(E) = m(\varphi(E)) \) and if the map \( \varphi \) is absolutely continuous with respect to \( m \), the Radon-Nikodym derivative for \( m \circ \varphi \) and \( m \) is denoted by \( \frac{dm \circ \varphi}{dm} \).

2. \( \alpha_\varphi \) denotes the *-endomorphism of \( L^\infty(X) = L^\infty(X, m) \) defined by \( \alpha_\varphi(f) = f(\varphi(x)) \) for \( f \) in \( L^\infty(X) \).

3. \( T_\varphi \) denotes the linear operator on the Hilbert space \( \mathcal{H} = L^2(X) = L^2(X, m) \) defined by \( (T_\varphi \xi)(x) = \xi(\varphi(x)) \) for \( \xi \) in \( \mathcal{H} \).

4. For a subset \( Y \) of \( X \), \( \chi_Y \) means the characteristic function of \( Y \).

5. For a measurable function \( f \) on \( X \), \( M_f \) denotes the multiplication operator on \( L^2(X) \) defined by \( M_f \xi = f \xi \) for \( \xi \) in \( L^2(X) \).
For $f$ in $L^{\infty}(X)$, $\pi(f)$ denotes the bounded multiplication operator on $L^{2}(X)$ defined by $\pi(f)\xi = f\xi$ for $\xi$ in $L^{2}(X)$.

Definition 2.1. Let $X$ be a measure space with measure $m$. A measurable map $\varphi$ of $X$ onto $X$ is said to be a map with $n$-laps, $M\text{W}n\text{L}$ for short, if there exists $n$ measurable subsets $\{X_{i}\}_{i=1}^{n}$ of $X$ such that

1. $\bigcup_{i=1}^{n}X_{i} = X$ and $X_{i} \cap X_{j} = \emptyset$ for $i \neq j$.

2. Each restriction $\varphi_{i}$ of $\varphi$ to $X_{i}$ is a bimeasurable map of $X_{i}$ onto $\varphi_{i}(X_{i})$ with $m(X \setminus \varphi_{i}(X_{i})) = 0$ and $\varphi_{i}^{-1}$ is measurable, too.

3. For each $i$, $\varphi_{i}$ and $\varphi_{i}^{-1}$ are absolutely continuous with respect to $m$ and non-singular in the sense that

$$\frac{dm \circ \varphi}{dm}(x) \neq 0, \text{ a.e.} x \quad \text{and} \quad \frac{dm \circ \varphi^{-1}}{dm}(x) \neq 0, \text{ a.e.} x.$$

For a measure space $(X, m)$ and a measurable map $\varphi$ of $X$ into itself, $M_{f}$ and $T_{\varphi}$ is not necessarily defined on the full space $\mathcal{H}$. Then each isometry $V_{i}$ in the following definition, if necessary, is considered as a uniquely extended bounded linear operator on the full Hilbert space $\mathcal{H}$.

Definition 2.2. Let $\varphi$ be a $M\text{W}n\text{L}$ on a measure space $(X, m)$. We define a family isometries $\{V_{i}(\varphi)\}_{i=1}^{n}$ associated with $\varphi$ as follows.

$$V_{i}(\varphi) = M_{\sqrt{dm \circ \varphi^{-1}/dm}} M_{\chi X_{i}} T_{\varphi} \quad (i = 1, \ldots n).$$

By the definition we can see that

1. $V_{i}(\varphi)^{*} = M_{\sqrt{dm \circ \varphi^{-1}/dm}} T_{\varphi^{-1}}^{-1} \quad (i = 1, \ldots n)$.

2. $V_{i}(\varphi)V_{i}(\varphi)^{*} = M_{\chi X_{i}} \quad (i = 1, \ldots n)$.

3. $\int f(\varphi(x))\eta(x)dm(x) = \sum_{i=1}^{n} \int_{X} \frac{dm \circ \varphi_{i}^{-1}}{dm} \eta(\varphi_{i}^{-1}(x))dm$ for $\eta$ in $L^{1}(X, m)$.

Proposition 2.3. Let $\varphi$ be a $M\text{W}n\text{L}$ on a measure space $(X, m)$ and $\{V_{i} = V_{i}(\varphi)\}_{i=1}^{n}$ a family isometries associated with $\varphi$ defined in Definition 2.2. Then it follows that

1. $\{V_{i}\}_{i=1}^{n}$ satisfies condition (C.1) in §1, that is, $\{V_{i}\}_{i=1}^{n}$ is a FIC on $L^{2}(X, m)$.

2. $\pi(\alpha_{\varphi}(f)) = \alpha_{\varphi}(\pi(f))$ for all $f$ in $L^{\infty}(X)$. 

Proposition 2.3 (2) implies that \( \alpha_V \) is a \*-endomorphism of the von Neumann algebra \( M_{L^\infty(X)} \) and we denote by \( A_\varphi \) the transpose of the restriction of \( \alpha_V \) to \( M_{L^\infty(X)} \). Then we have

\[
(A_\varphi \eta)(x) = \sum_{i=1}^{n} \frac{dm \circ \varphi^{-1}_i}{dm} \eta(\varphi^{-1}_i(x)).
\]

The transformation \( A_\varphi \) is known as Perron-Frobenius operator on \( L^1(x_m,\mathcal{H}) \).

Theorem 2.4. Let \( \phi \) be a MWnL on a measure space \( (X,m) \). Suppose that there exists a FIC \( \{W_i\}_{i=1}^{n} \) such that \( W_1 \) has eigenvalue \( 1 \) with eigenvector \( e \) and

\[
\alpha_V(T) = \alpha_W(T) \quad \text{for } T \text{ in } M,
\]

where \( M \) is a von Neumann algebra on \( \mathcal{H} \). Then for any state \( \omega \) of the form \( \omega = \sum_{k=1}^{\infty} \omega_k \), where \( \omega_k \)'s are in \( \mathcal{H} \), it follows that

\[
\lim_{n \to \infty} (\alpha_V^n)(\omega) = \omega_e \quad \text{(norm topology on } M_*)\).
\]

Moreover, this implies that

\[
\lim_{n \to \infty} \|A_\varphi^n(\eta) - |e|^2\|_1 = 0.
\]

where \( \eta = |\xi|^2 \) for \( \xi \) in \( \mathcal{H} \).

Proposition 2.5. Let \( \varphi \) be a MW2L on the interval \([0,1]\) with Lebesgue measure \( m \). Then the following conditions are equivalent.

(1) \( V_1(\varphi) \) has eigenvalue \( 1 \) with eigenvector \( e \).

(2) \( m(\{x \in [0,1] | \frac{d\circ \varphi}{dm}(x) = 1 \}) > 0 \).

Theorem 2.6. Let \( \varphi \) be a MWnL on a measure space \( (X,m) \) and \( e(x) = 1 \) for a.e. \( x \) in \( X \). Then following conditions are equivalent.

(1) There exists a FIC \( \{W_i\}_{i=1}^{n} \) such that \( \alpha_V(T) = \alpha_W(T) \) for \( T \) in \( M_{L^\infty(X)} \) and \( W_1 e = e \).

(2) \( T_\varphi \) is an isometry.

(3) \( \sum_{i=1}^{n} \frac{dm \circ \varphi^{-1}_i}{dm}(x) = 1 \) for a.e. \( x \) in \( X \).

Definition 2.7. Let \( \varphi \) and \( \psi \) be two MWnL's on \( (X,m) \). Two maps are said to be AC-topologically conjugate if there exists a bijective map \( h \) of \( X \) onto itself satisfying following conditions.

(1) \( \varphi = h \circ \psi \circ h^{-1} \).

(2) Both \( m \circ h \) and \( m \circ h^{-1} \) are absolutely continuous and non-singular with respect to \( m \).
Remark. Let $h$ be a absolutely continuous map satisfying (2) of the definition above. We put

$$U(h) = M \sqrt{d m h / d m} T_h.$$ 

Then $U(h)$ is a unitary operator on $\mathcal{H}$.

Theorem 2.8. Let $\varphi$ and $\psi$ be two MWnL's on $(X, m)$. Suppose that $\psi$ is AC-conjugate to $\varphi$ and there exists a FIC $\{W_i\}_{i=1}^n$ satisfying following conditions.

1. $W_1$ has eigenvalue 1 with unit eigenvector $e$.
2. $\alpha_{V(\varphi)}(T) = \alpha_{W}(T)$ for $T$ in $M$,

where $M$ is a von Neumann algebra on $\mathcal{H}$. Let $f = U(h^{-1}) e$. Then for any state $\omega$ of the form $\omega = \sum_{k=1}^{\infty} \omega_{\xi_k}$ where $\xi_k$'s are in $\mathcal{H}_f$, it follows that

$$\lim_{n \to \infty} (\alpha_{V})^n(\omega) = \omega_e \quad \text{(norm topology on $(U(h)MU(h)^*)_\omega$)}.$$ 

§3. Examples of MWnL. We give typical and interesting examples of map with $n$ laps. Each number in each example indicates the following.

1. Measure space $(X, m)$ on which a map is given.
2. Map $\varphi$ with $n$ laps on $X$.
3. Number $n$ and partition $\{X_i\}_{i=1}^n$ of $X$.
4. $\{V_i\}_{i=1}^n = \{V_i(\varphi)\}_{i=1}^n$ defined in Definition 2.2.
4-1 An eigenvector $e$ for eigenvalue 1 of $W_1$ and $ONS(e, V) = \{e_k\}_{k=1}^\infty$.
4-2 $ONS(e, V)$ is complete or not.
5. $\{W_i\}_{i=1}^n$ such that $\alpha_{V}(T) = \alpha_{W}(T)$ for $T$ in a von Neumann algebra $M$ on $L^2(X, m)$.
6. The von Neumann algebra $M$ on which $\alpha_{V} = \alpha_{W}$.
6-1 An eigenvector $e$ for eigenvalue 1 of $W_1$ and $ONS(e, W) = \{e_k\}_{k=1}^\infty$.
6-2 $ONS(e, W)$ is complete or not.
7. Perron-Frobenius operator $A_{\varphi}$.

Example 3.1. (Tent map)
1. $X = [0, 1]$, and $m = \text{Lebesgue measure}.$
2. $\varphi$ is the map $\tau$ defined by

$$\tau(x) = 1 - |1 - 2x|.$$ 

3. $n = 2$ and $X_1 = [0, 1/2), X_2 = [1/2, 1].$
4. $V_1 = \sqrt{2} M_{[0, 1/2]} T_{\tau}$, $V_1 = \sqrt{2} M_{[1/2, 1]} T_{\tau}$.
5. $(W_1, W_2) = (V_1, V_2) \begin{pmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ 1/\sqrt{2} & -1/\sqrt{2} \end{pmatrix}.$
(6) $M = B(L^2[0,1])$
(6-1) $e(x) = 1 \ (x \in [0,1])$ and $e_1 = e$, $e_2 = M_{[0,1/2]}e_1 - M_{[1/2,1]}e_1$.
(6-2) $ONS(e, W)$ is complete.
(7) $A_{\tau}(\eta)(x) = \frac{1}{2} \left( \eta \left( \frac{x}{2} \right) + \eta \left( 1 - \frac{x}{2} \right) \right)$.

Example 3.2. (Generalized tent map)
(1) $X = [0,1]$, and $m =$Lebesgue measure.
(2) $\varphi = \tau_c$, $(0 < c < 1)$ defined by
\[ \varphi_c(x) = \begin{cases} \frac{1}{c}x & \text{for } 0 \leq x \leq c, \\ \frac{1}{c-1}(x-1) & \text{for } c < x \leq 1. \end{cases} \]
(3) $n = 2$ and $X_1 = [0,1/2)$, $X_2 = [1/2,1]$.
(4) $V_1 = M_{\sqrt{1/c}M_{X[0,c]}T_{\tau}}$, $V_2 = M_{\sqrt{1/(1-c)}M_{X[c,1]}T_{\tau}}$.
(5) $(W_1, W_2) = (V_1, V_2) \left( \frac{\sqrt{c}}{\sqrt{1-c}}, \frac{\sqrt{1-c}}{-\sqrt{c}} \right)$.
(6) $M = B(L^2[0,1])$
(6-1) $e(x) = 1 \ (x \in [0,1])$ and $e_1 = e$, $e_2(x) = \begin{cases} \frac{1}{\sqrt{c}} & \text{for } 0 \leq x \leq c, \\ -\frac{1}{\sqrt{c-1}} & \text{for } c < x \leq 1. \end{cases}$
(6-2) $ONS(e, W)$ is complete.
(7) $A_{\tau}(\eta)(x) = c(\eta(cx)) + (1-c)\eta((c-1)x + 1))$.

Remark. $\tau_c$ and $\tau_c$ are topologically conjugate (cf.[6],[8]) but they are AC-conjugate only if $c = d$.

Example 3.3. (Logistic map) (cf.[9])
(1) $X = [0,1]$, and $m =$Lebesgue measure.
(2) $\varphi$ is the map $\lambda$ defined by
\[ \lambda(x) = 4x(1-x) \]
(3) $n = 2$ and $X_1 = [0,1/2)$, $X_2 = [1/2,1]$. 

\[ \boxed{\text{Example 3.2. (Generalized tent map)}} \]
\[ \boxed{\text{Example 3.3. (Logistic map) (cf.[9])}} \]
The logistic map is topologically conjugate to the tent map with conjugacy $h(x) = \sin^2(\pi x/2)$ (cf. [7]).

Example 3.4. (Typical map with 3 laps)

1. $X = [0, 1]$, and $m =$Lebesgue measure.

2. $\varphi$ is the map defined by

$$
\varphi(x) = \begin{cases} 
3x & \text{for } 0 \leq x < 1/3, \\
3x - 1 & \text{for } 1/3 \leq x < 2/3, \\
3x - 2 & \text{for } 2/3 \leq x \leq 1.
\end{cases}
$$

3. $n = 3$ and $X_1 = [0, 1/3), X_2 = [1/3, 2/3), X_2 = [2/3, 1]$.

4. $V_1 = \sqrt{3}M_{X_{[0,1/3]}}T_{\varphi}, V_2 = \sqrt{3}M_{X_{[1/3,2/3]}}T_{\varphi}, V_3 = \sqrt{3}M_{X_{[2/3,1]}}T_{\varphi}$.

5. $(W_1, W_2, W_3) = \left( \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}} \right)$.

6. $(W_1, W_2, W_3) = \left( \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}} \right)$.

7. $A_{\varphi}(\eta)(x) = \frac{1}{3} \left( \eta \left( \frac{x}{3} \right) + \eta \left( \frac{x + 1}{3} \right) + \eta \left( \frac{x + 2}{3} \right) \right)$. 

The logistic map is topologically conjugate to the tent map with conjugacy $h(x) = \sin^2(\pi x/2)$ (cf. [7]).

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3x - 1 & \text{for } 1/3 \leq x < 2/3, \\
3x - 2 & \text{for } 2/3 \leq x \leq 1.
\end{cases}
$$

3. $n = 3$ and $X_1 = [0, 1/3), X_2 = [1/3, 2/3), X_2 = [2/3, 1]$.

4. $V_1 = \sqrt{3}M_{X_{[0,1/3]}}T_{\varphi}, V_2 = \sqrt{3}M_{X_{[1/3,2/3]}}T_{\varphi}, V_3 = \sqrt{3}M_{X_{[2/3,1]}}T_{\varphi}$.

5. $(W_1, W_2, W_3) = \left( \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}} \right)$.

6. $(W_1, W_2, W_3) = \left( \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}} \right)$.

7. $A_{\varphi}(\eta)(x) = \frac{1}{3} \left( \eta \left( \frac{x}{3} \right) + \eta \left( \frac{x + 1}{3} \right) + \eta \left( \frac{x + 2}{3} \right) \right)$. 

The logistic map is topologically conjugate to the tent map with conjugacy $h(x) = \sin^2(\pi x/2)$ (cf. [7]).
Example 3.5. (MW2L on $[0,1]$ such that $V_1$ has an eigenvector for eigenvalue 1:a)

1. $X = [0,1]$ and $m$ is the Lebesgue measure.
2. $\varphi$ is the map defined by
   \[ \varphi(x) = \begin{cases} 
   x & \text{for } 0 \leq x < 1/4, \\
   (6x - 1)/2 & \text{for } 1/4 \leq x < 1/2, \\
   -2x + 2 & \text{for } 1/2 \leq x \leq 1 
   \end{cases} \]
3. $n = 2$ and $X_1 = [0,1/2), X_2 = [1/2,1]$.
4. $V_1 = M_{\chi[0,1/4)} + \sqrt{3}M_{\chi[1/4,1/2)}$, $V_2 = \sqrt{2}M_{\chi[1/2,1]}$.
5. $e_1 = e = 2\chi_{[0,1/4)}$, $e_2 = \sqrt{2}\chi_{(7/8,1)}$, $e_3 = 2\sqrt{6}\chi_{(1/24,1/2]}$ $e_4 = 4\chi_{[1/2,9/16]}$.
6. $\text{ONS}(e, V)$ is not complete.
7. $M = B(L^2[0,1])$.

Example 3.6. (MW2L on $[0,1]$ such that $V_1$ has an eigenvector for eigenvalue 1:b)

1. $X = [0,1]$ and $m$ is the Lebesgue measure.
2. $\varphi$ is the map defined by
   \[ \varphi(x) = \begin{cases} 
   -5x + 1 & \text{for } 0 \leq x < 1/8, \\
   -x + (1/2) & \text{for } 1/8 \leq x < 1/2, \\
   2x - 1 & \text{for } 1/2 \leq x \leq 1 
   \end{cases} \]
3. $n = 2$ and $X_1 = [0,1/2], X_2 = [1/2,1]$.
4. $V_1 = \sqrt{5}M_{\chi[0,1/8)} + M_{\chi[1/8,1/2)}$, $V_2 = \sqrt{2}M_{\chi[1/2,1]}$.
5. $e_1 = e = 2\chi_{[1/8,3/8)}$, $e_2 = \sqrt{2}\chi_{[9/16,11/16]}$, $e_3 = 2\sqrt{10}\chi_{[5/80,7/80]}$ $e_4 = 4\chi_{[25/32,27/32]}$.
6. $\text{ONS}(e, V)$ is not complete.
7. $M = B(L^2[0,1])$.
Example 3.7. (Square root map)

(1) $X = [0, 1]$ and $m =$ Lebesgue measure.

(2) $\varphi$ is the map defined by

$$
\varphi(x) = \begin{cases} 
\sqrt{2x} & \text{for } 0 \leq x < 1/2, \\
1 - \sqrt{2x - 1} & \text{for } 1/2 \leq x \leq 1.
\end{cases}
$$

(3) $n = 2$ and $X_1 = [0, 1/2)$, $X_2 = [1/2, 1]$.

(4) $V_1 = (1/\sqrt{2x})M_{x(0,1/2)}T_{\varphi}$, $V_2 = (1/\sqrt{2x - 1})M_{x(1/2,1)}T_{\varphi}$.

(5) $(W_1, W_2) = (V_1, V_2) \left( \frac{M_{\sqrt{2x}}}{M_{1/2-2x}}, \frac{M_{\sqrt{2x}}}{M_{2x}} \right)$.

(6) $M = M_{L^\infty[0,1]}$

(6-1) $e_1(x) = e(x) = 1$, $e_2(x) = \sqrt{1/\sqrt{2x}} - 1_{X(0,1/2)}(x) - \sqrt{1/\sqrt{2x - 1}} - 1_{X(1/2,1)}(x)$

(6-2) Now we cannot find whether $ONS(e, W)$ is complete or not.

(7) $A_{\varphi}(\eta)(x) = \frac{1}{x} \left( \eta \left( \frac{x^2}{2} \right) + \frac{1}{x-1} \eta \left( \frac{x^2 - 2x + 2}{2} \right) \right)$.

Example 3.8. (Map of broken line)

(1) $X = [0, 1]$ and $m =$ Lebesgue measure.

(2) $\varphi$ is the map defined by

$$
\varphi(x) = \begin{cases} 
8x/5 & \text{for } 0 \leq x < 1/4, \\
(12x - 1)/5 & \text{for } 1/4 \leq x < 1/2, \\
(-12x + 13)/7 & \text{for } 1/2 \leq x < 17/20, \\
(-8x + 8)/3 & \text{for } 7/20 \leq x \leq 1,
\end{cases}
$$

(3) $n = 2$ and $X_1 = [0, 1/2)$, $X_2 = [1/2, 1]$.

(4) $V_1 = (\sqrt{8/5}M_{X(0,1/4)} + \sqrt{12/5}M_{X(1/4,1/2)})T_{\varphi}$, $V_2 = (\sqrt{12/7}M_{X(1/2,17/20)} + \sqrt{8/3}M_{X(17/20,1)})T_{\varphi}$.

(5) $(W_1, W_2) = (V_1, V_2) \left( \frac{\sqrt{5/8}M_{X(0,2/5)}}{\sqrt{3/8}M_{X(2/5,5/2)}}, \frac{\sqrt{5/12}M_{X(1/2,5/1)}}{\sqrt{7/12}M_{X(2/5,5/1)}}, \frac{\sqrt{3/8}M_{X(0,2/5)}}{\sqrt{5/8}M_{X(0,2/5)} - \sqrt{7/12}M_{X(2/5,5/1)}}, \frac{\sqrt{7/12}M_{X(1/2,5/1)}}{\sqrt{3/8}M_{X(2/5,5/1)} - \sqrt{5/12}M_{X(2/5,5/1)}} \right)$.

(6) $M = B(L^2[0,2/5]) \oplus B(L^2[2/5,1])$

(6-1) $e_1(x) = e(x) = 1$, $e_2(x) = \begin{cases} 
\sqrt{3/5} & \text{for } 0 \leq x < 1/4, \\
\sqrt{7/5} & \text{for } 1/4 \leq x < 1/2, \\
-\sqrt{5/7} & \text{for } 1/2 \leq x < 17/20, \\
-\sqrt{5/3} & \text{for } 7/20 \leq x \leq 1.
\end{cases}$
now we cannot find whether $ONS(e, W)$ is complete or not.

(7) $A_\varphi(\eta)(x) = \frac{5}{8} \eta \left( \frac{5x}{8} \right) \chi_{[0,2/5]}(x) + \frac{5}{12} \eta \left( \frac{5x + 1}{12} \right) \chi_{[2/5,1]}(x) + \frac{3}{8} \eta \left( \frac{-3x + 8}{8} \right) \chi_{[0,2/5]}(x) + \frac{7}{12} \eta \left( \frac{-7x + 13}{12} \right) \chi_{[2/5,1]}(x).

Example 3.9. (Product of tent maps)

(1) $X = [0, 1] \times [0, 1]$ and $m = \text{Lebesgue measure}.$

(2) $\varphi$ is the map defined by

$\varphi(x, y) = (\tau(x), \tau(y))$

where $\tau$ is the tent maps defined in Example 3.1.

(3) $n = 4$ and $X_1 = [0, 1/2] \times [0, 1/2], X_2 = [1/2, 1] \times [0, 1/2], X_3 = [1/2, 1] \times [1/2, 1], X_4 = [0, 1/2] \times [1/2, 1].$

(4) $V_1 = 2M_{\chi_{[0,1/2]} \times [0,1/2]} T_\varphi, V_2 = 2M_{\chi_{[1/2,1]} \times [0,1/2]} T_\varphi, V_3 = 2M_{\chi_{[1/2,1]} \times [1/2,1]} T_\varphi, V_4 = 2M_{\chi_{[1/2,1]} \times [1/2,1]} T_\varphi.$

(5) $(W_1, W_2, W_3, W_4) = (V_1, V_2, V_3, V_4)$

(6) $M = B(L^2([0,1] \times [0,1]))$

(6-1) $e_1(x, y) = e(x, y) = 1 \ ((x, y) \in [0, 1] \times [0, 1])$ and

$e_2(x) = \chi_{[0,1/2]} \times [0,1/2] - \chi_{[1/2,1]} \times [0,1/2] + \chi_{[0,1/2]} \times [1/2,1] - \chi_{[1/2,1]} \times [1/2,1].$

(6-2) $ONS(e, W)$ is complete.

(7) $A_\varphi(\eta)(x) = \frac{1}{4} \left( \eta \left( \frac{x}{2}, \frac{x}{2} \right) + \eta \left( 1 - \frac{x}{2}, \frac{x}{2} \right) + \eta \left( \frac{x}{2}, 1 - \frac{x}{2} \right) + \eta \left( 1 - \frac{x}{2}, 1 - \frac{x}{2} \right) \right).$

Example 3.10. (Baker’s transformation)

(1) $X = [0, 1] \times [0, 1]$ and $m = \text{Lebesgue measure}.$

(2) $\varphi$ is the map $\beta$ defined by
\[ \beta(x, y) = \begin{cases} (2x, y/2) & \text{for } 0 \leq x < 1/2, \\ (2x - 1, (y + 1)/2) & \text{for } 1/2 \leq x \leq 1. \end{cases} \]

(3) \( n = 1 \) and \( X_1 = X \)

(4) \( V_1 = T_\beta \)

(4-1) \( e_1(x, y) = e(x, y) = 1 \)

(4-2) \( ONS(e, W) = \{e_1\} \) is not complete.

(6) \( M = B(L^2([0,1] \times [0,1])) \)

(7) \( A_\beta(\eta)(x) = \eta(\beta(x)) \)

Remark. Baker's transformation is strong-mixing but \( \{(\alpha_V^n)^\omega(\omega_\epsilon)\}_{n=1}^\infty \) does not converges to \( \omega_\epsilon \) in the norm topology in \( M_* \).

Example 3.11. (Unilateral shift map)

(1) \( X = \prod_{n=1}^\infty \{1, 2\} \) and \( m = \text{usual measure} \).

(2) \( \varphi \) is the map \( \sigma \) defined by
\( \sigma((x_1, x_2, x_3, \ldots)) = (x_2, x_3, x_4, \ldots) \),

(3) \( n = 2 \) and \( X_1 = X(1) = \{(x_n)_{n=1}^\infty \in X | x_1 = 1\} \), \( X_2 = X(2) = \{(x_n)_{n=1}^\infty \in X | x_1 = 2\} \)

(4) \( V_1 = \sqrt{2}M_{X(1)}T_\sigma, \quad V_1 = \sqrt{2}M_{X(2)}T_\sigma \).

(5) \( (W_1, W_2) = (V_1, V_2) \begin{pmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ 1/\sqrt{2} & -1/\sqrt{2} \end{pmatrix} \).

(6) \( M = B(L^2(X)) \)

(6-1) \( e(x) = 1 (x \in X) \) and \( e_1 = e, e_2 = \chi_{X(1)}e_1 - \chi_{X(2)}e_1 \).

(6-2) \( ONS(e, W) \) is complete.

(7) \( A_\sigma(\eta)(x) = \frac{1}{2}(\eta(\gamma_1) + \eta(\gamma_2)) \),

where \( \gamma_1((x_1, x_2, x_3, \ldots)) = (1, x_1, x_2, \ldots) \) and \( \gamma_2((x_1, x_2, x_3, \ldots)) = (2, x_1, x_2, \ldots) \).

Example 3.12. (MW2L on the set \( N \) of all natural numbers)

(1) \( X = N \) and \( m \) is the counting measure.

(2) \( \varphi \) is the map defined by
\( \varphi(2k - 1) = k \) and \( \varphi(2k) = k \) \( (k \in N) \)

(3) \( n = 2 \) and \( X_1 = 2N - 1, \quad X_2 = 2N \).

(4) \( V_1 = M_{X(2N-1)}T_\varphi, \quad V_2 = M_{X(2N)}T_\varphi \).

(4-1) \( e = e_1 = \chi_{\{1\}} \) and the sequence \( \{e_k\}_{k=1}^\infty \) is the canonical CONS of \( \ell^2(N) \).
(4-2) $ONS(e, V)$ is complete.

(6) $M = B(\ell^2(\mathbb{N}))$.

(7) $A_\varphi(\eta)(k) = \eta(2k - 1) + \eta(2k)$.

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References


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