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<th>Title</th>
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Stable rank for a pair of C*-algebras

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1 Introduction and Main Result

The (topological) stable rank of Rieffel[19] is noncommutative generalization of the dimension of a compact Hausdorff space. In fact, when $X$ is a compact Hausdorff space, the stable rank of $C(X)$ is $\left\lceil \frac{\dim X}{2} \right\rceil + 1$, where $\dim X$ is covering dimension of $X$. Recall that a unital C*-algebra $A$ has stable rank $n$ if for any element $a_1, a_2, \cdots, a_n$ and $\varepsilon > 0$ there exist $b_1, b_2, \cdots, b_n$ in $A$ such that

\begin{align*}
(1) & \quad ||a_i - b_i|| < \varepsilon \\
(2) & \quad \sum_{i=1}^{n} b_i^* b_i > 0.
\end{align*}

The condition (2) is equivalent to that there exist $c_1, c_2, \cdots, c_n$ in $A$ such that $\sum_{i=1}^{n} c_i b_i = 1$.

If $A$ has no unital, we define stable rank of $A$ as stable rank of the unitaization of $A$. Note that stable rank one condition is equivalent to that the set of invertible elements is dense in a given C*-algebra.

Many mathematicians tried to determine stable rank of interesting C*-algebras, in particular, simple unital C*-algebras ([5] [6] [8] [10] [11] [12] [13] [14] [15] [18] [20] [21] [22] etc). For examples, AF C*-algebras and non-commutative tori have stable rank one ([18]), Toeplitz algebra has stable rank two, and Cuntz algebra has an infinity ([19]).

It has been a problem of considerable interest to determine stable rank of a crossed product algebra $A \times_{\alpha} G$ of a unital C*-algebra $A$ with stable rank one by a finite group $G$. Blackadair presented this problem in the case that $A$ is an AF C*-algebra ([2]), and constructed a symmetry $\alpha$ on $A = C[0,1] \otimes UHF$ whose crossed product algebra $A \times_{\alpha} Z_2$ has stable rank two. So, to consider the above problem, we need the assumption of the simplicity on a given C*-algebra $A$.

In this direction Jeong and the author conclude ([10][11]) that a crossed product algebra $A \times_{\alpha} G$ has the cancellation property if $A$ is simple with stable rank one and the SP-property. Recall that a C*-algebra $A$ is said to have the SP-property if any non-zero hereditary subalgebra of $A$ has non-zero projection. For example, an AF C*-algebra has the SP-property. Therefore, we could conclude by [1] that a crossed product algebra $A \times_{\alpha} G$ has stable rank one if we add real rank zero condition to this crossed product algebra, that is, the set of self-adjoint elements with finite spectra in $A \times_{\alpha} G$ is dense in the set of self-adjoint elements. As Elliott presented a crossed product algebra $UHF \times_{\alpha} Z_2$ with real
rank one, however, we can not always hope that a given crossed product algebra has real rank zero.

In this talk we try to estimate stable rank of a given unital C*-algebra $B$ by stable rank of a C*-subalgebra $A$ with common unit. In case that $B$ is a crossed product algebra of $A$ by a finite group $G$, $sr(B) \leq sr(A) \times |G|$ ([11]). More generally, we have the following result:

**Theorem 1** Let $1 \in A \subset B$ be unital C*-algebras. Suppose that $B$ is a finitely generated left $A$-module, that is, there are some $n$ elements $v_1, v_2, \cdots, v_n$ in $B$ such that $\sum_{i=1}^{n} Av_i = B$. Then, $sr(B) \leq sr(A) \times n$.

## 2 Stable rank

We prove main theorem with using the technique of matrix algebras. To this end the following lemma is needed.

**Lemma 2** (Spatial case of Rieffel[19]) Let $n \in \mathbb{N}$.

$$sr(M_n(A)) \leq sr(A).$$

**Proof.** We will give a sketch of the proof. Suppose that $sr(A) = m$. Take $m$ elements $T_1, \cdots, T_m$ from $M_n(A)$. Set $S = (T_1, T_2, \cdots, T_m)^t$ in $M_{nm,m}(A)$. Let $(a_1, a_2, \cdots, a_m)$ be the first row in $S$. Since $sr(A) = m$, we may assume that there exist $c_2, \cdots, c_{m+1}$ such that

$$c_2a_2 + c_3a_3 + \cdots + c_{m+1}a_{m+1} = 1 - a_1.$$

Consider

$$
\begin{pmatrix}
1 & c_2 & \cdots & c_{m+1} & 0 & \cdots & 0 \\
1 & & & & & & \\
\vdots & & & & & & \\
\vdots & & & & & & \\
1 & & & & & & \\
\end{pmatrix} S.
$$

Then, the new first row is $(1, b_2, \cdots, b_{nm})^t$. Doing the iteration there is an invertible matrix $R \in M_{nm}(A)$ such that $RS = \text{diag}(1, S')$, $S' \in M_{n-1,n-1}$. By induction there is $U \in M_{n-1,nm-1}$ such that $US' = I_{n-1}$. Note that $||R^{-1}\text{diag}(1, S') - S||$ is small.

Write

$$R^{-1}\text{diag}(1, S') = (S_1, \cdots, S_m)^t$$

$$\text{diag}(1, U)R = (U_1, \cdots, U_m),$$

where $S_1, \cdots, S_m, U_1, \cdots, U_m$ are in $M_n(A)$. Then, we have $||T_i - S_i||$ is small, and $\sum_{i=1}^{m} U_iS_i = I_n$. $\square$
**Definition 3** Define

\[ Lg_n(A) = \{(a_1, a_2, \cdots, a_n) \in A^n | \sum_{i=1}^n Aa_i = A\}. \]

Then, \( sr(A) \leq n \) if and only if \( Lg_n(A) \) is dense in \( A^n \).

**Proof of Theorem 1.**

We give only the proof of the case of \( sr(A) = 1 \) and \( G = \mathbb{Z}_2 \). That is, we will show that \( sr(A \times_\alpha \mathbb{Z}_2) \leq 2 \) for a unital C*-algebra \( A \). In general case we can guess it from the proof of Lemma 2.

Take \( a_0 + a_1 u, b_0 + b_1 u \) in \( A \times_\alpha \mathbb{Z}_2 \), where \( u \) is a unitary implementing \( \alpha \). Let \( \varepsilon > 0 \) be given. Consider

\[
\begin{bmatrix}
  a_0 + a_1 u \\
  b_0 + b_1 u
\end{bmatrix} = \begin{bmatrix}
  a_0 & a_1 \\
  b_0 & b_1
\end{bmatrix} \begin{bmatrix}
  1 \\
  u
\end{bmatrix}.
\]

Since \( sr(M_2(A)) = 1 \) by Lemma 2, there exists an invertible element

\[
\begin{bmatrix}
  c_0 & c_1 \\
  d_0 & d_1
\end{bmatrix} \in M_n(A)
\]

such that

\[
\|\begin{bmatrix}
  a_0 & a_1 \\
  b_0 & b_1
\end{bmatrix} - \begin{bmatrix}
  c_0 & c_1 \\
  d_0 & d_1
\end{bmatrix}\| < \frac{\varepsilon}{2}.
\]

Consider

\[
\begin{bmatrix}
  c_0 & c_1 \\
  d_0 & d_1
\end{bmatrix} \begin{bmatrix}
  1 \\
  u
\end{bmatrix} = \begin{bmatrix}
  c_0 + c_1 u \\
  d_0 + d_1 u
\end{bmatrix}.
\]

Then, \( (c_0 + c_1 u, d_0 + d_1 u) \in Lg_2(A \times_\alpha \mathbb{Z}_2) \), and \( \|a_0 + a_1 u - (c_0 + c_1 u)\| < \varepsilon, \|b_0 + b_1 u - (d_0 + d_1 u)\| < \varepsilon \). Hence, \( sr(A \times_\alpha \mathbb{Z}_2) \leq 2 \). \( \square \)

**Corollary 4** Let \( 1 \in A \subset B \) be a pair of unital C*-algebras, and \( E : B \to A \) be a faithful conditional expectation of index-finite type. That is, there exists a quasi-basis \( \{v_i^*, v_i\}_{i=1}^n \) such that \( x = \sum_{i=1}^n E(xv_i^*)v_i, \forall x \in B \). Then, \( sr(B) \leq sr(A) \times n \).

**Corollary 5** Let \( 1 \in A \) be a unital C*-algebra and \( G \) be a finite group. Then,

\[ sr(A \times_\alpha G) \leq sr(A) \times |G|. \]

3 Application

Using Corollary 5 we can present an affirmative data to a question of Blackdar[2]:
Question 6 Let $A$ be a AF $C^*$-algebra and $G$ be a finite group. Then

$$sr(A \times_{\alpha} G) \leq 1.$$ 

Theorem 7 (Jeong-Osaka[11]) Let $A$ be a simple unital $C^*$-algebra with $sr(A) = 1$ and SP-property. If $G$ is a finite group and $\alpha$ is an action of $G$ on $A$ then the crossed product $A \times_{\alpha} G$ has cancellation.

Here, a $C^*$-algebra has SP-property if each of its non-zero hereditary $C^*$-subalgebras contains a non-zero projection.

In particular,

Corollary 8 Under the assumptions of the above theorem, if $A \times_{\alpha} G$ has real rank zero, that is, any self-adjoint element can be approximated by a self-adjoint element with finite spectra, then $sr(A \times_{\alpha} G) = 1$.

Remark 9 Generally, we can not hope that a given simple crossed product algebra $A \times_{\alpha} G$ has real rank zero, even if $A$ is UHF , and $G = \mathbb{Z}_2$ [7].

If one consider a crossed product by the integer group $\mathbb{Z}$ then there is no conditional expectation of index-finite type from the crossed product $A \times_{\alpha} \mathbb{Z}$ onto $A$, but we have the following cancellation theorem:

Theorem 10 (Jeong-Osaka[11]) Let $A$ be a simple unital $C^*$-algebra with $sr(A) = 1$ and SP-property. If $\alpha$ is an outer action of the integer group $\mathbb{Z}$ on $A$ such that $\alpha_* = id$ on $K_0(A)$ then the crossed product $A \times_{\alpha} \mathbb{Z}$ has cancellation.

Example 11 Simple AF $C^*$-algebras and non-commutative tori $A_{\theta}$ are examples for $C^*$-algebras in Theorems 7 and 10.

参考文献


