数理解析研究所講究録
1152巻 2000年 24-34

深い流体中の内部孤立波の不安定と崩壊

山口大学工学部　松野好雅 (Yoshimasa Matsuno)

Abstract

The effects of transverse perturbations on one-dimensional (1D) internal algebraic solitary waves are investigated on the basis of the 2D Benjamin-Ono equation. Applying Whitham's theory, we find that the 1D solitary waves are unstable in media with positive dispersion. We are particularly concerned here with the long-term evolution of instabilities in the long-wave limit. We show that the Whitham modulation equations reduce to the model equations describing the nonlinear development of the Rayleigh-Taylor instability in a shallow layer of incompressible fluid. Analytical solutions to the modulation equations reveal that the transverse instability of 1D solitary wave results in the formation of 2D collapsing clusters.

1. Introduction

The two-dimensional (2D) evolution of long internal waves in fluids of great depth is described by the following 2D Benjamin-Ono (BO) equation

$$u_t + 2uu_x + Hu_{xx} = \beta \int_{x_0}^{x} u_{yy} dx', \quad Hu(x,y,t) = \frac{1}{\pi} P \int_{-\infty}^{\infty} \frac{u(x',y,t)}{x'-x} dx', \tag{1}$$

where H is the Hilbert transform operator acting on the x variable, β is a parameter characterizing the property of the medium and x_0 is a constant. Equation (1) was first derived in a system of stratified fluid of great depth [1] and later in a two-layer fluid system with the depth of upper (or lower) layer being infinite [2, 3]. This equation is a deep-water analog of the Kadomtsev-Petviashvili (KP) equation [4] in the theory of shallow water waves.

Equation (1) exhibits the 1D algebraic solitary wave solution of the form

$$u_0 = \frac{2a}{a^2(x-\xi)^2+1}, \quad \xi = at + \xi_0, \tag{2}$$

where $a(>0)$ and ξ_0 are the velocity and initial position of the solitary wave, respectively. The solution (2) was shown to be neutrally stable with respect to 1D infinitesimal perturbations with use of the BO equation [5, 6]. Furthermore, the effect of long transverse perturbations on the 1D solitary wave (2) was investigated in the context of Eq. (1). It was found that for the negative dispersion case ($\beta < 0$), the solution is stable while for the positive dispersion case ($\beta > 0$), it is unstable [1, 7]. All the results mentioned above are based on the linear stability analysis and hence they say nothing about the nonlinear stage of the development of instability.

The purpose of this paper is to study the long-term evolution of the solitary wave (2) modulated by long transverse perturbations within the framework of the 2D BO equation (1). The nonlinear development of the wave instability is investigated by a variational method initiated by Whitham [8]. We derive a system of modulation equations for the velocity and position of the solitary wave. In the positive dispersion case ($\beta > 0$), we show that the 1D solitary waves are unstable for long transverse perturbations. We are concerned here with the development of the instability in the long-wave limit. We show in this limit that the modulation equations reduce to the model equations describing the Rayleigh-Taylor instability in a shallow layer of incompressible fluid. The exact analytical solutions for these equations are constructed using the hodograph transformation, showing that the instability leads to the 2D collapse of the solitary wave.

2. Modulation equations

The 2D BO equation can be derived from the variational principle

$$\delta S = 0, \quad S = \int_0^t dt \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} L dx dy, \tag{3}$$

where the Lagrangian L is given by

$$L = \frac{1}{2}\phi_t \phi_x + \frac{1}{3}\phi_x^3 + \frac{1}{2}\phi_x H \phi_{xx} - \frac{\beta}{2}\phi_y^2, \quad (u = \phi_x). \tag{4}$$

In order to apply Whitham's modulation theory [8], we assume that the parameters a and ξ are slowly varying functions of y and t, i.e., $a = a(y,t)$ and $\xi = \xi(y,t)$ and expand

u as $u = u_0(x-\xi, a)+u_1(x-\xi, a; y, t)+\cdots$ where u_0 represents (2) and u_1 is a correction. Substituting (2) into (4) and integrating over x, we obtain the averaged Lagrangian

$$\bar{L} \equiv \int_{-\infty}^{\infty} L dx = \pi \left[-a\xi_t + \frac{a^2}{2} - \beta\Big(\frac{a_y^2}{a^3} + a\xi_y^2\Big)\right]. \tag{5}$$

Then, variations of action S with respect to ξ and a yield the following system of modulation equations for a and ξ:

$$a_t = -2\beta(a\xi_y)_y, \tag{6a}$$

$$\xi_t = a + \beta\left(2\frac{a_{yy}}{a^3} - 3\frac{a_y^2}{a^4} - \xi_y^2\right). \tag{6b}$$

Before analyzing above equations, we shall make two remarks. We first note that these equations can also be obtained formally applying the direct soliton perturbation theory for the BO equation [9, 10] by regarding the right-hand side of (1) as a perturbation. The second remark is concerned with the conservation laws. It is easy to show that (6a) and (6b) have the three conserved quantities

$$P_x = \int_{-\infty}^{\infty} (a - a_0)dy, \quad P_y = \int_{-\infty}^{\infty} a\xi_y dy, \tag{7}$$

$$H = \int_{-\infty}^{\infty} \mathcal{H} dy, \tag{8a}$$

with

$$\mathcal{H}(y,t) = \frac{a^2}{2} - \beta\left(a\xi_y^2 + \frac{a_y^2}{a^3}\right) - \frac{a_0^2}{2}, \tag{8b}$$

where we have imposed the boundary conditions $a(\pm\infty, t) = a_0$ ($a_0 =$ const.) and $\xi_y(\pm\infty, t) = 0$. P_x and P_y are the x and y projections of the momentum, respectively and H is the energy (or Hamiltonian) averaged over x. It then turns out that the system of equations (6) and (7) can be written in the form of Hamilton's equation of motion

$$a_t = \frac{\partial}{\partial y}\frac{\delta H}{\delta q}, \tag{9a}$$

$$q_t = \frac{\partial}{\partial y}\frac{\delta H}{\delta a}, \tag{9b}$$

where $q = \xi_y$.

3. Initial evolution of the system

Let us now perform the linear stability analysis for (6) to study the initial evolution of the system where perturbations remain linear. To this end, we put $a = a_0 + \delta a$ and $\xi = a_0 t + \delta\xi$ and linearize (6) about the unperturbed state $a = a_0$ and $\xi = a_0 t$. The resulting linear equations for δa and $\delta\xi$ read in the form

$$\delta a_t = -2\beta a_0 \delta\xi_{yy}, \tag{10a}$$

$$\delta\xi_t = \delta a + \frac{2\beta}{a_0^3}\delta a_{yy}. \tag{10b}$$

Assuming that $\delta a \propto e^{i(py-\omega t)}, \delta\xi \propto e^{i(py-\omega t)}$, one finds from (10) the linear dispersion relation

$$\omega^2 = -2\beta a_0 p^2 \left(1 - \frac{2\beta}{a_0^3}p^2\right). \tag{11}$$

For the negative dispersion $\beta < 0$, (11) always yields real ω, implying that the solitary wave (2) is stable against the transverse perturbation of the plane-wave type. On the other hand, when $\beta > 0$ corresponding to the positive dispersion, (11) gives pure imaginary ω for the wavenumber within the range $0 < p < p_c$ with $p_c = \sqrt{\frac{a_0^3}{2\beta}}$, resulting in the instability. Note however, that the applicability of (11) should be restricted by the condition $p << p_c$, namely the second term in the parentheses is small compared with the first term. If one neglects the term of order p^4 in (11), one recovers the result of Ref. [7] which has been obtained by a different method using the completeness relation for the eigenfunctions of the linearized BO equation. It is interesting to note that an analogous result has been reported in the transverse instability problem described by the KP equation with the positive dispersion [11]. Unlike the BO case, however, the cutoff wavenumber has been derived exactly on the basis of the inverse scattering transform method.

The further development of the instability in the positive dispersion case must be investigated by taking account of nonlinearity. We shall briefly consider the weakly nonlinear case. For this purpose, we introduce the slow variables Y and T according to

$$Y = \epsilon y, \quad T = \epsilon t, \tag{12}$$

and expand ξ about the unperturbed state as

$$\xi = a_0 t + \epsilon \Theta(Y, T), \tag{13}$$

where ϵ is a small parameter. Substituting (12) and (13) into (6b), we find the asymptotic expansion for the velocity

$$a = a_0 + \epsilon^2 \Theta_T - \beta \epsilon^4 \left(\frac{2\Theta_{TYY}}{a_0^3} - \Theta_Y^2 \right) + O(\epsilon^6). \tag{14}$$

Lastly, substitution of (14) into (6a) yields the nonlinear evolution equation for Θ

$$\Theta_{TT} + 2\beta a_0 \Theta_{YY} + 2\beta\epsilon^2 \left(2\Theta_Y \Theta_{TY} + \Theta_T \Theta_{YY} + \frac{2\beta}{a_0^2} \Theta_{YYYY} \right) + O(\epsilon^4) = 0. \tag{15}$$

The leading term of (15) coincides perfectly with the modulation equation [1] derived in the linear stability analysis of the solitary wave (2) with respect to long transverse perturbations. In addition, Eq. (15) which takes account of the quadratic nonlinearities is found to be essentially identical to the weakly nonlinear evolution equation derived in the study of the stability problem of algebraic solitary waves on the basis of Shrira's model equation which describes the nonlinear evolution of 2D perturbations in parallel boundary-layer type shear flow [12]. It was shown in [12] that Eq. (15) can be recast into the integrable elliptic Boussinesq equation by means of appropriate transformation using Lagrangian coordinates. However, since the applicability of the weakly nonlinear theory would be limited by a finite time when the velocity a becomes zero [12], we shall not address this problem further and proceed to consider the fully nonlinear phenomenon described by (6).

4. Wave collapse

In order to study the long-term dynamics of unstable perturbations, one must solve the Whitham equation (6) itself. Here, we shall perform the asymptotic analysis in the long-wave limit $p \to 0$. While the analysis near the maximum growth rate $\Gamma_{max} = \frac{a_0^2}{2}\left(p = \sqrt{\frac{a_0^3}{4\beta}}\right)$ is still interesting, it will be discussed elsewhere. In the long-wave limit, we will be able to solve our system of equations analytically and predict the formation of singularity, i.e., the so-called wave collapse. In the following analysis, we consider the unstable case and put $\beta = 1$ in (6) without loss of generality.

First of all, introduce new variables θ and v by $\xi = \theta/\epsilon, v = \theta_Y$ as well as the slow variables (12), to reduce Eq. (6) into the form

$$a_T + 2(av)_Y = 0, \tag{16a}$$

$$v_T + 2vv_Y = a_Y, \tag{16b}$$

where we have neglected the terms of order ϵ^2. Note in this approximation that the small parameter ϵ may be identified with the small transverse wavenumber p. Remarkably, the first-order system of equations (16) is seen to be equivalent to the model long-wave equations for the Rayleigh-Taylor instability in a shallow layer of incompressible fluid [13]. The similar equations also have been derived in various physical contexts to explain the nonlinear evolution of instability phenomena [14]. Particular solutions to (16) have been constructed by means of the hodograph method [14]. We shall shortly summarize the method of solution and then present solutions relevant to the present instability problem.

The hodograph transformation assures the linearization. In fact, it enables us to transform (16) into the following system of linear partial differential equations for $Y = Y(a,v)$ and $T = T(a,v)$

$$Y_v = 2(vT_v - aT_a), \tag{17a}$$

$$Y_a = 2vT_a + T_v. \tag{17b}$$

Eliminating Y from (17), we obtain the second-order equation for T

$$T_{vv} + 2(aT_{aa} + 2T_a) = 0. \tag{18}$$

Furthermore, if we introduce new variables $r, z, \tilde{T}$ according to the relations

$$a = a_0 r^2, \ v = \sqrt{2a_0} z, \ T = \frac{\tilde{T}}{r}, \tag{19}$$

Eq. (18) is put into the form

$$\tilde{T}_{rr} + \frac{1}{r}\tilde{T}_r - \frac{1}{r^2}\tilde{T} + \tilde{T}_{zz} = 0. \tag{20}$$

It is worthwhile to notice that the Laplace equation $\nabla^2\Psi = 0$ expressed by cylindrical coordinates (r, z, ϕ) is reduced to (20) with the substitution $\Psi(r, z, \phi) = \tilde{T}(r, z)\cos\phi$. To solve (20), however, one must impose appropriate boundary conditions. We consider the boundary condition such that perturbations vanish at an initial time, $T = -\infty$, for example. This condition turns out to requiring $r = 1$ and $v = 0$ by (19). Then, the solutions to Eq. (20) can be constructed analytically with use of toroidal coordinates [14]

$$r = \frac{\sinh\mu}{\cosh\mu + \cos\eta}, \quad z = \frac{\sin\eta}{\cosh\mu + \cos\eta}. \tag{21}$$

With the new variables defined by (19) and (21), Eqs. (17) are rewritten in the form

$$Y_\mu = \sqrt{2a_0}(2zT_\mu + rT_\eta), \tag{22a}$$

$$Y_\eta = \sqrt{2a_0}(2zT_\eta - rT_\mu). \tag{22b}$$

Once the solutions T are constructed by solving (20), the solutions Y are obtained simply by integrating (22).

The solutions for T satisfying the boundary condition mentioned above are now expressed in a series of the associated Legendre functions $Q^n_{\frac{1}{2}}(\coth\mu)$ $(n = 0, 1, 2, ...)$, which are [14]

$$T = \frac{1}{r^{\frac{3}{2}}}\sum_{n=0}^{\infty} a_n Q^n_{\frac{1}{2}}(\coth\mu)\cos(n\eta + b_n), \tag{23}$$

where a_n and b_n are constants. Here, we shall restrict our consideration to the simplest solutions which exhibit the formation of singularities caused by periodic transverse perturbations. The relevant solution for T will be seen to be represented by the first term of the expansion (23). In terms of the original variable t, it reads in the form

$$\Gamma t = -\frac{1}{r^{\frac{3}{2}}} Q_{\frac{1}{2}}(\coth\mu). \tag{24a}$$

Substituting this expression into (22) and integrating, one obtains for y

$$py = -\left\{\tanh^{\frac{1}{2}}(\mu/2)F(\eta/2,s) + \frac{2\sin\eta}{r^{\frac{1}{2}}\sinh\mu}\right\} Q_{\frac{1}{2}}(\coth\mu)$$
$$+\coth^{\frac{1}{2}}(\mu/2)Q_{-\frac{1}{2}}(\coth\mu)E(\eta/2,s), \tag{24b}$$

where F and E are the elliptic integrals of the first and second kinds, given respectively by [15]

$$F(\phi,k) = \int_0^{\phi} \frac{d\alpha}{\sqrt{1-k^2\sin^2\alpha}}, \quad E(\phi,k) = \int_0^{\phi} \sqrt{1-k^2\sin^2\alpha}\,d\alpha, \tag{25a}$$

and

$$s = \operatorname{sech}(\mu/2), \;\; \Gamma = \sqrt{2a_0}p. \tag{25b}$$

Here, Γ defined by (25b) is the instability growth rate in the long-wave limit $p \to 0$ (see (11)). If we use (21), (24a) and the relations

$$Q_{\frac{1}{2}}(\coth\mu) = \frac{2}{k^{\frac{1}{2}}}\left(K(k) - E(k)\right), \; Q_{-\frac{1}{2}}(\coth\mu) = 2k^{\frac{1}{2}}K(k), \quad k = \tanh(\mu/2), \tag{26}$$

where $K(k) \equiv F(\frac{\pi}{2},k)$ and $E(k) \equiv E(\frac{\pi}{2},k)$ are the complete elliptic integrals of the first and second kinds, respectively, the solutions (24) can be rewritten in more transparent form as

$$\Gamma t = -\frac{2}{(kr^3)^{\frac{1}{2}}}\left(K(k) - E(k)\right), \tag{27a}$$

$$py = 2\left\{E(k) - K(k)\right\}F(\eta/2,s) + 2K(k)E(\eta/2,s) + 2z\Gamma t. \tag{27b}$$

We shall now describe the behavior of the solutions. In the initial stage of the evolution of the instability, $a \sim a_0$ which corresponds to taking $\mu \to \infty (k \to 1)$ by (19) and (21). It then follows from (27) that $\Gamma t \sim -\mu,\ py \sim \eta$. Using these relations in (19) and (21), we find

$$a \sim a_0 - 4a_0 e^{\Gamma t} \cos py. \tag{28}$$

This expression indicates that the velocity (or amplitude) of the solitary wave is modulated slowly in the transverse direction due to the action of periodic perturbations. The modulation of the wave profile is accelerated due to the instability and it will eventually lead to the collapse of the wave. We shall describe this process by focusing on the behavior of the maximum and minimum values of the amplitude of the solitary wave. Invoking the formulas [15]

$$F(\phi + n\pi, k) = F(\phi, k) + 2nK(k),\ E(\phi + n\pi, k) = E(\phi, k) + 2nE(k),\ (n = 0, 1, 2, ...), \tag{29a}$$

$$E(k)E(k') + E(k')K(k) - K(k)K(k') = \frac{\pi}{2},\ \left(k' = \sqrt{1-k^2}\right), \tag{29b}$$

we see from (19), (21) and (27) that

$$a_{max} = a_0 \coth^2(\mu/2) = \frac{a_0}{k^2} \ \text{ at } py = \pm(2n+1)\pi, \tag{30a}$$

$$a_{min} = a_0 \tanh^2(\mu/2) = a_0 k^2 \ \text{ at } py = \pm 2n\pi, \tag{30b}$$

where k is defined by (26). An inspection of (27) and (30) shows that a_{min} becomes zero when $\Gamma t = -\pi/2$. At this instant, a_{max} takes a finite value with k being determined by the equation $k(K(k) - E(k)) = \pi/4$, i.e. $k \simeq 0.8585, a_{max} \simeq 1.357a_0$. After this time, a_{max} grows indefinitely and it diverges as $t \to 0$. These observations show that in the case of the positive dispersion, long-wave transverse periodic perturbations destroy 1D algebraic solitary wave and lead to the formation of 2D periodic clusters with increasing amplitudes whose peak positions locate at $py = \pm(2n+1)\pi$ $(n = 0, 1, 2, ...)$.

In Fig. 1, a typical example is depicted which shows clearly the formation of the collapse. One can see that the collapse occurs at $py = \pm\pi$.

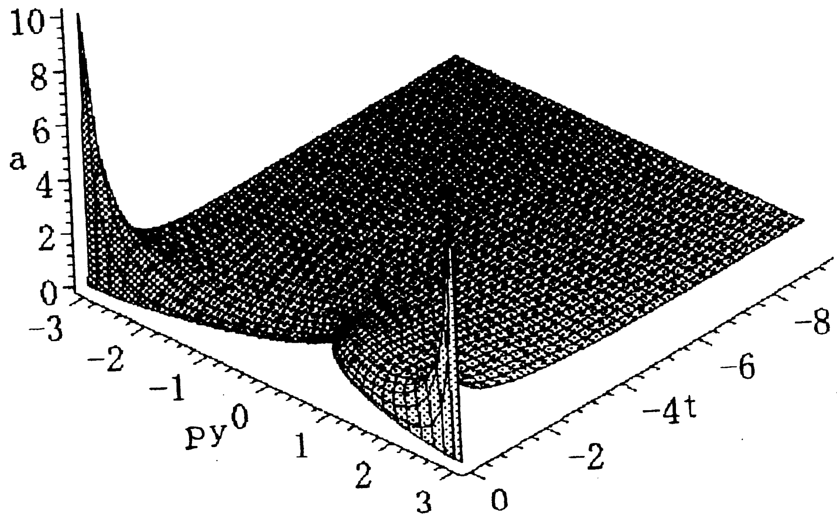

Fig. 1 : A typical example showing the formation of the collapse. The initial amplitude is specified as $a = 1.0$. The figure is depicted in one period $-\pi \leq py \leq \pi$.

In this paper, the collapse of a solitary wave solution of the BO equation was shown to occur on the basis of Eq. (16). However, a natural question arises whether the collapse will continue in the final stage of the nonlinear development where the higher-order terms neglected in (6) may become dominant and suppress the development of the collapse. To study this problem, one must solve Eq. (6) without any approximation. An analysis shows that there exists an exact stationary solution of the form $a = a(y-y_0)$ and $\xi = a_0t+\xi_0$ $(y_0, a_0, \xi_0 :$ const.) which is expressed in terms of an elliptic integral. Whether this solution is realized or not relies on its stability characteristics. This interesting problem will be dealt with in a future work.

Acknowledgement

The author is deeply grateful to Dr. D.E. Pelinovsky for valuable remarks concerning the soliton perturbation theory.

References

[1] M.J. Ablowitz and H. Segur, Stud. Appl. Math. **62** (1980) 249.

[2] M. Oikawa, Bull. Res. Inst. Appl. Mech. Kyushu Univ. **60** (1984) 467. (in Japanese.)

[3] Y. Matsuno, Proc. R. Soc. Lond. A **454** (1998) 835.

[4] B.B. Kadomtsev and V.I. Petviashvili, Sov. Phys. Dokl. **15** (1970) 539.

[5] H.H. Chen and D.J. Kaup, Phys. Fluids **23** (1980) 235.

[6] Y. Matsuno and D.J. Kaup, Phys. Lett. A **228** (1997) 176.

[7] M.D. Spector and T. Miloh, SIAM J. Appl. Math. **54** (1994) 688.

[8] G.B. Whitham, *Linear and Nonlinear Waves* (John Wiley & Sons, New York, 1974).

[9] Y. Matsuno, Phys. Rev. E **51** (1995) 1471.

[10] Y. Matsuno and D.J. Kaup, J. Math. Phys. **38** (1997) 5198.

[11] V.E. Zakharov, Sov. Phys. JETP Lett. **22** (1975) 174.

[12] D.E. Pelinovsky and V.I. Shrira, Phys. Lett. A**206** (1995) 195.

[13] D.L. Book, E. Ott and A.L. Sulton, Phys. Fluids **17** (1974) 676.

[14] B.A. Trubnikov and S.K. Zhdanov, Phys. Rep. **155** (1987) 137.

[15] I.S. Gradshteyn and I.M. Ryzhik, *Table of Integrals, Series, and Products* (Academic Press, New York, 1965).