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Rigidity of infinitely renormalizable polynomials of higher degree

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Abstract

The conjecture that hyperbolic rational maps are dense in the space of all rational maps of degree $d$ is one of the central problems in complex dynamics. It is known that no invariant line field conjecture implies the density of hyperbolicity (see [MS]).

In the case of quadratic polynomials, McMullen shows that a robust infinitely renormalizable quadratic polynomial carries no invariant line field on its Julia set [Mc].

In this paper, we give the extension of renormalization and the above theorem of McMullen to polynomial of any degree.

1 Notation and backgrounds

Notation. Let $f$ be a polynomial of degree $d$.

- The Fatou set $F(f)$ of $f$ is the maximal open set of $\mathbb{C}$ where $\{f^n\}$ is normal.
- The Julia set $J(f)$ of $f$ is the complement of $F(f)$.
- The filled Julia set $K(f)$ of $f$ is the set of all point in $\mathbb{C}$ whose forward orbit by $f$ does not tend to infinity. Note that $\partial K(f) = J(f)$.
- Let $C(f)$ be the set of critical points of $f$.
- The postcritical set $P(f)$ is the closure of the strict forward image of critical points by $f$:

$$P(f) = \overline{\bigcup_{n>1} f^n(C(f))}$$

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Definition. A polynomial-like map $f : U \to V$ is a proper holomorphic map with $\overline{U} \subset V$.

The filled Julia set $K(f)$ of a polynomial-like map $f : U \to V$ is the set of all point $z \in U$ such that $f^n(z) \in U$ for all $n \geq 0$. The Julia set $J(f)$ is the boundary of $K(f)$.

Two polynomial-like maps $f$ and $g$ are hybrid equivalent if there is a quasiconformal map $\phi$ from a neighborhood of $K(f)$ to a neighborhood of $K(g)$, such that $\phi \circ f = g \circ \phi$ and $\overline{\partial} \phi = 0$ on $K(f)$.

Theorem 1.1. Every polynomial-like map $f$ is hybrid equivalent to some polynomial $g$ of the same degree. Furthermore, if $K(f)$ is connected, $g$ is unique up to affine conjugacy.

See [DH, Theorem 1].

Lemma 1.1. Let $f_i : U_i \to V_i$ be polynomial-like maps of degree $d_i$ for $i = 1, 2$. Suppose $f_1 = f_2 = f$ on $U = U_1 \cap U_2$ and let $U'$ be a component of $U$ with $U' \subset f(U') = V'$. Then $f : U' \to V'$ is polynomial-like map of degree $d \leq \min(d_1, d_2)$, and

$$K(f) = K(f_1) \cap K(f_2) \cap U'.$$

Moreover, if $d = d_i$, then $K(f) = K(f_i)$.

See [Mc, Theorem 5.11].

Lemma 1.2. Let $f$ be a polynomial with connected Julia set. Let $f^n : U \to V$ be a polynomial-like restriction of degree more than 1 with connected filled Julia set $K$. Then:

1. The Julia set of $f^n : U \to V$ is contained in $J(f)$.

2. For any closed connected set $L$ contained in $K(f)$, $L \cap K$ is also connected.

See [Mc, Theorem 6.13].

Definition. A line field supported on $E \subset \mathbb{C}$ is the choice of a real line through the origin of $T_z \mathbb{C}$ at each $z \in E$. It is equivalent to take a Beltrami differential $\mu = \mu(z) d\overline{z}/dz$ supported on $E$ with $|\mu| = 1$.

We say $f$ carries an invariant line field on its Julia set if there exists a measurable Beltrami differential $\mu$ on $\mathbb{C}$ such that $f^* \mu = \mu$ and $|\mu| = 1$ on a set of positive measure contained in $J(f)$ and vanishes elsewhere.

Conjecture 1.1 (No invariant line fields). A polynomial carries no invariant line field on its Julia set.
If this conjecture is true, the following one is also true. Here the polynomial $f$ is hyperbolic if all critical points tend to attracting periodic cycles under iteration.

**Conjecture 1.2 (Density of hyperbolicity).** *Hyperbolic maps are dense in the family of polynomial of degree $d$.***

See [MS].

## 2 Renormalization

In this section, we give the definition of renormalization and describe some basic properties.

**Definition.** $f^n$ is called renormalizable if there exist open disks $U, V \subset \mathbb{C}$ satisfying the followings:

1. $U \cap C(f) \neq \phi$.

2. $f^n : U \to V$ is a polynomial-like map with connected filled Julia set.

3. For each $c \in C(f)$, there is at most one $i$, $0 < i \leq n$, such that $c \in f^i(U)$.

4. $n > 1$ or $U \nsubseteq C(f)$.

A renormalization is a polynomial-like restriction $f^n : U \to V$ as above.

**Notation.** Let $f^n : U \to V$ be a renormalization.

- The filled Julia set of a renormalization $f^n : U \to V$ is denoted by $K_n(U)$ and the postcritical set by $P_n(U)$.

- For $i = 1, \ldots, n$, the $i$th small filled Julia set is denoted by $K_n(U, i) = f^i(K_n(U))$.

- The $i$th small postcritical set is denoted by $P_n(U, i) = K_n(U, i) \cap P(f)$.

- $C_n(U, i) = K_n(U, i) \cap C(f)$. By definition, $C_n(U, n)$ is nonempty and $C_n(U, i)$ is empty with at most $d - 1$ exceptions.

- $\mathcal{K}_n(U) = \bigcup_{i=1}^n K_n(U, i)$ is the union of the small filled Julia sets.

- $C_n(U) = \bigcup_{i=1}^n C_n(U, i)$ is the set of critical points appear in the renormalization $f^n : U \to V$.

- Let $V_n(U, i) = f^i(U)$ and $U_n(U, i)$ be the component of $f^{i-n}(U)$ contained in $V_n(U, i)$ Then $f^n : U_n(U, i) \to V_n(U, i)$ is polynomial-like map of the same degree as $f^n : U \to V$. 
Now, when it is clear which $U$ we consider, we will simply write $K_n(i)$ instead of $K_n(U, i)$, and so on.

In this paper, we fix a critical point $c_0 \in C(f)$ and consider only renormalizations about $c_0$, i.e. $C_n(U) = C_n(U, n)$ contains $c_0$.

The next proposition implies that two renormalizations are essentially the same if their period and critical points are equal.

**Proposition 2.1.** Let $f^n : U^k \to V^k$ be renormalizations for $k = 1, 2$.

If for any $i$, $0 \leq i < n$, $C_n(U^1, i) = C_n(U^2, i)$, their filled Julia sets are equal.

**Proof.** Let $K^k$ be the filled Julia set of $f^n : U^k \to V^k$. By Lemma 1.2, $K = K^1 \cap K^2$ is connected.

Let $U$ be the component of $U^1 \cap U^2$ containing $K$. Let $V = f^n(U)$. Since $V$ contains $f(K) = K$, $V$ contains $U$. By Lemma 1.1, $f^n : U \to V$ is polynomial-like with filled Julia set $K$. Since critical points of these three maps are equal, we have $K = K^1 \cap K^2$. 

**Proposition 2.2.** Let $f^a : U_a \to V_a$ and $f^b : U_b \to V_b$ be renormalizations about $c_0$. Then there exists a renormalization $f^c : U \to V$ with filled Julia set $K_c = K_a \cap K_b$ where $c$ is the least common multiple of $a$ and $b$.

**Proof.** By Lemma 1.2, $K = K_a \cap K_b$ is connected.

Let

$$
\tilde{U}_a = \left\{ z \in U_a \mid f^{ja}(z) \in U_a \text{ for } j = 1, \ldots, \frac{c}{a} - 1 \right\}
$$

$$
\tilde{U}_b = \left\{ z \in U_b \mid f^{jb}(z) \in U_b \text{ for } j = 1, \ldots, \frac{c}{b} - 1 \right\}.
$$

Then $f^c : \tilde{U}_a \to V_a$ and $f^c : \tilde{U}_b \to V_b$ are polynomial-like. Let $U_c$ be a component of $\tilde{U}_a \cap \tilde{U}_b$ which contains $K$. Then by Lemma 1.1, $f^c : U_c \to f^c(U_c)$ is polynomial-like map with filled Julia set $K_c$.

Suppose $c \in C_c(i)$. then $c \in C_c(j)$ is equivalent to $j \equiv i \pmod{a}$ and $j \equiv i \pmod{b}$, which means $j = i$. Therefore, $f^c : U_c \to V_c$ is a renormalization with filled Julia set $K_c = K$.

Define the **intersecting set** of a renormalization $f^n : U \to V$ by

$$
I_n(U) = K_n(U) \cap \left( \bigcup_{i=1}^{n-1} K_n(U, i) \right).
$$

We say a renormalization is **intersecting** if $I_n(U) \neq \emptyset$.

**Proposition 2.3.** If a renormalization $f^n : U \to V$ is intersecting, then $I_n(U)$ consists of only one point which is a repelling fixed point of $f^n$. 
Proof. Suppose $E = K_n(U) \cap K_n(U, i) \neq \emptyset$ for some $0 < i < n$. By Lemma 1.2, $E$ is connected.

Let $U$ be the component of $U \cap U(i)$ containing $E$. By Lemma 1.1, $f^n : U \to f^n(U)$ is a polynomial-like map of degree 1. By the Schwarz lemma, $E$ consists of a single repelling fixed point $x$ of $f^n$.

Suppose $K_n(U) \cap K_n(U, j) = \{y\}$ with $y \neq x$. Then there is a sequence \{i_0, i_1, \ldots, i_K\} such that $K_n(U, i_k) \cap K_n(U, i_{k+1})$ is nonempty and $K_n(U, i_k) \cap K_n(U, i_{k+1}) \cap K_n(U, i_{k+2})$ is empty (where $K + 1$, $K + 2$ is interpreted as 0, 1, respectively).

Let

$$L = K_n(U, i_1) \cap \ldots K_n(U, i_K).$$

Then $L$ is a closed connected set in $K(f)$. But $L \cap K_n(U)$ consists of two points and it contradicts Lemma 1.2. □

Since a repelling fixed point separates filled Julia set into a finite number of components, components of $K_n(U) - I_n(U)$ are finite. We say a renormalization is simple if $K_n(U) - I_n(U)$ is connected, and crossed if it is disconnected.

**Theorem 2.1.** For $p > 0$, there are finitely many $n > 0$ such that there exists a renormalization $f^n : U_n \to V_n$ such that $K_n(U)$ contains a periodic point of period $p$.

Proof. Let $x$ be a periodic point of period $p$. Assume the filled Julia set of a renormalization $f^n : U \to V$ with $p < n$ contains $x$. Since $x$ is a repelling fixed point of $f^n$ (by Proposition 2.3), $p$ divides $n$ and the number $\rho$ of the components of $K_n(U_n) - \{x\}$ is finite.

Let $E$ be the component of $K_n(U)$ which contains $x$. $E - \{x\}$ has exactly $\rho n/p$ components. Let $q$ be the number of the components of $K(f) - \{x\}$. Since $x$ is a repelling periodic point of $f$, $q < \infty$.

Suppose a component $A$ of $K(f) - \{x\}$ contains two components $B_1, B_2$ of $E - \{x\}$. Then we can take a path in $A -(B_1 \cup B_2)$ from $x$ to some point in $B_1$. It contradicts Lemma 1.2.

Therefore each component of $K(f) - \{x\}$ can contain at most one component of $E - \{x\}$. So $q \geq \rho n/p$, it concludes $n \leq pq$.

There are finitely many periodic points of period $p$, the theorem follows. □

**Proposition 2.4.** Let $f^a : U_a \to V_a$ and $f^b : U_b \to V_b$ be renormalizations about $c_0$. Suppose that $f^b : U_b \to V_b$ is simple. Then either $a$ divides $b$ or $b$ divides $a$.

Proof. Let $c$ be the greatest common divisor of $a$ and $b$. If $c = a$ or $c = b$, the proposition follows. So suppose $c < a, b$. 

Since $K_a \cap K_b$ is nonempty (it contains $c_0$), $f^i(K_a) \cap f^i(K_b)$ is nonempty for any $i > 0$. Therefore $K_a(c) \cap K_b(c)$, $K_a(c) \cap K_b$ and $K_a \cap K_b(c)$ are all nonempty. Therefore $L = K_b \cup K_a(c) \cup K_b(c)$ is connected.

By Lemma 1.2, $K_a \cap L$ is connected. Since $K_a \cap K_a(c)$ is at most one point and $L$ is a closed connected set, $K_a \cap (K_b \cup K_b(c))$ is connected. So $K_a \cap K_b \cap K_b(c)$ is nonempty. By Proposition 2.3, $K_b \cap K_b(c) = \{ x \}$ where $x$ is a repelling fixed point of $f^b$, so $K_a \ni x$. Since $f^b : U_b \to V_b$ is simple, $x$ does not disconnect $K_b$.

By Proposition 2.2, there exists a renormalization $f^{ab/c} : U \to V$ with Julia set $K^{ab/c} = K_a \cap K_b$. But $K^{ab/c}$ cannot contain $x$ because $K_b - \{ x \}$ is connected and $ab/c > b$ (see the proof of Theorem 2.1), it is a contradiction.

Example. Let $f(z) = z^3 - \frac{3}{4}z - \frac{\sqrt{2}}{4}$. Then $C(f) = \{ \pm \frac{1}{2} \}$ and $\pm \frac{1}{2}$ are periodic of period 2. Let $W_{\pm}$ be the Fatou component which contains $\pm \frac{1}{2}$. They are superattracting basin of period 2.

Every renormalization $f^n : U \to V$ must satisfy $U \supset W_-$ or $W_+$. So $n \leq 2$ and by symmetry, we will consider only the case $U \supset W_-$.

**Type I.** Let $K$ be the connected component of the closure of $\bigcup_{n>0} f^{-n}(W_-)$ which contains $W_-$ and let $U_1$ be a small neighborhood of $K$.

Then $f : U_1 \to f(U_1)$ is a renormalization with filled Julia set $K(1, U) = K_1$ which is hybrid equivalent to $z \mapsto z^2 - 1$.

**Type II.** Let $U_2$ be a small neighborhood of $W_-$. Then $f^2 : U_2 \to f^2(U_2)$ is a renormalization with filled Julia set $K(2, U_2) = \overline{W_-}$, which is hybrid equivalent to $z \mapsto z^2$.

**Type III.** Let $K'_2$ be the connected component of $\bigcup_{n>0} f^{-2n}(W_- \cup W_+)$ which contains $W_-$ and let $U'_2$ be a small neighborhood of $K'_2$.

Then $f^2 : U'_2 \to f^2(U'_2)$ is a renormalization with filled Julia set $K'_2$, which is hybrid equivalent to $z \mapsto z^3 - \frac{3}{\sqrt{2}}z$.

**Type IV.** Let $K''_2$ be the connected component of $\bigcup_{n>0} f^{-2n}(W_- \cup f(W_+))$ which contains $W_-$ and let $U''_2$ be a small neighborhood of $K''_2$.

Then $f^2 : U''_2 \to f^2(U''_2)$ is a renormalization with filled Julia set $K''_2$ and of degree 4.

Similarly, consider $\bigcup_{n>0} f^{-2n}(W_- \cup f(W_-) \cup W_+)$ and then we can construct a polynomial-like map $f^2 : U \to V$ of degree 6. But it is not a renormalization because $-\frac{1}{2}$ is contained in both $U$ and $f(U)$.

## 3 Infinite renormalization

For a subset $C_R \subset C(f)$, let $\mathcal{R}(f, C_R)$ be the set of all $n > 0$ such that there exists a renormalization $f^n : U_n \to V_n$ about $c_0$ with $C_n(U_n) = C_R$. Let $\mathcal{S}\mathcal{R}(f, C_R)$ be the set of such $n \in \mathcal{R}(n, C_R)$ that $f^n : U_n \to V_n$ is simple.
Figure 1: The filled Julia set of $z \mapsto z^3 - \frac{3}{4}z - \frac{\sqrt{7}}{4}$.

Figure 2: Five types of Polynomial-like restrictions.
Proposition 3.1. Let \( n_1, n_2 \in \mathcal{S}\mathcal{R}(f, C_R) \). If \( n_1 < n_2 \), then \( n_1 \) divides \( n_2 \) and \( K_{n_1}(U_1) \supset K_{n_2}(U_2) \).

**Proof.** By Proposition 2.4, \( n_1 \) divides \( n_2 \).

Assume \( K_{n_1}(U_1) \nsubseteq K_{n_2}(U_2) \). By Proposition 2.2, there exists a renormalization \( f^{n_2}: U_{n_2}' \to V_{n_2}' \) with filled Julia set \( K_{n_2}(U_{n_2}') = K_{n_1}(U_{n_1}) \cap K_{n_2}(U_{n_2}) \).

For simplicity, we write \( K_{n_1} = K_{n_1}(U_{n_1}), K_{n_2} = K_{n_2}(U_{n_2}) \) and \( K_{n_2}' = K_{n_2}(U_{n_2}') \).

If \( C_{n_2}(U_{n_2}') = C_R \), then \( K_{n_2}' = K_{n_2} \). Therefore there exists a critical point \( c_1 \in C_R - C_{n_2}(U_{n_2}') \). Let \( i_k \) be a number which satisfies \( K_{n_1}(i_k) \ni c_1 \). Then \( i_1 \neq i_2 \) (mod \( n_1 \)). So there exists \( i_0 \) such that \( K_{n_1}(i_0) \) intersects \( K_{n_2} \).

Therefore let a closed connected subset \( L \) of \( K(f) \) as the following:

\[
    L = K_{n_1}(i_0) \cup K_{n_2}(i_0) \cup K_{n_1}(2i_0) \cup \cdots \cup K_{n_1}.
\]

Then \( L \cap K_{n_2} \) is disconnected and it contradicts Lemma 1.2. \( \square \)

Proposition 3.2. If \( f \) can be infinitely renormalizable, \( f \) has infinitely many simple renormalizations.

More precisely, if \( \mathcal{R}(n, C_R) \) is infinite for some \( C_R \subset C(f) \), then there exists some \( C, C_R \subset C \subset C(f) \), such that \( \mathcal{S}\mathcal{R}(f, C) \) is infinite.

**Proof.** For \( n \in \mathcal{R}(n, C_R) \), Let \( \kappa_n \) be the number of components of \( \mathcal{K}_n \). Since \( \kappa_n \) is equal to the minimum of the period of periodic point of \( f \) contained in \( K_n \), \( \kappa_n \to \infty \) by Theorem 2.1.

Now we show \( f^{\kappa_n} \) is simply renormalizable. For sufficiently large \( n \), choose a repelling periodic point \( x \) of \( f \) of period less than \( \kappa_n \). Then \( x \notin \mathcal{K}_n \). We construct the Yoccoz puzzle from the rays landing at \( x \) and some equipotential curve.

For any depth \( r \geq 0 \), the piece \( P_r(c_0) \) containing \( c_0 \) contains the component \( E \) of \( \mathcal{K}_n \) containing \( c_0 \). Thus the tableau \( P_r(f^k(c_0)) \) for \( c_0 \) is periodic of period \( p \) with \( p|\kappa_n \), i.e., for any \( r > 0 \), \( P_r(f^p(c_0)) = P_r(f^p(c_0)) \).

Then by slightly thickening the pieces, we can obtain a simple renormalization \( f^p: U_p \to V_p \) with \( K_p \supset E \) (more precisely, see [Mi2, Lemma 2]).

If \( p = \kappa_n \), we are done.

Otherwise, let \( g \) be the polynomial hybrid equivalent to \( f^p: U_p \to V_p \). There exists a renormalization \( g^{n/p}: \tilde{U}_{n/p} \to \tilde{V}_{n/p} \) corresponds to \( f^n: U_n \to V_n \).

Now apply the argument above to \( g \) and the renormalization \( g^{n/p}: \tilde{U}_{n/p} \to \tilde{V}_{n/p} \) and eventually we obtain a simple renormalization of \( f^{\kappa_n} \). \( \square \)

Now we assume that \( f \) is infinitely renormalizable. By the proposition above, \( \#\mathcal{S}\mathcal{R}(f, C_R) \) is infinite for some \( C_R \subset C(f) \).

Furthermore, suppose \( f(C_R) = f(C(f)) \), i.e., for any \( c' \in C(f) - C_R \), there exists some \( c \in C_R \) such that \( f(c) = f(c') \).
Remark. The above condition is satisfied for a polynomial which is hybrid equivalent to \( f^n : U_n \to V_n \) for \( n \in \mathcal{SR}(f, C_R) \).

So this assumption is to consider the polynomial hybrid equivalent to some renormalization instead of the original polynomial.

Definition. Let \( f \) as above. For each \( n \in \mathcal{SR}(f, C_R) \), let \( \delta_n(i) \) be a closed curve which separates \( K_n(i) \) from \( P(f) - P_n(i) \) (in our case, such a curve exists and its homotopy class is uniquely determined). Let \( \gamma_n(i) \) is the hyperbolic geodesic on \( \mathbb{C} - P(f) \) which is homotopic to \( \delta_n(i) \) on \( \mathbb{C} - P(f) \) and let \( \gamma_n = \gamma_n(n) \).

We say \( \mathcal{SR}(f, C_R) \) is robust if

\[
\lim_{n \to \infty} \ell(\gamma_n) < \infty,
\]

where \( \ell(\cdot) \) denotes the hyperbolic length on \( \mathbb{C} - P(f) \).

Let \( \Sigma = \text{proj lim}_{n \in \mathcal{SR}(f, C_R)} \mathbb{Z}/n \) and define \( \sigma : \Sigma \to \Sigma \) by:

\[
\sigma((i_n)_{n \in \mathcal{SR}(f, C_R)}) = (i_n + 1).
\]

Theorem 3.1. Let \( f \) as above. When \( \mathcal{SR}(f, C_R) \) is robust, then:

1. The postcritical set \( P(f) \) is a Cantor set of measure zero.
2. \( \lim_{n \in \mathcal{SR}(f, C_R)} \sup_{0 < i \leq n} \text{diam} P_n(i) \to 0. \)
3. \( f : P(f) \to P(f) \) is topologically conjugate to \( \sigma : \Sigma \to \Sigma \). Especially, \( f|_{P(f)} \) is a homeomorphism.

Proof. By the hyperbolic geometry, the geodesics \( \gamma_n(i) \) \((n \in \mathcal{SR}(f, C_R), 0 < i \leq n)\) are simple and mutually disjoint, and their length are comparable with \( \ell(\gamma_n) \).

Thus by the collar theorem, there is a standard collar \( A_n(i) \) about \( \gamma_n(i) \) in \( \mathbb{C} - P(f) \) such that they are mutually disjoint and mod(\( A_n(i) \)) is a decreasing function of \( \ell(\gamma_n(i)) \). Note that \( A_n(i) \) separates \( P_n(i) \) from the rest of the postcritical set.

Let \( E_n = \bigcup_{i=1}^{n} A_n(i) \) and \( F_n \) be the union of the bounded components of \( \mathbb{C} - E_n \).

Then \( F_n \) contains \( P(f) \) and each component of \( F_n \) meets \( P(f) \).

For any sequence \( \{A_n(i_n)\}_{n \in \mathcal{SR}(f, C_R)} \) of nested annuli,

\[
\sum_{n \in \mathcal{SR}(f, C_R)} \text{mod} A_n(i_n) = \infty,
\]

because \( \lim \inf \ell(\gamma_n) < \infty. \)
Therefore $F = \bigcap_{n \in \mathcal{SR}(f, C_R)} F_n$ is a Cantor set of measure zero. Since $F$ contains $P(f)$ and each component of $F_n$ contains $P_n(i)$ for some $i$, $F$ is equal to $P(f)$, so the postcritical set is measure zero and diameter of $P_n(i)$ tends to zero.

For $n \in \mathcal{SR}(f, C_R)$, we define $\phi_n : P(f) \rightarrow \mathbb{Z}/n$ by $\phi(z) = i \pmod{n}$ when $z \in P_n(i)$. Then $\phi(f(z)) = \phi(z) + 1 \pmod{n}$.

Therefore, define $\phi : P(f) \rightarrow \limproj \mathbb{Z}/n$ by $\phi(z) = (\phi_n(z))_{n \in \mathcal{SR}(f, C_R)}$ and it gives the conjugacy between $f|_{P(f)}$ and $\sigma$.

Corollary 3.1. Let $f$ as above. Suppose $\mathcal{SR}(f, C_R)$ is robust. Then for sufficiently large $n \in \mathcal{SR}(f, C_R)$ and any $i$, $0 < i \leq n$, $\#C_n(i) \leq 1$.

Proof. Suppose

$$\# \left( \bigcap_{n \in \mathcal{SR}(f, C_R)} C_n(n) \right) > 1.$$

By Theorem 3.1, $\bigcap P_n(1)$ consists of only one point $x$. Therefore, $f(C_n(n)) = \{x\}$. But it is impossible because there is no other critical point in $U_n$ for sufficiently large $n$. \qed

4 Robust rigidity

In this section, we prove the following theorem:

Theorem 4.1 (Robust rigidity). Let $f$ as above. If $\mathcal{SR}(f, C_R)$ is robust, then $f$ carries no invariant line field on its Julia set.

The proof depends on the following two lemmas.

Lemma 4.1. Let $f_n : (U_n, u_n) \rightarrow (V_n, v_n)$ be a sequence of holomorphic maps between disks and let $\mu_n$ is a sequence of $f_n$-invariant line field on $V_n$. Suppose $f_n$ converge to $f : (U, u) \rightarrow (V, v)$ in the Carathéodory topology and $\mu_n$ converge in measure to $\mu$ on $V$. Then $\mu$ is $f$-invariant.

See [Mc, Theorem 5.14].

Lemma 4.2. Let $\mu$ be a measurable line field on $\mathbb{C}$. Assume $\mu$ is almost continuous at $x$ and $|\mu(x)| = 1$. Let $(V_n, v_n) \rightarrow (V, v)$ be a convergent sequence of disks, and let $h_n : V_n \rightarrow \mathbb{C}$ be a sequence of univalent maps with $h_n'(v_n) \rightarrow 0$.

Suppose

$$\sup \frac{|x - h_n(v_n)|}{h_n'(v_n)} < \infty.$$

Then there exists a subsequence such that $h_n^*(\mu)$ converges in measure to a univalent line field on $V$. 

Now we give the summary of the proof of the theorem. We divide the proof into two cases: whether \(\liminf \ell(\gamma_n)\) is zero or not. But outline of these two proof are very similar. We assume there exists a measurable invariant line field \(\mu\) supported on \(J(f)\) and induce contradiction.

First, we take a point \(x \in J(f)\) where \(\mu\) is almost continuous, such that \(||(f^k)'(x)|| \to \infty\) with respect to hyperbolic metric on \(\mathbb{C} - P(f)\), and such that \(f^k(x)\) does not land in but tends to \(P(f)\).

Next we construct some critically compact proper map \(f^n : X_n \to Y_n\) from \(f^n : U_n \to V_n\). By assumption, \(f^n(x)\) eventually land in \(Y_n\). If we take disks \(X_n, Y_n\) properly, we can take a univalent inverse branch \(h_n\) of \(f^{-k}\) from \(Y_n\) to the region near \(x\). Note that \(h_n^*(\mu) = \mu\) is \(f^n\)-invariant line field on \(Y_n\).

By properly scaling \(f^n : X_n \to Y_n\) and taking a subsequence, they converge to a proper map \(g : U \to V\). Furthermore, by Lemma 4.2 and Lemma 4.1, \(g\) must have an invariant univalent line field \(\nu\) on \(V\).

But \(g\) have a critical point \(c \in U \cap V\), then, by invariance, \(\nu(c) = 0\), that is a contradiction.

### 4.1 Thin rigidity

**Definition.** A renormalization \(f^n : U_n \to V_n\) is unbranched if

\[
V_n \cap P(f) = P_n.
\]

Let \(f^n : U_n \to V_n\) be an unbranched renormalization. Let \(W\) be a component of \(f^{-1}(V_n(i+1))\) which is not \(V_n(i)\). Then any inverse branch of \(f^{-k}\) on \(W\) is univalent because \(W\) is disjoint from the postcritical set.

**Lemma 4.3.** There exists some \(M > 0\) such that if \(\ell(\gamma_n) < M\), we can choose \(U_n\) and \(V_n\) such that \(f^n : U_n \to V_n\) is unbranched renormalization and

\[
\mod(U_n, V_n) > m(\ell(\gamma_n)) > 0
\]

where \(m(\ell) \to \infty\) as \(\ell \to 0\).

**Proof.** Let \(A_n\) be the standard collar about \(\gamma_n\) with respect to the hyperbolic metric on \(\mathbb{C} - P(f)\). Let \(B_n\) be the component of \(f^{-n}(A_n)\) which is the same homotopy class as \(\gamma_n\). Let \(D_n\) (resp. \(E_n\)) be the union of \(B_n\) (resp. \(A_n\)) and the bounded component of the complement. \(f^n : D_n \to E_n\) is a critically compact proper map with postcritical set \(P_n\).

When \(\ell(\gamma_n)\) is sufficiently small, \(\mod(P_n, E_n) \geq \mod(A_n)\) is sufficiently large. Then we can choose \(U_n \subset D_n\) and \(V_n \subset E_n\) such that \(f^n : U_n \to V_n\) is a renormalization and \(\mod(U_n, V_n)\) is bounded below in terms of \(\mod(P_n, E_n)\).

The modulus of collar \(A_n\) depends only on \(\ell(\gamma_n)\) and tends to infinity as \(\ell(\gamma_n)\) tends to zero. Since \(\mod(P_n, E_n) \geq \mod(A_n)\), we are done. \(\square\)
Theorem 4.2. Let $f$ as above. Suppose for infinitely many $n \in \mathcal{SR}(f, C_R)$ there is a simple unbranched renormalization $f^n : U_n \to V_n$ with $\text{mod}(U_n, V_n) > m$ for a constant $m > 0$.

Then $f$ carries no invariant line field on its Julia set.

By the previous lemma, the following corollary is trivial.

Corollary 4.1 (Thin rigidity). There is $L > 0$ such that if

$$\liminf_{\mathcal{SR}(f, C_R)} \ell(\gamma_n) < L,$$

then $f$ carries no invariant line field on its Julia set.

Proof of Theorem 4.2. Let $\mathcal{USR}(f, C_R, m)$ be a set of $n \in \mathcal{SR}(f, C_R)$ such that there is an unbranched simple renormalization $f^n : U_n \to V_n$ with $\text{mod}(U_n, V_n) > m$.

For $n \in \mathcal{USR}(f, C_R, m)$, there is an annulus of definite modulus separating $J_n(i)$ from $P(f) - P_n(i)$. So $\mathcal{SR}(f, C_R)$ is robust and

$$\bigcap_{n \in \mathcal{SR}(f, C_R)} J_n = P(f).$$

Therefore, by the fact that a forward orbit of almost every point in $J(f)$ tends to $P(f)$, almost every $x$ in $J(f)$ satisfies the followings:

1. The forward orbit of $x$ does not meet the postcritical set.
2. $||(f^k)'(x)|| \to \infty$ in the hyperbolic metric on $\mathbb{C} - P(f)$.
3. For any $n \in \mathcal{SR}(f, C_R)$, there is a $k > 0$ with $f^k(x) \in J_n$.
4. For any $k > 0$, there is an $n \in \mathcal{SR}(f, C_R)$ such that $f^k(x) \notin J_n$.

(Nota the condition 2 is satisfied every point which satisfies the condition 1.)

Suppose that $f$ carries an invariant line field $\mu$ on $J(f)$. Let $x$ be a point in $J(f)$ at which $\mu$ is almost continuous, $|\mu(x)| = 1$ and satisfies the above condition 1-4. For each $n \in \mathcal{SR}(f, C_R)$, let $k(n) \geq 0$ be the least integer such that $f^{k(n)+1}(x) \in J_n$. By the condition 3, such $k(n)$ exists and tends to infinity by the condition 4. Now $f^{k(n)+1}(x)$ is contained in $J_n(i(n) + 1)$ for some $0 \leq i(n) < n$.

For $n$ sufficiently large, $k(n) > 0$ and $f^{k(n)}(x) \notin J_n$. So $f^{k(n)}(x)$ is contained in some component $W_n$ of $f^{-1}(V_n(i(n) + 1))$ which is not $V_n(i(n))$. $W_n$ is disjoint from the postcritical set. Furthermore, $W_n$ contains no critical point for sufficiently large $n$ (actually, it is true if $k(n) > k(n_0)$ where $n_0 = \min(\mathcal{USR}(f, C_R, m)))$. 

Let \( j(n) > i(n) \) be the least number such that \( C_n(j(n)) \) is nonempty, so that \( f^{j(n)-i(n)}: W_n \to V_n(j(n)) \) is univalent. Then there exists an univalent branch \( h_n \) of \( f^{i(n)-j(n)-k(n)} \) defined on \( V_n(j(n)) \) which maps \( f^{j(n)-i(n)+k(n)}(x) \) to \( x \).

Let \( J_n^* = h_n(J_n(j(n))) \). Since there is an annulus of definite modulus in \( \mathbb{C} - P(f) \) enclosing it, the diameter of \( f^k(J_n^*) (= f^{-1}(J_n(i(n) + 1)) \cap W_n) \) is bounded with respect to the hyperbolic metric on \( \mathbb{C} - P(f) \). Therefore, by the condition 2, the diameter of \( J_n^* \) in the hyperbolic metric on \( \mathbb{C} - P(f) \) tends to zero.

Let \( c \in C_R \) be a critical point such that for infinitely many \( n \in USR(f, C_R, m) \), \( C_n(j(n)) \) contains \( c \). By taking a subsequence and replacing \( f^n : U_n \to V_n \) by \( f^n : U_n(j(n)) \to V_n(j(n)) \), we may assume \( c = c_0 \) and \( j(n) = n \), so \( h_n \) is defined on \( V_n \). (Note that \( \text{mod}(U_n(j(n)), V_n(j(n))) \geq \frac{1}{d_R} \text{mod}(U_n, V_n) > \frac{m}{d_R} \), where \( d_R \) is the degree of renormalization \( f^n : U_n \to V_n \). Thus we should replace \( m \) by \( \frac{m}{d_R} \).)

Let

\[
A_n(z) = \frac{z - c_0}{\text{diam}(J_n)},
\]

\[
g_n = A_n \circ f^n \circ A_n^{-1},
\]

\[
y_n = A_n(h_n^{-1}(x)).
\]

Then

\[
g_n : (A_n(U_n), 0) \to (A_n(V_n), A_n(f^n(c_0)))
\]

is a polynomial-like map with \( \text{diam}(J(g_n)) = 1 \) and \( \text{mod}(A_n(U_n), A_n(V_n)) > m \).

Thus, by taking a subsequence, \( g_n \) converges to some polynomial-like map (or polynomial) \( g : (U, 0) \to (V, g(0)) \) with \( \text{mod}(U, V) > m \) in the Carathéodory topology (see [Mc, Theorem 5.8]).

Let \( k_n = h_n \circ A_n^{-1} : A_n(V_n) \xrightarrow{A_n^{-1}} V_n \xrightarrow{h_n} \mathbb{C} \) and \( \nu_n = k_n^*(\mu) \). Then \( \nu_n \) is \( g_n \)-invariant line field on \( A_n(V_n) \) because \( \mu = h_n^*(\mu) \) is \( f \)-invariant. Since \( \text{diam}(J(g_n)) = 1 \) and \( \text{diam}(J^*_n) \to 0 \), \( k_n(y_n) \to 0 \).

Now we take a further subsequence of \( n \) so that \( (A_n(V_n), y_n) \to (V, y) \). Then by Lemma 4.2, after passing a further subsequence, \( \nu_n \) converges to an univalent \( g \)-invariant line field \( \nu \) on \( V \).

For \( f^n : U_n \to V_n \) have connected Julia set, so does \( g \). Thus the critical point and critical value lie in \( V \). But it contradicts the fact that \( g \) has a univalent invariant line field \( \nu \).

\[ \square \]

4.2 Thick rigidity

Theorem 4.3 (Thick rigidity). Let \( f \) as above. Suppose

\[ 0 < \liminf_{n \in \mathcal{R}(f, C_R)} \ell(\gamma_n) < \infty, \]

Then \( f \) carries no invariant line field on its Julia set.
Notation. For $n \in \mathcal{S}\mathcal{R}(f, C_{R})$,

- Let $\delta_{n}$ be the component of $f^{-n}(\gamma_{n})$ which is homotopic to $\gamma_{n}$ on $C - P(f)$.
- Let $X_{n}$ (resp. $Y_{n}$) be the disk bounded by $\delta_{n}$ (resp. $\gamma_{n}$). Then $f^{n} : X_{n} \to Y_{n}$ is a proper map whose degree is the same as that of $f^{n} : U_{n} \to V_{n}$.
- $Y_{n}(i) = f^{i}(X_{n})$ for $0 < i \leq n$. Then $Y_{n}(i) \cap P(f) = P_{n}(i)$.
- $y_{n} = \bigcap_{i=1}^{n} Y_{n}(i)$. Then $\mathcal{Y}_{n}$ contains $P(f)$.
- Let $B_{n}$ be the largest Euclidean ball centered at $c_{0}$ and contained in $X_{n} \cap Y_{n}$.

Lemma 4.4.

\[ \bigcap_{n \in \mathcal{S}\mathcal{R}(f, C_{R})} \mathcal{Y}_{n} = P(f) . \]

Proof. When $n$ is sufficiently large, the diameter of $P_{n}(i)$ is small. But for $m > n$, $\gamma_{m}(i)$ separates $P_{n}(i)$ into two pieces, so $\gamma_{m}(i)$ passes very close to $P(f)$. Since the hyperbolic length of $\gamma_{m}(i)$ on $C - P(f)$ is bounded for infinitely many $m$, the Euclidean diameter of $Y_{n}(i)$ is also small. \qed

Thus just as the proof of the thin rigidity, we obtain the following.

Lemma 4.5. Almost every $x$ in $J(f)$ satisfies the followings:

1. The forward orbit of $x$ does not meet the postcritical set.
2. $\|(f^{k})'(x)\| \to \infty$ in the hyperbolic metric on $C - P(f)$.
3. For any $n \in \mathcal{S}\mathcal{R}(f, C_{R})$, there is a $k > 0$ with $f^{k}(x) \in \mathcal{Y}_{n}$.
4. For any $n > 0$, there is an $n \in \mathcal{S}\mathcal{R}(f, C_{R})$ such that $f^{k}(x) \notin \mathcal{Y}_{n}$.

Let

\[ \mathcal{S}\mathcal{R}(f, C_{R}, \lambda) = \{ n \in \mathcal{S}\mathcal{R}(f, C_{R}) \mid 1/\lambda < \ell(\gamma_{n}) < \lambda \} . \]

When $0 < \lim \inf \ell(\gamma_{n}) < \infty$, $\mathcal{S}\mathcal{R}(f, C_{R}, \lambda)$ is infinite for some $\lambda > 0$.

By using the collar theorem, we obtain the Euclidean diameters of $X_{n}$, $Y_{n}$ and $B_{n}$ are comparable for $n \in \mathcal{S}\mathcal{R}(f, C_{R}, \lambda)$. So let $A_{n}(z) = \frac{z-c_{0}}{\text{diam}(B_{n})}$ and then after passing a subsequence,

\[ (A_{n}(X_{n}), 0) \to (X, 0), \]
\[ (A_{n}(Y_{n}), A_{n}(f^{n}(0))) \to (Y, g(0)), \]
\[ A_{n} \circ f^{n} \circ A_{n}^{-1} \to g, \]

where $g : (X, 0) \to (Y, g(0))$ is a proper map, $0 \in X \cap Y$ and $g'(0) = 0$. 72
Lemma 4.6. For each $n \in \mathcal{SR}(f, C_R, \lambda)$, there exists a disk $Z_n \in \mathbb{C} - P(f)$ and an integer $m$, $0 < m < 2n$ such that

1. $f^m : Z_n \to Y_n(j)$ is a univalent map for some $j$ with $0 < j \leq n$ and $C_n(j) \neq \emptyset$;
2. $d(\partial X_n, \partial Z_n)$ is bounded above in terms of $\lambda$;
3. $\ell(\partial Z_n) < \lambda$;
4. $\text{area}(Z_n)$ is bounded below in terms of $\lambda$.

in the hyperbolic metric on $\mathbb{C} - P(f)$.

Proof. By the lower bound of $\gamma_n(i)$, there exist $\gamma_n(i)$ and $\gamma_n(j)$ such that $d(\gamma_n(i), \gamma_n(j))$ is bounded above in terms of $\lambda$. Furthermore, $\gamma_n(k)$ and $\partial Y_n(k)$ is uniformly close. So $d(\partial Y_n(i), \partial Y_n(j))$ is bounded above.

Considering backward images of $Y_n(i)$ and $Y_n(j)$, there is a disk $Z_n$ close to $X_n$ and maps to $Y_n(k)$ ($k = i$ or $j$) univalently by $f^m$.

Since $\text{mod}(P_n, Y_n)$ is bounded below and $\|\alpha f'(z)\|$ is not so expanding near $\partial X_n$, $\text{area}(Z_n)$ is bounded below.

Proof of Theorem 4.3. Suppose $\mu$ is an $f$-invariant line field supported on $J(f)$. Let $x$ be a point at which $\mu$ is almost continuous and satisfies the condition 1-4 of Lemma 4.5.

For each $n \in \mathcal{SR}(f, C_R, \lambda)$, let $k(n) \geq 0$ be the least integer such that $f^{k(n)+1}(x) \in Y_n$. For $k(n) \to \infty$, we consider $n$ sufficiently large so that $k(n) > 0$ (so $f^{k(n)}(x) \notin Y_n$).

Now we construct univalent maps $h_n : Y_n(j(n)) \to T_n \subset \mathbb{C}$. Let $i(n), 0 \leq i(n) \leq n$, be the number such that $Y_n(i(n) + 1)$ contains $f^{k(n)+1}(x)$.

Case I. $i(n) > 0$. Then $f^{k(n)}(x)$ is contained in a component $W_n$ of $f^{-1}(Y_n(i(n) + 1))$, which is not $Y_n(i(n))$. $W_n$ does not meet the postcritical set. Furthermore, for $n$ sufficiently large, $W_n$ contains no critical points.

So let $j(n) \geq i(n)$ be the least integer such that $C_n(j(n)) \neq \emptyset$ and define $h_n$ be the following:

$$Y_n(j(n)) \xrightarrow{f^{(n)-j(n)}} W_n \xrightarrow{f^{-k(n)}} T_n \subset \mathbb{C}.$$ where the branch of $f^{-k(n)}$ is chosen to maps $f^{k(n)}(x)$ to $x$.

Case II. $i(n) = 0$ and $f^{k(n)}(x) \notin X_n - Y_n$. Since $f^{k(n)}(x) \notin X_n$, define $h_n$ just the same as Case I.

Case III. $i(n) = 0$ and $f^{k(n)}(x) \in X_n - Y_n$. Since $\partial X_n$ is close to $\partial Y_n$, $f^{k(n)}(x)$ is close to $Z_n$. So let $\zeta_n$ be a path joining $f^{k(n)}(x)$ to $Z_n$ with length bounded above in terms of $\lambda$. 
Then by the previous lemma, there is a univalent map $f^n : Z_n \to Y_n(j(n))$. So define $h_n$ by:

$$Y_n(j(n)) \xrightarrow{f^{-m}} Z_n \xrightarrow{f^{-k(n)}} T_n \subset \mathbb{C}.$$ 

We choose the inverse branch of $f^{-k(n)}$ so that the extension to $Z_n \cap \zeta_n$ maps $f^{k(n)}(x)$ to $x$.

By the estimates for the derivative $\|(f^{k(n)})'(z)\|$ on $\partial T_n$ in terms of $\|(f^{k(n)})'(x)\|$ and $\lambda$, $\text{diam}(T_n) \to 0$ and $d(x, T_n) \leq C_1 \text{diam}(T_n)$ where $C_1$ is a constant which depends only on $\lambda$.

Let $k_n = h_n \circ A_n^{-1}$. Then $|k_n'(0)| \to 0$. Therefore,

$$\frac{|x - k_n(0)|}{|k_n'(0)|} \leq C_2 \frac{d(x, T_n) + \text{diam}(T_n)}{\text{diam}(T_n)} \leq C_3,$$

where $C_2$ and $C_3$ depend only on $\lambda$.

Thus we can apply Lemma 4.2 and deduce the contradiction. \qed

References


