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Rigidity of infinitely renormalizable polynomials of higher degree

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Abstract
The conjecture that hyperbolic rational maps are dense in the space of all rational maps of degree $d$ is one of the central problems in complex dynamics. It is known that no invariant line field conjecture implies the density of hyperbolicity (see [MS]).

In the case of quadratic polynomials, McMullen shows that a robust infinitely renormalizable quadratic polynomial carries no invariant line field on its Julia set [Mc].

In this paper, we give the extension of renormalization and the above theorem of McMullen to polynomial of any degree.

1 Notation and backgrounds

Notation. Let $f$ be a polynomial of degree $d$.

- The Fatou set $F(f)$ of $f$ is the maximal open set of $\mathbb{C}$ where $\{f^n\}$ is normal.
- The Julia set $J(f)$ of $f$ is the complement of $F(f)$.
- The filled Julia set $K(f)$ of $f$ is the set of all points in $\mathbb{C}$ whose forward orbit by $f$ does not tend to infinity. Note that $\partial K(f) = J(f)$.
- Let $C(f)$ be the set of critical points of $f$.
- The postcritical set $P(f)$ is the closure of the strict forward image of critical points by $f$:

\[ P(f) = \bigcup_{n>1} f^n(C(f)) \]

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Definition. A polynomial-like map $f : U \to V$ is a proper holomorphic map with $\overline{U} \subset V$.

The filled Julia set $K(f)$ of a polynomial-like map $f : U \to V$ is the set of all point $z \in U$ such that $f^n(z) \in U$ for all $n \geq 0$. The Julia set $J(f)$ is the boundary of $K(f)$.

Two polynomial-like map $f$ and $g$ are hybrid equivalent if there is a quasiconformal map $\phi$ from a neighborhood of $K(f)$ to a neighborhood of $K(g)$, such that $\phi \circ f = g \circ \phi$ and $\overline{\partial} \phi = 0$ on $K(f)$.

Theorem 1.1. Every polynomial-like map $f$ is hybrid equivalent to some polynomial $g$ of the same degree. Furthermore, if $K(f)$ is connected, $g$ is unique up to affine conjugacy.

See [DH, Theorem 1].

Lemma 1.1. Let $f_i : U_i \to V_i$ be polynomial-like maps of degree $d_i$ for $i = 1, 2$. Suppose $f_1 = f_2 = f$ on $U = U_1 \cap U_2$ and let $U'$ be a component of $U$ with $U' \subset f(U') = V'$. Then $f : U' \to V'$ is polynomial-like map of degree $d \leq \min(d_1, d_2)$, and

$$K(f) = K(f_1) \cap K(f_2) \cap U'.$$

Moreover, if $d = d_i$, then $K(f) = K(f_i)$.

See [Mc, Theorem 5.11].

Lemma 1.2. Let $f$ be a polynomial with connected Julia set. Let $f^n : U \to V$ be a polynomial-like restriction of degree more than 1 with connected filled Julia set $K$. Then:

1. The Julia set of $f^n : U \to V$ is contained in $J(f)$.

2. For any closed connected set $L$ contained in $K(f)$, $L \cap K$ is also connected.

See [Mc, Theorem 6.13].

Definition. A line field supported on $E \subset \mathbb{C}$ is the choice of a real line through the origin of $T_z \mathbb{C}$ at each $z \in E$. It is equivalent to take a Beltrami differential $\mu = \mu(z)dz/d\bar{z}$ supported on $E$ with $|\mu| = 1$.

We say $f$ carries an invariant line field on its Julia set if there exists a measurable Beltrami differential $\mu$ on $\mathbb{C}$ such that $f^* \mu = \mu$ and $|\mu| = 1$ on a set of positive measure contained in $J(f)$ and vanishes elsewhere.

Conjecture 1.1 (No invariant line fields). A polynomial carries no invariant line field on its Julia set.
If this conjecture is true, the following one is also true. Here the polynomial \( f \) is \textit{hyperbolic} if all critical points tend to attracting periodic cycles under iteration.

**Conjecture 1.2 (Density of hyperbolicity).** \textit{Hyperbolic maps are dense in the family of polynomial of degree \( d \).}

See [MS].

## 2 Renormalization

In this section, we give the definition of renormalization and describe some basic properties.

**Definition.** \( f^n \) is called \textit{renormalizable} if there exist open disks \( U, V \subset \mathbb{C} \) satisfying the followings:

1. \( U \cap C(f) \neq \phi \).

2. \( f^n : U \to V \) is a polynomial-like map with connected filled Julia set.

3. For each \( c \in C(f) \), there is at most one \( i, 0 < i \leq n \), such that \( c \in f^i(U) \).

4. \( n > 1 \) or \( U \not\supset C(f) \).

A \textit{renormalization} is a polynomial-like restriction \( f^n : U \to V \) as above.

**Notation.** Let \( f^n : U \to V \) be a renormalization.

- The filled Julia set of a renormalization \( f^n : U \to V \) is denoted by \( K_n(U) \) and the postcritical set by \( P_n(U) \).

- For \( i = 1, \ldots, n \), the \textit{i-th small filled Julia set} is denoted by \( K_n(U, i) = f^i(K_n(U)) \).

- The \textit{i-th small postcritical set} is denoted by \( P_n(U, i) = K_n(U, i) \cap P(f) \).

- \( C_n(U, i) = K_n(U, i) \cap C(f) \). By definition, \( C_n(U, n) \) is nonempty and \( C_n(U, i) \) is empty with at most \( d - 1 \) exceptions.

- \( \mathcal{K}_n(U) = \bigcup_{i=1}^n K_n(U, i) \) is the union of the small filled Julia sets.

- \( \mathcal{C}_n(U) = \bigcup_{i=1}^n C_n(U, i) \) is the set of critical points appear in the renormalization \( f^n : U \to V \).

- Let \( V_n(U, i) = f^i(U) \) and \( U_n(U, i) \) be the component of \( f^{i-n}(U) \) contained in \( V_n(U, i) \). Then \( f^n : U_n(U, i) \to V_n(U, i) \) is polynomial-like map of the same degree as \( f^n : U \to V \).
Now, when it is clear which $U$ we consider, we will simply write $K_n(i)$ instead of $K_n(U,i)$, and so on.

In this paper, we fix a critical point $c_0 \in C(f)$ and consider only renormalizations about $c_0$, i.e. $C_n(U) = C_n(U,n)$ contains $c_0$.

The next proposition implies that two renormalizations are essentially the same if their period and critical points are equal.

**Proposition 2.1.** Let $f^n : U^k \to V^k$ be renormalizations for $k = 1, 2$.
If for any $i$, $0 \leq i < n$, $C_n(U^1, i) = C_n(U^2, i)$, their filled Julia sets are equal.

**Proof.** Let $K^k$ be the filled Julia set of $f^n : U^k \to V^k$. By Lemma 1.2, $K = K^1 \cap K^2$ is connected.

Let $U$ be the component of $U^1 \cap U^2$ containing $K$. Let $V = f^n(U)$. Since $V$ contains $f(K) = K$, $V$ contains $U$. By Lemma 1.1, $f^n : U \to V$ is polynomial-like with filled Julia set $K$. Since critical points of these three maps are equal, we have $K = K^1 \cap K^2$. \qed

**Proposition 2.2.** Let $f^a : U_a \to V_a$ and $f^b : U_b \to V_b$ be renormalizations about $c_0$.
Then there exists a renormalization $f^c : U \to V$ with filled Julia set $K_c = K_a \cap K_b$ where $c$ is the least common multiple of $a$ and $b$.

**Proof.** By Lemma 1.2, $K = K_a \cap K_b$ is connected.

Let
\[
\tilde{U}_a = \{ z \in U_a \mid f^{ja}(z) \in U_a \text{ for } j = 1, \ldots, \frac{c}{a} - 1 \} \\
\tilde{U}_b = \{ z \in U_b \mid f^{jb}(z) \in U_b \text{ for } j = 1, \ldots, \frac{c}{b} - 1 \}.
\]

Then $f^c : \tilde{U}_a \to V_a$ and $f^c : \tilde{U}_b \to V_b$ are polynomial-like. Let $U_c$ be a component of $\tilde{U}_a \cap \tilde{U}_b$ which contains $K$. Then by Lemma 1.1, $f^c : U_c \to f^c(U_c)$ is polynomial-like map with filled Julia set $K$.

Suppose $c \in C_c(i)$. then $c \in C_c(j)$ is equivalent to $j \equiv i \pmod{a}$ and $j \equiv i \pmod{b}$, which means $j = i$. Therefore, $f^c : U_c \to V_c$ is a renormalization with filled Julia set $K_c = K$. \qed

Define the *intersecting set* of a renormalization $f^n : U \to V$ by
\[
I_n(U) = K_n(U) \cap \left( \bigcup_{i=1}^{n-1} K_n(U,i) \right).
\]

We say a renormalization is intersecting if $I_n(U) \neq \emptyset$.

**Proposition 2.3.** If a renormalization $f^n : U \to V$ is intersecting, then $I_n(U)$ consists of only one point which is a repelling fixed point of $f^n$. 

Proof. Suppose $E = K_n(U) \cap K_n(U, i) \neq \emptyset$ for some $0 < i < n$. By Lemma 1.2, $E$ is connected.

Let $U$ be the component of $U \cap U(i)$ containing $E$. By Lemma 1.1, $f^n : U \to f^n(U)$ is a polynomial-like map of degree 1. By the Schwarz lemma, $E$ consists of a single repelling fixed point $x$ of $f^n$.

Suppose $K_n(U) \cap K_n(U, j) = \{y\}$ with $y \neq x$. Then there is a sequence $\{i_0, i_1, \ldots, i_K\}$ such that $K_n(U, i_k) \cap K_n(U, i_{k+1})$ is nonempty and $K_n(U, i_k) \cap K_n(U, i_{k+1}) \cap K_n(U, i_{k+2})$ is empty (where $K + 1$, $K + 2$ is interpreted as 0, 1, respectively).

Let

$$L = K_n(U, i_1) \cap \ldots K_n(U, i_K).$$

Then $L$ is a closed connected set in $K(f)$. But $L \cap K_n(U)$ consists of two points and it contradicts Lemma 1.2.

Since a repelling fixed point separates filled Julia set into a finite number of components, components of $K_n(U) - I_n(U)$ are finite. We say a renormalization is simple if $K_n(U) - I_n(U)$ is connected, and crossed if it is disconnected.

Theorem 2.1. For $p > 0$, there are finitely many $n > 0$ such that there exists a renormalization $f^n : U_n \to V_n$ such that $K_n(U)$ contains a periodic point of period $p$.

Proof. Let $x$ be a periodic point of period $p$. Assume the filled Julia set of a renormalization $f^n : U \to V$ with $p < n$ contains $x$. Since $x$ is a repelling fixed point of $f^n$ (by Proposition 2.3), $p$ divides $n$ and the number $\rho$ of the components of $K_n(U_n) - \{x\}$ is finite.

Let $E$ be the component of $K_n(U)$ which contains $x$. $E - \{x\}$ has exactly $\rho n/p$ components. Let $q$ be the number of the components of $K(f) - \{x\}$. Since $x$ is a repelling periodic point of $f$, $q < \infty$.

Suppose a component $A$ of $K(f) - \{x\}$ contains two components $B_1, B_2$ of $E - \{x\}$. Then we can take a path in $A - (B_1 \cup B_2)$ from $x$ to some point in $B_1$. It contradicts Lemma 1.2.

Therefore each component of $K(f) - \{x\}$ can contain at most one component of $E - \{x\}$. So $q \geq \rho n/p$, it concludes $n \leq pq$.

There are finitely many periodic points of period $p$, the theorem follows.

Proposition 2.4. Let $f^a : U_a \to V_a$ and $f^b : U_b \to V_b$ be renormalizations about $c_0$. Suppose that $f^b : U_b \to V_b$ is simple. Then either $a$ divides $b$ or $b$ divides $a$.

Proof. Let $c$ be the greatest common divisor of $a$ and $b$. If $c = a$ or $c = b$, the proposition follows. So suppose $c < a, b$. 
Since $K_a \cap K_b$ is nonempty (it contains $c_0$), $f^i(K_a) \cap f^i(K_b)$ is nonempty for any $i > 0$. Therefore $K_a(c) \cap K_b(c)$, $K_a(c) \cap K_b$ and $K_a \cap K_b(c)$ are all nonempty. Therefore $L = K_b \cup K_a(c) \cup K_b(c)$ is connected.

By Lemma 1.2, $K_a \cap L$ is connected. Since $K_a \cap K_a(c)$ is at most one point and $L$ is a closed connected set, $K_a \cap (K_b \cup K_b(c))$ is connected. So $K_a \cap K_b \cap K_b(c)$ is nonempty. By Proposition 2.3, $K_b \cap K_b(c) = \{x\}$ where $x$ is a repelling fixed point of $f^b$, so $K_b \ni x$. Since $f^b : U_b \to V_b$ is simple, $x$ does not disconnect $K_b$.

By Proposition 2.2, there exists a renormalization $f_{ab/c} : U \to V$ with Julia set $K_{ab/c} = K_a \cap K_b$. But $K_{ab/c}$ cannot contain $x$ because $K_b \setminus \{x\}$ is connected and $ab/c > b$ (see the proof of Theorem 2.1), it is a contradiction.

Example. Let $f(z) = z^3 - \frac{3}{4}z - \sqrt{\frac{5}{4}}$. Then $C(f) = \{\pm \frac{1}{2}\}$ and $\pm \frac{1}{2}$ are periodic of period 2. Let $W_\pm$ be the Fatou component which contains $\pm \frac{1}{2}$. They are superattracting basin of period 2.

Every renormalization $f^n : U \to V$ must satisfy $U \supset W_-$ or $W_+$. So $n \leq 2$ and by symmetry, we will consider only the case $U \supset W_-$.

**Type I.** Let $K$ be the connected component of the closure of $\bigcup_{n>0} f^{-n}(W_-)$ which contains $W_-$ and let $U_1$ be a small neighborhood of $K$.

Then $f : U_1 \to f(U_1)$ is a renormalization with filled Julia set $K(1, U) = K_1$ which is hybrid equivalent to $z \mapsto z^2 - 1$.

**Type II.** Let $U_2$ be a small neighborhood of $W_-$. Then $f^2 : U_2 \to f^2(U_2)$ is a renormalization with filled Julia set $K(2, U_2) = \overline{W_-}$, which is hybrid equivalent to $z \mapsto z^2$.

**Type III.** Let $K'_2$ be the connected component of $\bigcup_{n>0} f^{-2n}(W_- \cup W_+)$ which contains $W_-$ and let $U'_2$ be a small neighborhood of $K'_2$.

Then $f^2 : U'_2 \to f^2(U'_2)$ is a renormalization with filled Julia set $K'_2$, which is hybrid equivalent to $z \mapsto z^3 - \frac{3}{\sqrt{2}}z$.

**Type IV.** Let $K''_2$ be the connected component of $\bigcup_{n>0} f^{-2n}(W_- \cup f(W_+))$ which contains $W_-$ and let $U''_2$ be a small neighborhood of $K''_2$.

Then $f^2 : U''_2 \to f^2(U''_2)$ is a renormalization with filled Julia set $K''_2$ and of degree 4.

Similarly, consider $\bigcup_{n>0} f^{-2n}(W_- \cup f(W_-) \cup W_+)$ and then we can construct a polynomial-like map $f^2 : U \to V$ of degree 6. But it is not a renormalization because $-\frac{1}{2}$ is contained in both $U$ and $f(U)$.

## 3 Infinite renormalization

For a subset $C_R \subset C(f)$, let $\mathcal{R}(f, C_R)$ be the set of all $n > 0$ such that there exists a renormalization $f^n : U_n \to V_n$ about $c_0$ with $C_n(U_n) = C_R$. Let $\mathcal{SR}(f, C_R)$ be the set of such $n \in \mathcal{R}(n, C_R)$ that $f^n : U_n \to V_n$ is simple.
Figure 1: The filled Julia set of $z \mapsto z^3 - \frac{3}{4} z - \frac{\sqrt{7}}{4}$.

Figure 2: Five types of Polynomial-like restrictions.
Proposition 3.1. Let $n_1, n_2 \in \mathcal{S}\mathcal{R}(f, C_R)$. If $n_1 < n_2$, then $n_1$ divides $n_2$ and $K_{n_1}(U_1) \supset K_{n_2}(U_2)$.

Proof. By Proposition 2.4, $n_1$ divides $n_2$.

Assume $K_{n_1}(U_1) \not\supset K_{n_2}(U_2)$. By Proposition 2.2, there exists a renormalization $f^{n_2} : U'_{n_2} \to V'_{n_2}$ with filled Julia set $K_{n_2}(U'_{n_2}) = K_{n_1}(U_{n_1}) \cap K_{n_2}(U_{n_2})$.

For simplicity, we write $K_{n_1} = K_{n_1}(U_{n_1})$, $K_{n_2} = K_{n_2}(U_{n_2})$ and $K'_{n_2} = K_{n_2}(U'_{n_2})$.

If $C_{n_2}(U'_{n_2}) = C_R$, then $K'_{n_2} = K_{n_2}$. Therefore there exists a critical point $c_1 \in C_R - C_{n_2}(U'_{n_2})$. Let $i_k$ be a number which satisfies $K_{n_1}(i_k) \ni c_1$. Then $i_1 \neq i_2$ (mod $n_1$). So there exists $i_0$ such that $K_{n_1}(i_0)$ intersects $K_{n_2}$.

Therefore let a closed connected subset $L$ of $K(f)$ as the following:

$$L = K_{n_1}(i_0) \cup K_{n_2}(i_0) \cup K_{n_1}(2i_0) \cup \cdots \cup K_{n_1}.$$

Then $L \cap K_{n_2}$ is disconnected and it contradicts Lemma 1.2. $\square$

Proposition 3.2. If $f$ can be infinitely renormalizable, $f$ has infinitely many simple renormalizations.

More precisely, if $\mathcal{R}(n, C_R)$ is infinite for some $C_R \subset C(f)$, then there exists some $C, C_R \subset C \subset C(f)$, such that $\mathcal{S}\mathcal{R}(f, C)$ is infinite.

Proof. For $n \in \mathcal{R}(n, C_R)$, Let $\kappa_n$ be the number of components of $\mathcal{K}_n$. Since $\kappa_n$ is equal to the minimum of the period of periodic point of $f$ contained in $K_n$, $\kappa_n \to \infty$ by Theorem 2.1.

Now we show $f^{\kappa_n}$ is simply renormalizable. For sufficiently large $n$, choose a repelling periodic point $x$ of $f$ of period less than $\kappa_n$. Then $x \not\in \mathcal{K}_n$. We construct the Yoccoz puzzle from the rays landing at $x$ and some equipotential curve.

For any depth $r \geq 0$, the piece $P_r(c_0)$ containing $c_0$ contains the component $E$ of $\mathcal{K}_n$ containing $c_0$. Thus the tableau $P_r(f^k(c_0))$ for $c_0$ is periodic of period $p$ with $p|\kappa_n$, i.e. for any $r > 0$, $P_r(f^p(c_0)) = P_r(f^p(c_0))$.

Then by slightly thickening the pieces, we can obtain a simple renormalization $f^p : U_p \to V_p$ with $K_p \supset E$ (more precisely, see [Mi2, Lemma 2]).

If $p = \kappa_n$, we are done.

Otherwise, let $g$ be the polynomial hybrid equivalent to $f^p : U_p \to V_p$. There exists a renormalization $g^{n/p} : \tilde{U}_{n/p} \to \tilde{V}_{n/p}$ corresponds to $f^n : U_n \to V_n$.

Now apply the argument above to $g$ and the renormalization $g^{n/p} : \tilde{U}_{n/p} \to \tilde{V}_{n/p}$ and eventually we obtain a simple renormalization of $f^{\kappa_n}$. $\square$

Now we assume that $f$ is infinitely renormalizable. By the proposition above, $\#\mathcal{S}\mathcal{R}(f, C_R)$ is infinite for some $C_R \subset C(f)$.

Furthermore, suppose $f(C_R) = f(C(f))$, i.e. for any $c' \in C(f) - C_R$, there exists some $c \in C_R$ such that $f(c) = f(c')$. 

Remark. The above condition is satisfied for a polynomial which is hybrid equivalent to $f^n : U_n \to V_n$ for $n \in \mathcal{SR}(f, CR)$.

So this assumption is to consider the polynomial hybrid equivalent to some renormalization instead of the original polynomial.

**Definition.** Let $f$ as above. For each $n \in \mathcal{SR}(f, CR)$, let $\delta_n(i)$ be a closed curve which separates $K_n(i)$ from $P(f) - P_n(i)$ (in our case, such a curve exists and its homotopy class is uniquely determined). Let $\gamma_n(i)$ is the hyperbolic geodesic on $\mathbb{C} - P(f)$ which is homotopic to $\delta_n(i)$ on $\mathbb{C} - P(f)$ and let $\gamma_n = \gamma_n(n)$.

We say $\mathcal{SR}(f, CR)$ is robust if

$$\liminf_{n \to \infty} \ell(\gamma_n) < \infty,$$

where $\ell(\cdot)$ denotes the hyperbolic length on $\mathbb{C} - P(f)$.

Let $\Sigma = \text{proj lim } \mathbb{Z}/n$ and define $\sigma : \Sigma \to \Sigma$ by:

$$\sigma((i_n)_{n \in \mathcal{SR}(f, CR)}) = (i_n + 1).$$

**Theorem 3.1.** Let $f$ as above. When $\mathcal{SR}(f, CR)$ is robust, then:

1. The postcritical set $P(f)$ is a Cantor set of measure zero.
2. $\limsup_{n \in \mathcal{SR}(f, CR), 0 < i \leq n} \text{diam } P_n(i) \to 0$.
3. $f : P(f) \to P(f)$ is topologically conjugate to $\sigma : \Sigma \to \Sigma$. Especially, $f|_{P(f)}$ is a homeomorphism.

**Proof.** By the hyperbolic geometry, the geodesics $\gamma_n(i)$ ($n \in \mathcal{SR}(f, CR)$, $0 < i \leq n$) are simple and mutually disjoint, and their length are comparable with $\ell(\gamma_n)$.

Thus by the collar theorem, there is a standard collar $A_n(i)$ about $\gamma_n(i)$ in $\mathbb{C} - P(f)$ such that they are mutually disjoint and $\text{mod}(A_n(i))$ is a decreasing function of $\ell(\gamma_n(i))$. Note that $A_n(i)$ separates $P_n(i)$ from the rest of the postcritical set.

Let $E_n = \bigcup_{i=1}^{n} A_n(i)$ and $F_n$ be the union of the bounded components of $\mathbb{C} - E_n$.

Then $F_n$ contains $P(f)$ and each component of $F_n$ meets $P(f)$.

For any sequence $\{A_n(i_n)\}_{n \in \mathcal{SR}(f, CR)}$ of nested annuli,

$$\sum_{n \in \mathcal{SR}(f, CR)} \text{mod } A_n(i_n) = \infty,$$

because $\liminf \ell(\gamma_n) < \infty$. 


Therefore $F = \bigcap_{n \in \mathcal{SR}(f, C_R)} F_n$ is a Cantor set of measure zero. Since $F$ contains $P(f)$ and each component of $F_n$ contains $P_n(i)$ for some $i$, $F$ is equal to $P(f)$, so the postcritical set is measure zero and diameter of $P_n(i)$ tends to zero.

For $n \in \mathcal{SR}(f, C_R)$, we define $\phi_n : P(f) \to \mathbb{Z}/n$ by $\phi(z) = i \pmod{n}$ when $z \in P_n(i)$. Then $\phi(f(z)) = \phi(z) + 1 \pmod{n}$.

Therefore, define $\phi : P(f) \to \lim_{n \in \mathbb{Z}/n} \mathbb{Z}/n$ by $\phi(z) = (\phi_n(z))_{n \in \mathbb{Z}/n}$ for some $i$. $F$ is equal to $P(f)$, so the postcritical set is measure zero and diameter of $P_n(i)$ tends to zero.

For $\lambda \in \mathcal{SR}(f, C_R)$, we define $\phi_n : P(f) \to \mathbb{Z}/n$ by $\phi(z) = i \pmod{n}$ when $z \in P_n(i)$. Then $\phi(f(z)) = \phi(z) + 1 \pmod{n}$.

Therefore, define $\phi : P(f) \to \lim_{n \in \mathbb{Z}/n} \mathbb{Z}/n$ by $\phi(z) = (\phi_n(z))_{n \in \mathbb{Z}/n}$ for some $i$. $F$ is equal to $P(f)$, so the postcritical set is measure zero and diameter of $P_n(i)$ tends to zero.

Corollary 3.1. Let $f$ as above. Suppose $\mathcal{SR}(f, C_R)$ is robust. Then for sufficiently large $n \in \mathcal{SR}(f, C_R)$ and any $i$, $0 < i \leq n$, \#$C_n(i) \leq 1$.

Proof. Suppose \# \left( \bigcap_{n \in \mathcal{SR}(f, C_R)} C_n(n) \right) > 1.

By Theorem 3.1, $\bigcap P_n(1)$ consists of only one point $x$. Therefore, $f(C_n(n)) = \{x\}$. But it is impossible because there is no other critical point in $U_n$ for sufficiently large $n$.

4 Robust rigidity

In this section, we prove the following theorem:

Theorem 4.1 (Robust rigidity). Let $f$ as above. If $\mathcal{SR}(f, C_R)$ is robust, then $f$ carries no invariant line field on its Julia set.

The proof depends on the following two lemmas.

Lemma 4.1. Let $f_n : (U_n, u_n) \to (V_n, v_n)$ be a sequence of holomorphic maps between disks and let $\mu_n$ be a sequence of $f_n$-invariant line field on $V_n$. Suppose $f_n$ converge to $f : (U, u) \to (V, v)$ in the Carathéodory topology and $\mu_n$ converge in measure to $\mu$ on $V$. Then $\mu$ is $f$-invariant.

See [Mc, Theorem 5.14].

Lemma 4.2. Let $\mu$ be a measurable line field on $\mathbb{C}$. Assume $\mu$ is almost continuous at $x$ and $|\mu(x)| = 1$. Let $(V_n, v_n) \to (V, v)$ be a convergent sequence of disks, and let $h_n : V_n \to \mathbb{C}$ be a sequence of univalent maps with $h_n'(v_n) \to 0$.

Suppose \[ \sup \frac{|x - h_n(v_n)|}{h_n'(v_n)} < \infty. \]

Then there exists a subsequence such that $h_n^*(\mu)$ converges in measure to a univalent line field on $V$. 

Now we give the summary of the proof of the theorem. We divide the proof into two cases: whether \( \liminf \ell(\gamma_n) \) is zero or not. But outline of these two proof are very similar. We assume there exists a measurable invariant line field \( \mu \) supported on \( J(f) \) and induce contradiction.

First, we take a point \( x \in J(f) \) where \( \mu \) is almost continuous, such that \( \| (f^k)'(x) \| \to \infty \) with respect to hyperbolic metric on \( C - P(f) \), and such that \( f^k(x) \) does not land in but tends to \( P(f) \).

Next we construct some critically compact proper map \( f^n : X_n \to Y_n \) from \( f^n : U_n \to V_n \). By assumption, \( f^k(x) \) eventually land in \( Y_n \). If we take disks \( X_n, Y_n \) properly, we can take a univalent inverse branch \( h_n \) of \( f^{-k} \) from \( Y_n \) to the region near \( x \). Note that \( h_n^*(\mu) = \mu \) is \( f^n \)-invariant line field on \( Y_n \).

By properly scaling \( f^n : X_n \to Y_n \) and taking a subsequence, they converge to a proper map \( g : U \to V \). Furthermore, by Lemma 4.2 and Lemma 4.1, \( g \) must have an invariant univalent line field \( \nu \) on \( V \).

But \( g \) have a critical point \( c \in U \cap V \), then, by invariance, \( \nu(c) = 0 \), that is a contradiction.

### 4.1 Thin rigidity

**Definition.** A renormalization \( f^n : U_n \to V_n \) is *unbranched* if

\[
V_n \cap P(f) = P_n.
\]

Let \( f^n : U_n \to V_n \) be an unbranched renormalization. Let \( W \) be a component of \( f^{-1}(V_n(i+1)) \) which is not \( V_n(i) \). Then any inverse branch of \( f^{-k} \) on \( W \) is univalent because \( W \) is disjoint from the postcritical set.

**Lemma 4.3.** There exists some \( M > 0 \) such that if \( \ell(\gamma_n) < M \), we can choose \( U_n \) and \( V_n \) such that \( f^n : U_n \to V_n \) is unbranched renormalization and

\[
\text{mod}(U_n, V_n) > m(\ell(\gamma_n)) > 0
\]

where \( m(\ell) \to \infty \) as \( \ell \to 0 \).

**Proof.** Let \( A_n \) be the standard collar about \( \gamma_n \) with respect to the hyperbolic metric on \( C - P(f) \). Let \( B_n \) be the component of \( f^{-n}(A_n) \) which is the same homotopy class as \( \gamma_n \). Let \( D_n \) (resp. \( E_n \)) be the union of \( B_n \) (resp. \( A_n \)) and the bounded component of the complement. \( f^n : D_n \to E_n \) is a critically compact proper map with postcritical set \( P_n \).

When \( \ell(\gamma_n) \) is sufficiently small, \( \text{mod}(P_n, E_n) \geq \text{mod}(A_n) \) is sufficiently large. Then we can choose \( U_n \subset D_n \) and \( V_n \subset E_n \) such that \( f^n : U_n \to V_n \) is a renormalization and \( \text{mod}(U_n, V_n) \) is bounded below in terms of \( \text{mod}(P_n, E_n) \).

The modulus of collar \( A_n \) depends only on \( \ell(\gamma_n) \) and tends to infinity as \( \ell(\gamma_n) \) tends to zero. Since \( \text{mod}(P_n, E_n) \geq \text{mod}(A_n) \), we are done. \( \square \)
Theorem 4.2. Let $f$ as above. Suppose for infinitely many $n \in \mathcal{SR}(f, C_R)$ there is a simple unbranched renormalization $f^n : U_n \to V_n$ with $\text{mod}(U_n, V_n) > m$ for a constant $m > 0$.

Then $f$ carries no invariant line field on its Julia set.

By the previous lemma, the following corollary is trivial.

Corollary 4.1 (Thin rigidity). There is $L > 0$ such that if

$$\liminf_{\mathcal{SR}(f, C_n)} \ell(\gamma_n) < L,$$

then $f$ carries no invariant line field on its Julia set.

Proof of Theorem 4.2. Let $\mathcal{USR}(f, C_R, m)$ be a set of $n \in \mathcal{SR}(f, C_R)$ such that there is an unbranched simple renormalization $f^n : U_n \to V_n$ with $\text{mod}(U_n, V_n) > m$.

For $n \in \mathcal{USR}(f, C_R, m)$, there is an annulus of definite modulus separating $J_n(i)$ from $P(f) - P_n(i)$. So $\mathcal{SR}(f, C_R)$ is robust and

$$\bigcap_{n \in \mathcal{SR}(f, C_n)} J_n = P(f).$$

Therefore, by the fact that a forward orbit of almost every point in $J(f)$ tends to $P(f)$, almost every $x$ in $J(f)$ satisfies the followings:

1. The forward orbit of $x$ does not meet the postcritical set.
2. $\|f^k(x)\| \to \infty$ in the hyperbolic metric on $\mathbb{C} - P(f)$.
3. For any $n \in \mathcal{SR}(f, C_R)$, there is a $k > 0$ with $f^k(x) \in J_n$.
4. For any $k > 0$, there is an $n \in \mathcal{SR}(f, C_R)$ such that $f^k(x) \notin J_n$.

(Note that the condition 2 is satisfied every point which satisfies the condition 1.)

Suppose that $f$ carries an invariant line field $\mu$ on $J(f)$. Let $x$ be a point in $J(f)$ at which $\mu$ is almost continuous, $|\mu(x)| = 1$ and satisfies the above condition 1-4. For each $n \in \mathcal{SR}(f, C_R)$, let $k(n) \geq 0$ be the least integer such that $f^{k(n)}(x) \in J_n$. By the condition 3, such $k(n)$ exists and tends to infinity by the condition 4. Now $f^{k(n)+1}(x)$ is contained in $J_n(i(n) + 1)$ for some $0 \leq i(n) < n$.

For $n$ sufficiently large, $k(n) > 0$ and $f^{k(n)}(x) \notin J_n$. So $f^{k(n)}(x)$ is contained in some component $W_n$ of $f^{-1}(V_n(i(n) + 1))$ which is not $V_n(i(n))$. $W_n$ is disjoint from the postcritical set. Furthermore, $W_n$ contains no critical point for sufficiently large $n$ (actually, it is true if $k(n) > k(n_0)$ where $n_0 = \min(\mathcal{USR}(f, C_R, m))$).
Then univalent univalent with map point the defined diam on where nonempty, condition maps. Since then exists an annulus of definite modulus in $C - P(f)$ enclosing it, the diameter of $f^{k(n)}(J^*_n) (= f^{-1}(J_n(i(n) + 1)) \cap W_n)$ is bounded with respect to the hyperbolic metric on $C - P(f)$. Therefore, by the condition 2, the diameter of $J^*_n$ in the hyperbolic metric on $C - P(f)$ tends to zero.

Let $c \in C_R$ be a critical point such that for infinitely many $n \in USR(f, C_R, m)$, $C_n(j(n))$ contains $c$. By taking a subsequence and replacing $f^n : U_n \to V_n$ by $f^n : U_n(j(n)) \to V_n(j(n))$, we may assume $c = c_0$ and $j(n) = n$, so $h_n$ is defined on $V_n$. (Note that $\text{mod}(U_n(j(n)), V_n(j(n))) \geq \frac{1}{d_R} \text{mod}(U_n, V_n) > \frac{m}{d_R}$, where $d_R$ is the degree of renormalization $f^n : U_n \to V_n$. Thus we should replace $m$ by $\frac{m}{d_R}$.)

Let

$$A_n(z) = \frac{z - c_0}{\text{diam}(J_n)},$$

$$g_n = A_n \circ f^n \circ A^{-1}_n,$$

$$y_n = A_n(h^{-1}_n(x)).$$

Then

$$g_n : (A_n(U_n), 0) \to (A_n(V_n), A_n(f^n(c_0)))$$

is a polynomial-like map with $\text{diam}(J(g_n)) = 1$ and $\text{mod}(A_n(U_n), A_n(V_n)) > m$.

Thus, by taking a subsequence, $g_n$ converges to some polynomial-like map (or polynomial) $g : (U, 0) \to (V, g(0))$ with $\text{mod}(U, V) > m$ in the Carathéodory topology (see [Mc, Theorem 5.8]).

Let $k_n = h_n \circ A^{-1}_n : A_n(V_n) \xrightarrow{A^{-1}_n} V_n \xrightarrow{h_n} C$ and $\nu_n = k_n^*(\mu)$. Then $\nu_n$ is $g_n$-invariant line field on $A_n(V_n)$ because $\mu = h_n^*(\mu)$ is $f$-invariant. Since $\text{diam}(J(g_n)) = 1$ and $\text{diam}(J^*_n) \to 0$, $k_n^*(y_n) \to 0$.

Now we take a further subsequence of $n$ so that $(A_n(V_n), y_n) \to (V, y)$. Then by Lemma 4.2, after passing a further subsequence, $\nu_n$ converges to an univalent $g$-invariant line field $\nu$ on $V$.

For $f^n : U_n \to V_n$ have connected Julia set, so does $g$. Thus the critical point and critical value lie in $V$. But it contradicts the fact that $g$ has a univalent invariant line field $\nu$.

\[ \square \]

4.2 Thick rigidity

Theorem 4.3 (Thick rigidity). Let $f$ as above. Suppose

$$0 < \lim\inf_{n \in SR(f, C_R)} \ell(\gamma_n) < \infty,$$

Then $f$ carries no invariant line field on its Julia set.
Notation. For \( n \in \mathcal{SR}(f, C_R) \),

- Let \( \delta_n \) be the component of \( f^{-n}(\gamma_n) \) which is homotopic to \( \gamma_n \) on \( \mathbb{C} - P(f) \).
- Let \( X_n \) (resp. \( Y_n \)) be the disk bounded by \( \delta_n \) (resp. \( \gamma_n \)). Then \( f^n : X_n \to Y_n \) is a proper map whose degree is the same as that of \( f^n : U_n \to V_n \).
- \( Y_n(i) = f^i(X_n) \) for \( 0 < i \leq n \). Then \( Y_n(i) \cap P(f) = P_n(i) \).
- \( \mathcal{Y}_n = \bigcap_{i=1}^{n} Y_n(i) \). Then \( \mathcal{Y}_n \) contains \( P(f) \).
- Let \( B_n \) be the largest Euclidean ball centered at \( c_0 \) and contained in \( X_n \cap Y_n \).

Lemma 4.4.

\[
\bigcap_{n \in \mathcal{SR}(f, C_R)} \mathcal{Y}_n = P(f).
\]

Proof. When \( n \) is sufficiently large, the diameter of \( P_n(i) \) is small. But for \( m > n \), \( \gamma_m(i) \) separates \( P_n(i) \) into two pieces, so \( \gamma_m(i) \) passes very close to \( P(f) \). Since the hyperbolic length of \( \gamma_m(i) \) on \( \mathbb{C} - P(f) \) is bounded for infinitely many \( m \), the Euclidean diameter of \( Y_n(i) \) is also small. \( \square \)

Thus just as the proof of the thin rigidity, we obtain the following.

Lemma 4.5. Almost every \( x \) in \( J(f) \) satisfies the followings:

1. The forward orbit of \( x \) does not meet the postcritical set.
2. \( \| (f^k)'(x) \| \to \infty \) in the hyperbolic metric on \( \mathbb{C} - P(f) \).
3. For any \( n \in \mathcal{SR}(f, C_R) \), there is a \( k > 0 \) with \( f^k(x) \in \mathcal{Y}_n \).
4. For any \( k > 0 \), there is an \( n \in \mathcal{SR}(f, C_R) \) such that \( f^k(x) \not\in \mathcal{Y}_n \).

Let

\[
\mathcal{SR}(f, C_R, \lambda) = \{ n \in \mathcal{SR}(f, C_R) | 1/\lambda < \ell(\gamma_n) < \lambda \}.
\]

When \( 0 < \liminf \ell(\gamma_n) < \infty \), \( \mathcal{SR}(f, C_R, \lambda) \) is infinite for some \( \lambda > 0 \).

By using the collar theorem, we obtain the Euclidean diameters of \( X_n, Y_n \) and \( B_n \) are comparable for \( n \in \mathcal{SR}(f, C_R, \lambda) \). So let \( A_n(z) = \frac{z-c_0}{\text{diam}(B_n)} \) and then after passing a subsequence,

\[
(A_n(X_n), 0) \to (X, 0),
\]

\[
(A_n(Y_n), A_n(f^n(0))) \to (Y, g(0)),
\]

\[
A_n \circ f^n \circ A_n^{-1} \to g,
\]

where \( g : (X, 0) \to (Y, g(0)) \) is a proper map, \( 0 \in X \cap Y \) and \( g'(0) = 0 \).
Lemma 4.6. For each \( n \in \mathcal{SR}(f, C_R, \lambda) \), there exists a disk \( Z_n \subset \mathbb{C} - P(f) \) and an integer \( m, 0 < m < 2n \) such that

1. \( f^m : Z_n \to Y_n(j) \) is a univalent map for some \( j \) with \( 0 < j \leq n \) and \( C_n(j) \neq \emptyset \);
2. \( d(\partial X_n, \partial Z_n) \) is bounded above in terms of \( \lambda \);
3. \( \ell(\partial Z_n) < \lambda \);
4. \( \text{area}(Z_n) \) is bounded below in terms of \( \lambda \).

in the hyperbolic metric on \( \mathbb{C} - P(f) \).

Proof. By the lower bound of \( \gamma_n(i) \), there exist \( \gamma_n(i) \) and \( \gamma_n(j) \) such that \( d(\gamma_n(i), \gamma_n(j)) \) is bounded above in terms of \( \lambda \). Furthermore, \( \gamma_n(k) \) and \( \partial Y_n(k) \) is uniformly close. So \( d(\partial Y_n(i), \partial Y_n(j)) \) is bounded above.

Considering backward images of \( Y_n(i) \) and \( Y_n(j) \), there is a disk \( Z_n \) close to \( X_n \) and maps to \( Y_n(k) \) \((k = i \text{ or } j)\) univalently by \( f^m \).

Since \( \text{mod}(P_n, Y_n) \) is bounded below and \( \| (f^n)'(z) \| \) is not so expanding near \( \partial X_n \), \( \text{area}(Z_n) \) is bounded below.

Proof of Theorem 4.3. Suppose \( \mu \) is an \( f \)-invariant line field supported on \( J(f) \). Let \( x \) be a point at which \( \mu \) is almost continuous and satisfies the condition 1-4 of Lemma 4.5.

For each \( n \in \mathcal{SR}(f, C_R, \lambda) \), let \( k(n) \geq 0 \) be the least integer such that \( f^{k(n)+1}(x) \in Y_n \). For \( k(n) \to \infty \), we consider \( n \) sufficiently large so that \( k(n) > 0 \) (so \( f^{k(n)}(x) \notin Y_n \)).

Now we construct univalent maps \( h_n : Y_n(j(n)) \to T_n \subset \mathbb{C} \). Let \( i(n), 0 \leq i(n) < n \), be the number such that \( Y_n(i(n) + 1) \) contains \( f^{k(n)+1}(x) \).

Case I. \( i(n) > 0 \). Then \( f^{k(n)}(x) \) is contained in a component \( W_n \) of \( f^{-1}(Y_n(i(n) + 1)) \), which is not \( Y_n(i(n)) \). \( W_n \) does not meet the postcritical set. Furthermore, for \( n \) sufficiently large, \( W_n \) contains no critical points.

So let \( j(n) \geq i(n) \) be the least integer such that \( C_n(j(n)) \neq \emptyset \) and define \( h_n \) be the following:

\[
Y_n(j(n)) \xrightarrow{f^{i(n)-j(n)}} W_n \xrightarrow{f^{-k(n)}} T_n \subset \mathbb{C}.
\]

where the branch of \( f^{-k(n)} \) is chosen to maps \( f^{k(n)}(x) \) to \( x \).

Case II. \( i(n) = 0 \) and \( f^{k(n)}(x) \notin X_n - Y_n \). Since \( f^{k(n)}(x) \notin X_n \), define \( h_n \) just the same as Case I.

Case III. \( i(n) = 0 \) and \( f^{k(n)}(x) \in X_n - Y_n \). Since \( \partial X_n \) is close to \( \partial Y_n \), \( f^{k(n)}(x) \) is close to \( Z_n \). So let \( \zeta \) be a path joining \( f^{k(n)}(x) \) to \( Z_n \) with length bounded above in terms of \( \lambda \).
Then by the previous lemma, there is a univalent map $f^n : Z_n \rightarrow Y_n(j(n))$. So define $h_n$ by:

$$Y_n(j(n)) \xrightarrow{f^{-m}} Z_n \xrightarrow{f^{-k(n)}} T_n \subset \mathbb{C}.$$  

We choose the inverse branch of $f^{-k(n)}$ so that the extension to $Z_n \cap \zeta_n$ maps $f^{k(n)}(x)$ to $x$.

By the estimates for the derivative $\| (f^{k(n)})'(z) \|$ on $\partial T_n$ in terms of $\| (f^{k(n)})'(x) \|$ and $\lambda$, $\text{diam}(T_n) \rightarrow 0$ and $d(x, T_n) \leq C_1 \text{diam}(T_n)$ where $C_1$ is a constant which depends only on $\lambda$.

Let $k_n = h_n \circ A_n^{-1}$. Then $|k_n'(0)| \rightarrow 0$. Therefore,

$$\frac{|x - k_n(0)|}{|k_n'(0)|} \leq C_2 \frac{d(x, T_n) + \text{diam}(T_n)}{\text{diam}(T_n)} \leq C_3,$$

where $C_2$ and $C_3$ depend only on $\lambda$.

Thus we can apply Lemma 4.2 and deduce the contradiction. \(\square\)

References


