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Action of mapping class group on extended Bers slice

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1 Introduction

Let $S$ be an oriented closed surface of genus $g \geq 2$. Put

$$V(S) = \text{Hom}(\pi_1(S), \text{PSL}_2(\mathbb{C}))/\text{PSL}_2(\mathbb{C}).$$

Let $X$ be an element of Teichmüller space $\text{Teich}(S)$ of $S$ and $C_X$ be the subset of $V(S)$ consisting of function groups which uniformize $X$. We define the action of mapping class group $\text{Mod}(S)$ on $C_X$ and investigate the distribution of elements of $C_X$.

2 Preliminaries

A compact 3-manifold $M$ is called compression body if it is constructed as follows: Let $S_1, \ldots, S_n$ be oriented closed surfaces of genus $\geq 1$ (possibly $n = 0$). Let $I = [0, 1]$ be a closed interval. $M$ is obtained from $S_1 \times I, \ldots, S_n \times I$ and a 3-ball $B^3$ by glueing a disk of $S_j \times \{0\}$ to a disk of $\partial B^3$ or a disk of $\partial B^3$ to a disk of $\partial B^3$ orientation reversingly. A component of $\partial M$ which intersects $\partial B^3$ is denoted by $\partial_0 M$ and is called the exterior boundary of $M$.

A Kleinian group is a discrete subgroup of $\text{PSL}_2(\mathbb{C}) = \text{Isom}^+ \mathbb{H}^3 = \text{Aut}(\hat{\mathbb{C}})$. We always assume that a Kleinian group is torsion-free and finitely generated. We denote by $\Omega(G)$ the region of discontinuity of a Kleinian group $G$. For a Kleinian group $G$, $\mathbb{H}^3/G$ is a hyperbolic 3-manifold and each component of $\Omega(G)/G$ is a Riemann surface. $N_G := \mathbb{H}^3 \cup \Omega(G)/G$ is called a Kleinian manifold.

A Kleinian group $G$ is called a function group if there is a $G$-invariant component $\Omega_0(G)$ of $\Omega(G)$. A function group $G$ is called a quasi-Fuchsian group if there are two $G$-invariant component of $\Omega(G)$. A Kleinian group $G$ is called geometrically finite if it has a finite sided convex polyhedron in $\mathbb{H}^3$.

Let $S$ be an oriented closed surface of genus $g \geq 2$. Put

$$CB(S) = \{M | M \text{ is a compression body s.t. } \partial_0 M \cong S\}.$$

If $G$ is a function group with invariant component $\Omega_0(G)$ such that $\Omega_0(G)/G \cong S$, then $\mathbb{H}^3/G$ is homeomorphic to the interior $\text{int} M$ of some $M \in CB(S)$ (i.e. function group is topologically tame).
If $G$ is a quasi-Fuchsian group such that each component of $\Omega(G)/G$ is homeomorphic to $S$, then $N_G = \mathbb{H}^3 \cup \Omega(G)/G \cong S \times I$.

Let $M \in CB(S)$. Let

$$V(M) = \text{Hom}(\pi_1(M), \text{PSL}_2(\mathbb{C}))/\text{PSL}_2(\mathbb{C})$$

be the representation space equipped with algebraic topology. We denote the conjugacy class of $\rho : \pi_1(M) \to G \subset \text{PSL}_2(\mathbb{C})$ by $[\rho, G]$ or $[\rho]$. Let

$$AH(M) = \{ [\rho] \in V(M) | \rho \text{ is discrete and faithful} \}$$

and $MP(M) = \text{int}AH(M)$. Any element $[\rho, G] \in MP(M)$ is geometrically finite and minimally parabolic, that is, any parabolic element $\gamma \in G$ is contained in $\rho(\pi_1(T))$ for some torus component $T$ of $\partial M$.

**Remark.**
- It is conjectured that $\overline{MP(M)} = AH(M)$ (Bers-Thurston conjecture).
- If $M \in CB(S)$, $MP(M)$ is connected.

Put

$$V(S) = \text{Hom}(\pi_1(S), \text{PSL}_2(\mathbb{C}))/\text{PSL}_2(\mathbb{C}).$$

Then $MP(S \times I) \subset AH(S \times I) \subset V(S)$. For any $[\rho, G] \in MP(S \times I)$, $G$ is a quasi-Fuchsian group. $MP(S \times I)$ is called the quasi-Fuchsian space.

Let $M \in CB(S)$. If an embedding $f : S \hookrightarrow M$ is homotopic to an orientation preserving homeomorphism $S \to \partial_0 M$, $f$ is called an admissible embedding. For an admissible embedding $f : S \hookrightarrow M$, the map

$$f^* : V(M) \to V(S)$$

defined by $[\rho] \mapsto [\rho] \circ f_*$ is a proper embedding.

Let $M_1, M_2 \in CB(S)$ and $f_j : S \hookrightarrow M_j (j = 1, 2)$ be admissible embeddings. Then the following holds:

- $\ker(f_1)_* = \ker(f_2)_* \iff (f_1)^*(AH(M_1)) = (f_2)^*(AH(M_2))$,
- $\ker(f_1)_* \neq \ker(f_2)_* \iff (f_1)^*(AH(M_1)) \cap (f_2)^*(AH(M_2)) = \emptyset$.

Let $M \in CB(S)$. Put

$$AH(M) = \bigcup_j f_j^*(AH(M)) \subset V(S)$$

and

$$MP(M) = \bigcup_j f_j^*(MP(M)) \subset V(S),$$

where the union is taken over all admissible embeddings $f : S \hookrightarrow M$. 
Remark. In general, $\mathcal{MP}(M)$ consists of infinitely many connected components. On the other hand, $\mathcal{MP}(S \times I) = MP(S \times I)$ is connected.

Let $\text{Teich}(S)$ be the Teichmüller space of $S$. Then

$$\mathcal{MP}(S \times I) = \text{Teich}(S) \times \text{Teich}(S).$$

We always fix $X \in \text{Teich}(S)$ in the following. Let

$$C_X = \{[\rho, G]|G \text{ is a function group s.t. } \Omega_0(G)/G \cong X\}.$$

More precisely, if $\rho : \pi_1(S) \to G \cong \pi_1(N_G)$ is induced by $S \to X \cong \Omega_0(G)/G \cong N_G$ for some function group $G$, then $[\rho, G]$ is an element of $C_X$. $C_X$ is called an extended Bers slice.

**Lemma 1.** $C_X$ is compact.

Put

$$\mathcal{AH}_X(M) := \mathcal{AH}(M) \cap C_X$$

$$\mathcal{MP}_X(M) := \mathcal{MP}(M) \cap C_X.$$ 

$B_X := \mathcal{MP}_X(S \times I) = \mathcal{MP}(S \times I) \cap C_X$ is called a Bers slice. Obviously

$$C_X = \bigcup_{M \in CB(s)} \mathcal{AH}_X(M).$$

### 3 Action of $\text{Mod}(S)$ on $C_X$

Let $\text{Mod}(S)$ denote the mapping class group of $S$. Let $[\rho, G] \in C_X$. Let $\text{Belt}(X)_1$ denote the set of Beltrami differentials $\mu = \mu(z)\overline{dz}/dz$ on $X$ such that $||\mu||_\infty < 1$.

$$\text{Belt}(X)_1 \xrightarrow{\cong} \text{Belt}(\Omega_0(G)/G)_1 \xrightarrow{\Psi_\rho} QC_0(\rho).$$

$QC_0(\rho)$ consists of the qc-deformations of $[\rho, G]$ whose Beltrami differentials are supported on $\Omega_0(G)$.

The action of $\sigma \in \text{Mod}(S)$ on $C_X$ is defined by

$$[\rho] \mapsto [\rho]^\sigma := \Psi_\rho(\sigma^{-1}X) \circ \sigma^{-1},$$

where $\sigma_*$ is the group automorphism of $\pi_1(S)$ induced by $\sigma$. 

4 Continuity of the action

Theorem 2. Let $[\rho, G] \in C_X$. If all components of $\Omega(G)/G$ except for $X = \Omega_0(G)/G$ are thrice-punctured spheres, then the action of $\text{Mod}(S)$ is continuous at $[\rho]$; that is, if $[\rho_n] \rightarrow [\rho]$ in $C_X$ then $[\rho_n]^{\sigma} \rightarrow [\rho]^{\sigma}$ for all $\sigma \in \text{Mod}(S)$.

Remark. In general, the action of $\text{Mod}(S)$ is not continuous at $\partial B_X = \overline{B_X} - B_X$ (Kerckhoff-Thurston).

5 Maximal cusps

Put $\partial M_{pX}(M) = \overline{M_{rX}(M)} - M_{pX}(M)$.

Definition. An element $[\rho, G] \in \partial M_{pX}(M)$ is called a maximal cusp if $G$ is geometrically finite and all components of $\Omega(G)/G$ except for $X = \Omega_0(G)/G$ are thrice-punctured spheres.

Theorem 3 (McMullen). The set of maximal cusps is dense in $\partial B_X$.

Proposition 4. For any $M \in CB(S)$, the set of maximal cusps is dense in $\partial M_{pX}(M)$.

The set of maximal cusps in $\partial M_{pX}(M)$ decomposes into finitely many orbit. The following theorem implies that “each” orbit is dense in $\partial M_{pX}(M)$.

Theorem 5. For any maximal cusp $[\rho] \in \partial M_{pX}(M)$, its orbit $\{[\rho]^{\sigma}\}_{\sigma \in \text{Mod}(S)}$ is dense in $\partial M_{pX}(M)$.

6 Statement of main theorem

Let $M_1, M_2 \in CB(S)$. An embedding $f : M_1 \hookrightarrow M_2$ is said to be admissible if $f$ is homotopic to an embedding $g : M_1 \hookrightarrow M_2$ such that $g|\partial M_1 : \partial M_1 \hookrightarrow M_2$ is a homeomorphism.

Theorem 6. Let $M \in CB(S)$ and $\{M_n\} \subset CB(S)$. If $\{[\rho_n] \in \mathcal{AH}_X(M_n)\}$ converges algebraically to $[\rho_{\infty}] \in \mathcal{AH}_X(M)$, then for large enough $n$ there exist admissible embeddings $f_n : M \hookrightarrow M_n$.

This can be easily seen from the fact that $\ker \rho_n \supseteq \ker \rho_{\infty}$ for large enough $n$.

Lemma 7. Let $M_1, M_2 \in CB(S)$ and $[\rho] \in \mathcal{AH}(M_2)$. If there is a sequence $\{\sigma_n\}$ of $\text{Mod}(S)$ such that $[\rho]^{\sigma_n}$ converges algebraically to $[\rho_{\infty}] \in \mathcal{AH}_X(M_1)$, then there exist an admissible embedding $f : M_1 \hookrightarrow M_2$. 
Conversely, the following holds.

**Theorem 8.** Let $M_1, M_2 \in CB(S)$. Suppose that there exists an admissible embedding $f : M_1 \hookrightarrow M_2$. Then for any geometrically finite element $[\rho] \in A\mathcal{H}_X(M_2)$, the set of accumulation points of $\{[\rho]^\sigma\}_{\sigma \in \text{Mod}(S)}$ contains $\partial \mathcal{M}P_x(M_1)$.

Recall that $S$ is a closed surface of genus $g \geq 2$. Let $H_g$ be a handle body of genus $g$. Note that for any $M \in CB(S)$, there are embeddings

$$S \times I \hookrightarrow M, M \hookrightarrow H_g$$

which preserve the exterior boundaries.

**Corollary 9.** (1) For any $[\rho] \in A\mathcal{H}_X(H_g)$, the set of accumulation points of $\{[\rho]^\sigma\}_{\sigma \in \text{Mod}(S)}$ contains $\bigcup_{M \in CB(S)} \partial \mathcal{M}P_x(M)$.

(2) For any $M \in CB(S)$ and any geometrically finite $[\rho] \in A\mathcal{H}_X(M)$, the set of accumulation points of $\{[\rho]^\sigma\}_{\sigma \in \text{Mod}(S)}$ contains $\partial B_X = \partial \mathcal{M}P_x(S \times I)$.

**Remark (Hejhal, Matsuzaki).** Let $[\rho] \in C_X$. $[\rho] \in A\mathcal{H}_X(H_g)$ if and only if $[\rho]$ is geometrically finite and isolated in $C_X$. 