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Local Langlands correspondence について(紹介)

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局所 Langlands 予想が、標数 0 の場合にも (一般の次数 n に対して) 最近 M. Harris と R. Taylor [HT] によって証明され、G. Henniart のより簡単な証明 [He] も出ましたので、正確に何が証明されたのか、という所を中心に紹介させていただきます。尚、ここでは紹介し切れませんが、[HT] を見ると過去の日本人の仕事 — 特に志村五郎、井草準一、本田 平 (故人)、藤原一宏各氏によるもの — も重要なところで使われていることがわかります。

F を標数 0 の局所体 ($\mathbb{Q}_p$ の有限次拡大) とするとき、主要結果は、大まかに云うと、1 対 1 対応

$$Gal(\bar{F}/F) \text{の } n \text{ 次複素表現} \underset{1:1}{\leftrightarrow} GL_n(F) \text{ の既約表現 (一般に無限次元)}$$

が存在する、という事ですが、左辺 (Galois side) で $Gal(\bar{F}/F)$ は Weil 群 W_F に置換え、その表現は"Φ-semisimple なもの" とし、右辺 (Automorphic side) では smooth な表現としなくてはなりません。それらの定義から復習します。その前に主な文献を列挙しましょう。

1. Main references

《"決定的"文献》

[HT] M. Harris, R. Taylor, "On the geometry and cohomology of some simple Shimura varieties" (Preprint, 1998, 99) (99, Aug 30 version を参照しました)

[He] G. Henniart, "Une preuve simple des conjectures des Langlands pour $GL(n)$ sur un corps p-adique", To appear in Inv. Math.

《基本的文献》
(Weil(-Deligne) 群については)

[De] P. Deligne, "Les constantes des équations fonctionelles des fonctions L" In: "Modular functions of one variable" II, SLN 349.

[Ta] J. Tate, "Number Theoretic Background" AMS Proc. Symp. Pure Math. 33-2 (1979).

(Automorphic representations については) AMS Proc. Symp. Pure Math. 33-1 の Cartier 等の Survey, 及び

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[Z] A. V. Zelevinskii, "Induced representations of reductive p-adic groups II" Ann. Sci. ENS **13** (1980), 等.

($n = 2$ 源泉)

[JL] H. Jacquet, R. P. Langlands, "Automorphic forms on $GL(2)$" SLN 114.

(比較的最近の部分的結果)

[He 1] G. Henniart, "La conjecture de Langlands locale numérique pour $GL(n)$" Ann. Sci. ENS **21** (1988).

[He 2] G. Henniart, "Caractérisation de la correspondance de Langlands locale par les facteurs ε de paires", Inv. Math. **113** (1993).

[Ha 1] M. Harris, "The local Langlands conjecture for $GL(n)$ over a p-adic field, $n < p$", Inv. Math. **134** (1998).

(Char $p > 0$ の場合の証明は)

[LRS] G. Laumon, M. Rapoport, U. Stuhler, "$\mathcal{D}$-elliptic sheaves and the Langlands correspondence", Inv. Math. **113** (1993).

2. Main definitions

《Galois side》

p : a prime number, $[F : \mathbb{Q}_p] < \infty$ (a commutative field), $\bar{F}$: an algebraic closure of F, $q = \#$ (the residue field of F).

- W_F (the Weil group of F) $:= \{w \in Gal(\bar{F}/F);\ w$ acts as $* \to *^{|w|}$ on the residue field, with some $|w| \in q^{\mathbb{Z}}\}$ $= \Phi^{\mathbb{Z}} \ltimes I_F$ (locally compact),

I_F : the inertia group (open compact),

Φ: a geometric Frobenius element, i.e., $\Phi \in W_F$ *s.t.* $|\Phi| = q^{-1}$.

Remark. W_F is a dense subgroup of $Gal(\bar{F}/F)$, but we consider W_F as a topological group in such a way that I_F is open (and hence $W_F/I_F(\cong \mathbb{Z})$ is discrete).

- The abelianization W_F^{ab} of W_F is canonically isomorphic with the multiplicative group $F^\times$, via local classfield theory;

$$W_F^{ab} \xrightarrow{\sim} F^\times.$$

The induced homomorphism $W_F \longrightarrow F^\times$ maps Φ to a prime element of $F^\times$, and $|w|$ on W_F corresponds with the standard valuation $|\ |_F$ of $F^\times$.

- $\rho : W_F \longrightarrow GL_n(\mathbb{C})$: a continuous representation, i.e., ρ is a group homomorphism such that its restriction to I_F factors through a finite quotient.

ρ : completely reducible $\leftrightarrow$ $\rho(\Phi)$ semi-simple for one Φ $\leftrightarrow$ for all Φ.

- Def Φ-semi-simple representation of $W_F' = W_F \ltimes \mathbb{G}_a$ (the Weil-Deligne group) of deg $n := \{(\rho, N) | \rho : W_F \longrightarrow GL_n(\mathbb{C})$, continuous completely reducible representation, $N \in M_n(\mathbb{C})$, nilpotent, $\rho(w)N\rho(w)^{-1} = |w|N$ $(^\forall w \in W_F)\}$

Remark (i) $\oplus, \otimes$ defined; (ii) ρ : *irreducible* $\longrightarrow N = 0$; (iii) each l-adic $(l \neq p)$ representation ρ_l of W_F gives rise to a Φ-semi-simple representation (ρ, N) of W_F'. (N is determined by its restriction to $\mathbb{Z}_l \subset I_F$, and ρ is a certain modification of ρ_l.)

- Examples of (ρ, N).

(I) Each continuous homomorphism $\chi : F^\times \longrightarrow \mathbb{C}^\times$ (a "quasi-character" of $F^\times$) can be regarded as a 1-dimensional representation of W_F, via $W_F \longrightarrow W_F^{ab} \xrightarrow{\sim} F^\times$. The unramified quasi-characters are those of the form χ_s $(s \in \mathbb{C})$ defined by $\chi_s(a) = |a|_F^s$.

(II) Given any irreducible $\rho_0 : W_F \longrightarrow GL_{n_0}(\mathbb{C})$ and an integer $d \geq 1$, the "Steinberg representation" $St_d(\rho_0) = (\rho, N)$ is by definition:

$$\rho: \qquad W_F \ni w \to \begin{pmatrix} \rho_0(w) & & & 0 \\ & \rho_0(w)|w|^{-1} & & \\ & & \ddots & \\ & 0 & & \rho_0(w)|w|^{-(d-1)} \end{pmatrix} \in GL_{n_0 d}(\mathbb{C}),$$

$$N = \begin{pmatrix} 0 & I_{n_0} & & 0 \\ & 0 & \ddots & \\ & & \ddots & I_{n_0} \\ 0 & & & 0 \end{pmatrix}$$

It is known that a Φ-semisimple representation of W'_F is of the form $St_d(\rho_0)$ if and only if it is indecomposable, and that $St_d(\rho_0)$ determines both d and ρ_0.

(III) Each Φ-semisimple representation of W'_F can be decomposed uniquely as a direct sum of indecomposable representations.

Thus, $(\rho, 0)$ with ρ: irreducible are the most fundamental Φ-semi-simple representations of W'_F.

《Automorphic representation side》

$n \geq 1, \quad G_n = GL_n(F) \qquad$ (locally compact groups).

- Def A smooth representation (π, V) of G_n is:

  V : a complex vector space (in most cases infinite dimensional),

  $\pi : G_n \to Aut_{\mathbb{C}} V$, a group homomorphism s.t. the stabilizer of each $v \in V$ in G_n is open.

  All representations will be assumed smooth.

- Def π: admissible $\leftrightarrow$ for each open compact subgroup $U \subset G_n$, $V^U := \{v \in V | \pi(u)v = v(\forall u \in U)\}$ is finite dimensional.

  π: irreducible $\to$ admissible (in the case of representations of G_n), and

  $\lambda \in End_{\mathbb{C}} V$, $\lambda \circ \pi(g) = \pi(g) \circ \lambda (\forall g \in G_n) \Rightarrow \lambda$: scalar.

  esp. $z \in F^\times \cdot I_n$ (the center of G_n) $\Rightarrow \pi(z)$: scalar.

  This defines the central character $\omega_\pi : F^\times \to \mathbb{C}^\times$ of π.

- The smooth dual $(\pi^\vee, V^\vee)$ of (π, V) for π: irreducible. $V^\vee$:={linear $V \to \mathbb{C}$ which annihilates the unique G_n-stable complement of V^U for some U},

$$\langle \pi(g)v, v^\vee \rangle = \langle v, \pi^\vee(g)^{-1}v^\vee \rangle$$

$(\pi^\vee)^\vee = \pi$.

- Def π: an irreducible smooth representation is called supercuspidal, if for any $v \in V, v^\vee \in V^\vee$, the support of $\langle \pi(g)v, v^\vee \rangle$ (a $\mathbb{C}$-valued function on G_n) is compact modulo the center $F^\times \cdot I_n$.

- Induced representations. $P \underset{\text{closed subgp}}{\subset} G = G_n$, (π_P, U): representation of P, given.

  Then: $\underset{P\uparrow G}{\operatorname{Ind}}(\pi_P, U) = (\pi_G, V)$: defined by

  $V = \{f : U\text{-valued fctns on } G;\ f(pg) = \delta_P^{1/2}(p)\pi_P(p)f(g)\}$.
  $(p \in P, g \in G)$

  $(\pi_G(g)f)(x) = f(xg)$, where δ_P: $G \to (\mathbb{R}^+)^\times$ is a character defined by the formula:

  $\delta_P(p)^{-1}\times$(right inv. Haar measure of P) =left inv. Hear measure of P.

- Examples.

  (I) Each quasi-character $\chi : F^\times \to \mathbb{C}^\times$ can be regarded as an irreducible representation of $GL_1(F)$. It is supercuspidal. (But the composition of χ with the determinant det: $GL_n(F) \to F^\times$ is not supercuspidal if $n > 1$.)

  (II) Given any irreducible supercuspidal representation ρ_0 of $GL_{n_0}(F)$ and an integer $d \geq 1$, put $n = n_0 d$ and let P be the parabolic subgroup of $GL_n(F)$ generated by the block-diagonal matrices $GL_{n_0}(F) \times \cdots \times GL_{n_0}(F)$ (d copies) and the upper triangular unipotent matrices. Consider the representation

$$(\rho_0 \otimes |det|_F^{d-1}) \otimes \cdots \otimes (\rho_0)$$

of

$$GL_{n_0}(F) \times \cdots \times GL_{n_0}(F).$$

Regard this as a representation $\rho_0^{(d,P)}$ of P by passage to the quotient, and take $\operatorname*{Ind}_{P\uparrow G} \rho_0^{(d,P)}$. Then, this has a unique irreducible <u>subrepresentation</u>, called $St_d(\rho_0)$. It is known that an irreducible representation of G_n is of the form $St_d(\rho_0)$ if and only if it is quasi-square integrable i.e., a tensor product of a quasi-character of $F^\times$ (regarded as a 1-dimensional representation of G_n via the determinant) and a square integrable (modulo the center) representation of G_n. They are also precisely those irreducible representations of G_n that correspond naturally with some irreducible representation of $D_n^\times$, where D_n is the central division algebra over F of degree n^2 and the Hasse invariant $1/n$ (Deligne-Kazhdan-Vigneras). $St_d(\rho_0)$ determines each of d and ρ_0 uniquely.

(III) An arbitrary irreducible representation π of G_n can be expressed uniquely as the "Langlands sum" $\boxplus$ of representations of the form treated in (II).

The sum $\rho_1 \boxplus \cdots \boxplus \rho_r$, where each ρ_i is of the type treated in (II), is defined by using the induced representation $\operatorname*{Ind}_{P\uparrow G}(\rho_1 \otimes \cdots \otimes \rho_r)$ analogous to the situation in (II). If $n = n_1 + \cdots + n_r$ and ρ_i is a representation of $GL_{n_i}(F)$ $(1 \leq i \leq r), P$ is now the semi-direct product of

$$GL_{n_1}(F) \times \cdots \times GL_{n_r}(F) \subset GL_n(F)$$

and the upper triangular unipotent matrices. But the ordering of $\rho_1, \ldots, \rho_r$ and whether we choose the unique irreducible subrepresentation of $\operatorname*{Ind}_{P\uparrow G}(\rho_1 \otimes \cdots \otimes \rho_r)$ or the quotient representation is a delicate point (whose choice should be reversed if we take as P the lower block-triangular matrices). I could not find an appropriate explicit reference for this, but the content of [Z] shows that it should be, here, the unique irreducible, <u>quotient</u> of $\operatorname*{Ind}_{P\uparrow G}(\rho_1 \otimes \cdots \otimes \rho_r)$ if the ordering of $\rho_1, \ldots, \rho_r$ is <u>admissible</u> in the following sense. Write $\rho_i = St_{d_i}(\rho_{i,0})$. The ordering is admissible if for each $i < j$, there does <u>not</u> exist any positive integer a such that

$$\rho_{j,0} = \rho_{i,0} \otimes |det|^a, \quad 1 + d_i - d_j \leq a \leq d_i.$$

Admissible orderings exist, and the resulting (unique irreducible) quotient of the induced representation is independent of the choice of such orderings.

3. The main result ([HT] [He])

Theorem (Harris-Taylor, Henniart) $\left(\begin{array}{ll} n=1: & \text{classical, } n=2: \cdots, \text{ Kutzko} \\ n=3 & \text{(almost) Henniart, } n<p\text{: Harris} \\ \text{char } p>0 & \cdots \text{Laumon-Rapoport-Stuhler} \end{array}\right)$

There exists a unique system of bijections $LG_n (n \geq 1)$ satisfying (1)~(5) below.

$$\{\rho' = (\rho, N); \quad \begin{array}{l} \rho : W_F \to GL_n(\mathbb{C}) \text{ continuous } \underline{\text{completely reducible}} \text{ representation,} \\ N \in M_n(\mathbb{C}) \text{ nilpotent, s.t. } \rho(w)N\rho(w)^{-1} = |w|N({}^\forall w \in W_F).\}/\simeq \end{array}$$

$\overset{LG_n}{\leftrightarrow}$ { smooth <u>irreducible</u> representations π of $GL_n(F)\}/\simeq$.
This system of bijection also satisfies:

$$\begin{array}{rcl} \rho : \text{irreducible}(\to N=0) & \leftrightarrow & \pi : \text{supercuspidal,} \\ St_d(\rho) & \leftrightarrow & St_d(\pi), \\ (i.e., \rho : \text{indecomposable} & \leftrightarrow & \pi : \text{quasi square-integrable)} \\ \oplus St_d(\rho_i) & \leftrightarrow & \boxplus St_d(\pi_i). \end{array}$$

(1) $n=1$, LG_1 : local classfield theory correspondence via $W_F^{ab} \xrightarrow{\sim} F^\times$.

(2) $\underset{(\rho,N)}{\overset{\rho'}{\parallel}} \leftrightarrow \pi \Rightarrow det\rho \leftrightarrow \omega_\pi$,

(3) $\Rightarrow (\chi \circ ab) \otimes \rho' \leftrightarrow (\chi \circ det) \otimes \pi$,

(4) $\Rightarrow (\rho')^\vee \leftrightarrow \pi^\vee$,

(5) $\rho'_i \underset{(i=1,2)}{\leftrightarrow} \pi_i \Rightarrow L(\rho'_1 \otimes \rho'_2, s) = L(\pi_1 \times \pi_2, s)$,

$$\varepsilon(\rho'_1 \otimes \rho'_2, s, \psi) = \varepsilon(\pi_1 \times \pi_2, s, \psi),$$

($\psi : F \to \mathbb{C}^\times$: a non-trivial additive character).

As for these local L-functions and local ε-factors, cf. e.g. [Ta] for those on the left side, and [JPS] for those on the right side. Each $L(*, s)$ is a finite product of $(1 - \chi(\pi)q^{-s})^{-1}$, where π is a prime element of F and χ runs over a finite number of unramified quasi-characters of $F^\times$ determined by $*$. Each $\varepsilon(*, s, \phi)$ is a function of s of the form $a \cdot \exp(bs)$ $(a \in \mathbb{C}^\times, b \in \mathbb{C})$, where b is essentially the conductor of $*$.

The basic questions:

(1) What should correspond to $\otimes$ of Galois representations on the automorphic side?

(2) What should correspond to $\operatorname*{Ind}_{F' \uparrow F}(\chi)$ (more precisely, from $W_{F'}$ to W_F) on the automorphic side, where χ is any quasi-character of $(F')^\times$?

They seem still open.

4. On proofs

Both in [HT] and [He], the proofs are based on constructions of global correspondences. They construct appropriate automorphic representations on GL_n over some CM-fields associated with some Shimura varieties, find corresponding global Galois representations, and compare their local factors. In [HT], study of Shimura varieties at primes with bad reductions are used, but in [He], good reductions are sufficient for the purpose. The proof in [He] is simpler, but [HT] contains more information on the global correspondences.

In [HT], the authors start with the big universal l-adic representation of $D_F^\times \times GL_n(F) \times W_F$ constructed by Drinfeld, and study its decomposition in greater detail. In [He], by using the Brauer theorem on representations of finite groups, the author reduces the problem to construction of a supercuspidal representation of $GL_m(F)$ corresponding to those irreducible representations of W_F that are induced from a (multiplicative) quasi-character of a finite extension of F (of degree m). Special types of Shimura varieties over a CM-field (in this case, a composite of a totally real field and an imaginary quadratic field, which has F as a non-archimedean completion), and cohomology groups of certain l-adic sheaves on such varieties are used effectively.