ALGEBRAIC K-THEORY OF HENSELIAN PAIRS

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ABSTRACT. In this note we study the relationship between $\mathcal{F}(R)$ and $\mathcal{F}(R/I)$ for a Henselian pair $(R, I)$ and various functors $\mathcal{F}$. For example, in the case of algebraic K-theory, if $m$ is invertible in $R$, then $K_i(R, \mathbb{Z}/m) \cong K_i(R/I, \mathbb{Z}/m)$.

If $p$ is not invertible in $R$, then in general $K_i(R, \mathbb{Z}/p)$ is not determined by the inverse system $\{K_i(R/p^j, \mathbb{Z}/p)\}_j$. However, if $p$ is not a zero-divisor and $R$ is $p$-complete, then we have $K_i(R, \mathbb{Z}/p) \cong \lim K_i(R/p^j, \mathbb{Z}/p)$. We sketch the proofs of these results.

1. HENSELIAN PAIRS

Definition 1.1. Let $R$ be a commutative ring with unit and $I$ an ideal of $R$. Then $(R, I)$ is called a Henselian pair, if the following equivalent conditions are satisfied [1][2]:

1. $I$ is contained in the Jacobsen radical of $R$, and for all monic polynomials $f \in R[T]$ and factorizations $f = \overline{g}\overline{h}$ mod $I$ with $\overline{g}, \overline{h} \in R/I[T]$ monic and relatively prime, there is a lifting $f = gh$ of the factorization with $g, h \in R[T]$ monic.

2. For any finite $R$-algebra $B$, there is a bijection of idempotents

$$\text{Idem}(B) \cong \text{Idem}(B/IB).$$

3. For any étale $R$-algebra $B$, any $R$-map $B \rightarrow R/I$ lifts uniquely to a map $B \rightarrow R$.

4. For any $f \in R[T]$ and any simple root $\overline{\alpha} \in R/I$ of $f$ mod $I$ there is lifting to a root $\alpha \in R$ of $f$ (note that $\alpha$ is called a simple root of the polynomial $f$ if and only if $f(\alpha) = 0$ and $f'(\alpha)$ is a unit).
In this note we want to study the relationship between $\mathcal{F}(R)$ and $\mathcal{F}(R/I)$ for various functors $\mathcal{F}$. As a first example, we have the following theorem of Gabber [1]:

**Theorem 1.2.** Let $(R, I)$ be a Henselian pair, and $\mathcal{F}$ a torsion sheaf on $(\text{Spec } R)_{\text{et}}$. Then

$$H^q_{\text{et}}(\text{Spec } R, \mathcal{F}) \cong H^q_{\text{et}}(\text{Spec } R/I, i^*\mathcal{F}).$$

Note that if $(R, I)$ is a Henselian local ring, then the theorem holds without the assumption that $\mathcal{F}$ is torsion and is an elementary property of étale cohomology. On the other hand, the theorem is wrong without the hypothesis that $\mathcal{F}$ is torsion. It is a general phenomenon that theorems as above only hold with torsion coefficients.

### 2. K-theory

To get interesting invariants of a ring $R$, generalizing for example the group of units $R^\times$ and the ideal class group $\text{Pic } R$, one can study the group of matrices $\text{GL}_n(R)$, or the direct limit $\text{GL}(R) = \varinjlim \text{GL}_n(R)$. A good method to analyze a group is to study its group homology

$$H_i(\text{GL}(R), \mathbb{Z}) := H_i(\text{BGL}(R), \mathbb{Z}).$$

Here the right hand side is the ordinary singular homology of a topological space, and for a group $G$, the topological space $BG$ is characterized by the property that it has only one nontrivial higher homotopy group, $\pi_1(BG) = G$. In particular, we have the Hurewicz homomorphism

$$\pi_1(\text{BGL}(R)) = \text{GL}(R) \to H_1(\text{GL}(R), \mathbb{Z}) = \text{GL}(R)^{ab}.$$

Since a good invariant of the ring $R$ should consist of abelian groups, the idea is to make the homotopy group abelian without changing the homology groups. There is a universal construction to achieve this, called $+$-construction. In other words, there is a topological space $\text{BGL}(R)^+$ characterized by the properties that for any abelian group $A$ we have

$$H_i(\text{BGL}(R), A) = H_i(\text{BGL}(R)^+, A)$$

and

$$\pi_1(\text{BGL}(R))^{ab} = \pi_1(\text{BGL}(R)^+).$$

This changes the higher homotopy groups, and we define K-theory as the higher homotopy groups of this space

$$K_i(R) := \pi_i(\text{BGL}(R)^+),$$

and similar with coefficients in an abelian groups $A$,

$$K_i(R, A) := \pi_i(\text{BGL}(R)^+, A).$$
If \( A = \mathbb{Z}/m \), then there is an exact sequence
\[
0 \rightarrow K_i(R)/m \rightarrow K_i(R, \mathbb{Z}/m) \rightarrow mK_{i-1}(R) \rightarrow 0,
\]
and if \( A = \mathbb{Q} \) then \( K_i(R, \mathbb{Q}) \cong K_i(R) \otimes \mathbb{Q} \). Note that it is possible to recover \( K_*(R) \) from \( K_*(R, \mathbb{Q}) \) and \( K_*(R, \mathbb{Z}/m) \) for all \( m \).

For example, by definition \( K_1(R) = \text{GL}(R)^{ab} \), and the determinant homomorphism \( \text{GL}(R) \rightarrow \mathbb{R}^\times \) together with the inclusion \( \mathbb{R}^\times \rightarrow \text{GL}(R) \), sending a unit \( r \) to a matrix with \( r \) in the upper left corner, shows that the units \( \mathbb{R}^\times \) form a split direct summand of \( K_1(R) \). In fact, \( K_1(R) = \mathbb{R}^\times \) if \( R \) is local.

As another example, for a field \( F \), \( K_2(F) \) has generators \( F^\times \otimes F^\times \) and relations \( a \otimes (1-a) = 0 \).

3. K-theory and group homology of Henselian pairs

The K-theory and group homology of Henselian pairs have been studied by Gabber, Suslin and Panin. Let \( (R, I) \) be a Henselian pair. Define \( \text{GL}(R, I) \) as the kernel of the (surjective) reduction map \( \text{GL}(R) \rightarrow \text{GL}(R/I) \), and let \( \bar{H}_* \) be reduced homology (i.e. removing the copy of \( \mathbb{Z} \) in degree zero). By definition, K-theory and homology of \( \text{GL}(R) \) are closely related. For a Henselian pair, this takes the following form:

**Proposition 3.1.** a) [5] Let \( (R, I) \) be a Henselian pair and \( m \) invertible in \( R \). Then the following statements are equivalent:

1. \( K_*(R, \mathbb{Z}/m) = K_*(R/I, \mathbb{Z}/m) \).
2. \( H_*(\text{GL}(R), \mathbb{Z}/m) = H_*(\text{GL}(R/I), \mathbb{Z}/m) \).
3. \( \bar{H}_*(\text{GL}(R, I), \mathbb{Z}/m) = 0 \).

b) [4] If \( m \) is not invertible in \( R \), then we still have an equivalence:

1. The pro-system \( \{K_*(R/I^j, \mathbb{Z}/m)\}_j \) is isomorphic to the constant pro-system \( \{K_*(R, \mathbb{Z}/m)\}_j \).
2. The pro-system \( \{H_*(\text{GL}(R/I^j), \mathbb{Z}/m)\}_j \) is isomorphic to the constant pro-system \( \{H_*(\text{GL}(R), \mathbb{Z}/m)\}_j \).

Recall that a map \( (\phi_i : (X_i) \rightarrow (Y_i)) \) of pro-systems is an isomorphism if and only if for all \( i \) there is a \( j > i \) and a map \( s : Y_j \rightarrow X_i \) making the obvious diagram commutative.

It is a deep theorem in algebraic K-theory that the conditions of (a) are satisfied:

**Theorem 3.2.** (Gabber, Suslin) Let \( (R, I) \) be a Henselian pair, and \( m \) invertible in \( R \). Then
\[
K_i(R, \mathbb{Z}/m) \cong K_i(R/I, \mathbb{Z}/m).
\]

The proof consists of three equally difficult steps:
1. The special case $R$ is the Henselian local ring of a smooth variety over a field $F$ in an $F$-rational point $[2][3]$. 
2. Apply this to the Henselization of $\text{GL}^x_n$ in the unit section to prove the theorem for $R$ containing a field $[5]$. 

Here is the main idea of the second step:

Let $(R, I)$ be a Henselian pair containing a field $F$. The group homology $H_*(G, \mathbb{Z}/m)$ can be calculated using the complex $C_*(G, \mathbb{Z}/m)$ which in degree $i$ is the free abelian group generated by $i$-tuples of elements of $G$, $[g_1, \ldots, g_i]$. Since every element of $\text{GL}(R, I)$ is contained in $\text{GL}_n(R, I)$ for some $n$, it suffices to show according to Proposition 3.1 that the inclusion

$$C_*(\text{GL}_n(R, I), \mathbb{Z}/m) \rightarrow C_*(\text{GL}(R, I), \mathbb{Z}/m)$$

induces the zero map on homology for all $n$. This holds if the map is null-homotopic, i.e. if we can construct maps

$$C_i(\text{GL}_n(R, I), \mathbb{Z}/m) \rightarrow C_{i+1}(\text{GL}(R, I), \mathbb{Z}/m)$$

such that $d \circ s + s \circ d = \iota$.

Consider the algebraic variety $\text{GL}_n(F)^x$. It is smooth over $F$ and has a distinguished $F$-rational point $e$, the unit section. Let $\mathcal{O}_{n,i}^h$ be the Henselian local ring at $e$, and $m$ the corresponding maximal ideal.

Let $\beta = [\beta_1, \ldots, \beta_i] \in C_i(\text{GL}_n(R, I), \mathbb{Z}/m)$. Each element $\beta_j$ defines a morphism $\text{Spec } R \rightarrow \text{GL}_n$ sending the subvariety defined by $I$ to the unit section. Hence $\beta$ defines a morphism $\text{Spec } R \rightarrow \text{GL}_n^x$ with the same property. Since $(R, I)$ is Henselian, this induces a homomorphism $\mathcal{O}_{n,i}^h \rightarrow R$ sending $m$ to $I$. In particular, we get a map

$$\beta^*: C_*(\text{GL}(\mathcal{O}_{n,i}^h, m), \mathbb{Z}/m) \rightarrow C_*(\text{GL}(R, I), \mathbb{Z}/m).$$

Since by step (1) the theorem is known for $\mathcal{O}_{n,i}^h$, we have

$$K_*(\mathcal{O}_{n,i}^h, \mathbb{Z}/m) \cong K_*(\mathcal{O}_{n,i}^h/m, \mathbb{Z}/m)$$

and hence by Proposition 3.1

$$\bar{H}_*(\text{GL}(\mathcal{O}_{n,i}^h, m), \mathbb{Z}/m) = 0.$$ 

This property can be used to construct inductively elements $c_{n,i} \in C_{i+1}(\text{GL}(\mathcal{O}_{n,i}^h, m), \mathbb{Z}/m)$ independent of $R$ such that $s(\beta) := \beta^*(c_{n,i})$ is the desired null-homotopy.
4. The general case

If $m$ is not invertible in $R$, then in general $K_*(R, \mathbb{Z}/m) \neq K_*(R/I, \mathbb{Z}/m)$. One can ask if at least $K_*(R, \mathbb{Z}/m)$ is determined by the pro-system $\{K_*(R/I^j, \mathbb{Z}/m)\}_j$. But this is also wrong in general. For example, let $R = \mathbb{F}_p[X]^h$ be the Henselization of the affine line over a finite field in the origin and $\hat{R} = \mathbb{F}_p[[X]]$ its completion. Then $R/I^j = \hat{R}/I^j$ for all $j$, but

$$K_1(R, \mathbb{Z}/p) = R^\times /p \neq \hat{R}^\times /p = K_1(\hat{R}, \mathbb{Z}/p)$$

because the former is countable and the latter is uncountable.

However, the idea of the proof of Theorem 3.2 can be used to prove the following:

**Theorem 4.1.** Let $R$ be a noetherian ring such that $p$ is not a zero divisor, such that the map from $R$ to the $p$-completion $\hat{R}$ is regular, and such that $(R, p)$ is a Henselian pair. Then

$$K_i(R, \mathbb{Z}/p) \cong \lim_{\rightarrow} K_i(R/p^j, \mathbb{Z}/p).$$

An integral domain of characteristic 0 which is complete for the $p$-adic topology, or the Henselization at a point of the closed fiber of a reduced variety of finite type over a discrete valuation ring satisfies the hypothesis of this theorem. The theorem is a generalization of the special case $R$ a Henselian valuation ring of mixed characteristic. Except the following essential new ingredient, the proof goes back to [5].

Let $S = R[\frac{1}{p}]$, equipped with the $p$-adic topology. The hypothesis implies that $S$ contains $\mathbb{Q}$.

**Proposition 4.2.** Let $e \in X$ a pointed topological space and $\mathcal{F}$ be the ring of germs of continuous functions from $X$ to $S$. Let $\mathcal{I} \subset \mathcal{F}$ be the ideal of germs of functions vanishing at $e$. Then $(\mathcal{F}, \mathcal{I})$ is a Henselian pair.

The proof of the proposition will be published in a forthcoming paper. To continue the proof of the theorem, consider $GL_n(S)^{\times i}$ as a topological space with the $p$-adic topology. Let $\mathcal{F}_{n,i}$ be the ring of germs of continuous $S$-valued functions defined in a neighborhood of the unit element $e$, and let $\mathcal{I}_{n,i}$ be the ideal of germs of functions vanishing at $e$. We are going to construct a homotopy as above.

Every chain $c \in C_{i+1}(GL_r(\mathcal{F}_{n,i}, \mathcal{I}_{n,i}), \mathbb{Z}/p)$ defines a map of some neighborhood of $e \in GL_n(S)^{\times i}$ to $C_{i+1}(GL_r(S), \mathbb{Z}/p)$ which is continuous, i.e. for each $t$ there is an $s$ such that $c$ is defined on $GL_n(R, p^s)^{\times i}$ and maps it to $C_{i+1}(GL_r(R, p^t), \mathbb{Z}/p)$. Let $\overline{c}$ be the $\mathbb{Z}/p$-linear extension

$$C_i(GL_n(R, p^s), \mathbb{Z}/p) \rightarrow C_{i+1}(GL_r(R, p^t), \mathbb{Z}/p).$$
Consider the algebraic variety \( X_{n,i} = \mathrm{GL}_{n}^{*}/S \) over \( S \), with affine coordinate ring \( S[X_{n,i}] \). Let \( \mathcal{I}_{n,i} \subset S[X_{n,i}] \) be the ideal of functions vanishing at the unit section. Then there is a map \( S[X_{n,i}] \to \mathcal{I}_{n,i} \), sending a polynomial on \( X_{n,i} \) to its associated function, sending \( \mathcal{I}_{n,i} \) to \( \mathcal{I}_{n,i} \), and which induces an isomorphism

\[
\frac{S[X_{n,i}]}{\mathcal{I}_{n,i}} \cong \mathcal{F}_{n,i}/\mathcal{I}_{n,i} \cong S.
\]

Since the pair \((\mathcal{F}_{n,i}, \mathcal{I}_{n,i})\) is Henselian, this induces a map of the Henselization \((\mathcal{O}_{n,i}^{h}, \mathcal{L}_{n,i})\) of \( S[X_{n,i}] \) at \( \mathcal{I}_{n,i} \) to \((\mathcal{F}_{n,i}, \mathcal{I}_{n,i})\).

Because \( S \) contains the field \( \mathbb{Q} \), there are the elements from above

\[
c_{n,i} \in C_{i+1}(\mathrm{GL}(\mathcal{O}_{n,i}^{h}, \mathcal{L}_{n,i}), \mathbb{Z}/p).
\]

Let \( c'_{n,i} \) be their image in \( C_{i+1}(\mathrm{GL}(\mathcal{F}_{n,i}, \mathcal{I}_{n,i}), \mathbb{Z}/p) \). For fixed \( N \), we can find \( r \geq n \) such that all chains \( c'_{n,i} \) for \( i \leq N \) lie in \( C_{i+1}(\mathrm{GL}(\mathcal{F}_{n,i}, \mathcal{I}_{n,i}), \mathbb{Z}/p) \), and then we can find \( s \geq t \) such that \( c'_{n,i} \) are defined on \( \mathrm{GL}_{n}(R,p^{S})^{*} \) and map it to \( C_{i+1}(\mathrm{GL}(R,p^{t}), \mathbb{Z}/p) \). Using the universal construction from above, we get a null-homotopy

\[
C_{i}(\mathrm{GL}_{n}(R,p^{S}), \mathbb{Z}/p) \to C_{i+1}(\mathrm{GL}_{r}(R,p^{t}), \mathbb{Z}/p).
\]

This proves

**Proposition 4.3.** Let \( N, n, t \in \mathbb{N} \). Then there exist \( r \geq n \) and \( s \geq t \) such that the embedding \( \mathrm{GL}_{n}(R,p^{s}) \to \mathrm{GL}_{r}(R,p^{t}) \) induces the zero homomorphism on reduced homology \( \tilde{H}_{i}(\cdot, \mathbb{Z}/p) \) for \( i \leq N \).

Finally, one analyzes spectral sequences of the form

\[
H_{a}(\mathrm{GL}_{n}(R/p^{t}), H_{b}(\mathrm{GL}_{n}(R,p^{t}), \mathbb{Z}/p)) \Rightarrow H_{a+b}(\mathrm{GL}_{n}(R), \mathbb{Z}/p)
\]

for \( n \) and \( t \) going to infinity to show that the the constant pro-system \( \{H_{*}(\mathrm{GL}(R), \mathbb{Z}/p)\}_{j} \) and the pro-system \( \{H_{*}(\mathrm{GL}(R/I^{j}), \mathbb{Z}/p)\}_{j} \) are isomorphic.

**References**


