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ALGEBRAIC K-THEORY OF HENSELIAN PAIRS

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Abstract. この稿では,Hensel 対 (R, I) と様々な関手 $\mathcal{F}$ に対して $\mathcal{F}(R)$ と $\mathcal{F}(R/I)$ の関係について述べる. 例えば代数的 K-理論に対しては, m が R で可逆な時 $K_i(R, \mathbb{Z}/M) \cong K_i(R/I, \mathbb{Z}/m)$ である (Gabber, Suslin).

もし p が R で可逆でないとすると, $K_i(R, \mathbb{Z}/p)$ は一般的には逆係 $\{K_i(R/p^j, \mathbb{Z}/p)\}_j$ では決定されない. しかし, もし p が R の零因子でなく R が p-完備ならば, $K_i(R, \mathbb{Z}/p) \cong \varprojlim K_i(R/p^j, \mathbb{Z}/p)$ である. これらの定理の証明を概説する.

Abstract. In this note we study the relationship between $\mathcal{F}(R)$ and $\mathcal{F}(R/I)$ for a Henselian pair (R, I) and various functors $\mathcal{F}$. For example, in the case of algebraic K-theory, if m is invertible in R, then $K_i(R, \mathbb{Z}/m) \cong K_i(R/I, \mathbb{Z}/m)$.

If p is not invertible in R, then in general $K_i(R, \mathbb{Z}/p)$ is not determined by the inverse system $\{K_i(R/p^j, \mathbb{Z}/p)\}_j$. However, if p is not a zero-divisor and R is p-complete, then we have $K_i(R, \mathbb{Z}/p) \cong \varprojlim K_i(R/p^j, \mathbb{Z}/p)$. We sketch of the proofs of these results.

1. Henselian pairs

Definition 1.1. *Let R be a commutative ring with unit and I an ideal of R. Then (R, I) is called a Henselian pair, if the following equivalent conditions are satisfied* [1][2]*:*

1. *I is contained in the Jacobsen radical of R, and for all monic polynomials $f \in R[T]$ and factorizations $f = \bar{g}\bar{h} \mod I$ with $\bar{g}, \bar{h} \in R/I[T]$ monic and relatively prime, there is a lifting $f = gh$ of the factorization with $g, h \in R[T]$ monic.*
2. *For any finite R-algebra B, there is a bijection of idempotents*
$$\mathrm{Idem}(B) \cong \mathrm{Idem}(B/IB).$$
3. *For any étale R-algebra B, any R-map $B \to R/I$ lifts uniquely to a map $B \to R$.*
4. *For any $f \in R[T]$ and any simple root $\bar{\alpha} \in R/I$ of $f \mod I$ there is lifting to a root $\alpha \in R$ of f (note that α is called a simple root of the polynomial f if and only if $f(\alpha) = 0$ and $f'(\alpha)$ is a unit).*

In this note we want to study the relationship between $\mathcal{F}(R)$ and $\mathcal{F}(R/I)$ for various functors $\mathcal{F}$. As a first example, we have the following theorem of Gabber [1]:

Theorem 1.2. *Let (R, I) be a Henselian pair, and $\mathcal{F}$ a torsion sheaf on $(\operatorname{Spec} R)_{\text{ét}}$. Then*

$$H^q_{\text{ét}}(\operatorname{Spec} R, \mathcal{F}) \cong H^q_{\text{ét}}(\operatorname{Spec} R/I, i^*\mathcal{F}).$$

Note that if (R, I) is a Henselian local ring, then the theorem holds without the assumption that $\mathcal{F}$ is torsion and is an elementary property of étale cohomology. On the other hand, the theorem is wrong without the hypothesis that $\mathcal{F}$ is torsion. It is a general phenomenom that theorems as above only hold with torsion coefficients.

2. K-THEORY

To get interesting invariants of a ring R, generalizing for example the group of units $R^\times$ and the ideal class group $\operatorname{Pic} R$, one can study the group of matrices $\mathrm{GL}_n(R)$, or the direct limit $\mathrm{GL}(R) = \varinjlim \mathrm{GL}_n(R)$. A good method to analyze a group is to study its group homology

$$H_i(\mathrm{GL}(R), \mathbb{Z}) := H_i(\mathrm{BGL}(R), \mathbb{Z}).$$

Here the right hand side is the ordinary singular homology of a topological space, and for a group G, the topological space BG is characterized by the property that it has only one nontrivial higher homotopy group, $\pi_1(BG) = G$. In particular, we have the Hurewicz homomorphism

$$\pi_1(\mathrm{BGL}(R)) = \mathrm{GL}(R) \to H_1(\mathrm{GL}(R), \mathbb{Z}) = \mathrm{GL}(R)^{ab}.$$

Since a good invariant of the ring R should consist of abelian groups, the idea is to make the homotopy group abelian without changing the homology groups. There is a universal construction to achieve this, called +-construction. In other words, there is a topological space $\mathrm{BGL}(R)^+$ characterized by the properties that for any abelian group A we have

$$H_i(\mathrm{BGL}(R), A) = H_i(\mathrm{BGL}(R)^+, A)$$

and

$$\pi_1(\mathrm{BGL}(R))^{ab} = \pi_1(\mathrm{BGL}(R)^+).$$

This changes the higher homotopy groups, and we define K-theory as the higher homotopy groups of this space

$$K_i(R) := \pi_i(\mathrm{BGL}(R)^+)$$

and similar with coefficients in an abelian groups A,

$$K_i(R, A) := \pi_i(\mathrm{BGL}(R)^+, A).$$

If $A = \mathbb{Z}/m$, then there is an exact sequence

$$0 \longrightarrow K_i(R)/m \longrightarrow K_i(R, \mathbb{Z}/m) \longrightarrow {}_mK_{i-1}(R) \longrightarrow 0,$$

and if $A = \mathbb{Q}$ then $K_i(R, \mathbb{Q}) \cong K_i(R) \otimes \mathbb{Q}$. Note that it is possible to recover $K_*(R)$ from $K_*(R, \mathbb{Q})$ and $K_*(R, \mathbb{Z}/m)$ for all m.

For example, by definition $K_1(R) = \mathrm{GL}(R)^{ab}$, and the determinant homomorphism $\mathrm{GL}(R) \longrightarrow R^\times$ together with the inclusion $R^\times \longrightarrow \mathrm{GL}(R)$, sending a unit r to a matrix with r in the upper left corner, shows that the units $R^\times$ form a split direct summand of $K_1(R)$. In fact, $K_1(R) = R^\times$ if R is local.

As another example, for a field F, $K_2(F)$ has generators $F^\times \otimes F^\times$ and relations $a \otimes (1 - a) = 0$.

3. K-theory and group homology of Henselian pairs

The K-theory and group homology of Henselian pairs have been studied by Gabber, Suslin and Panin. Let (R, I) be a Henselian pair. Define $\mathrm{GL}(R, I)$ as the kernel of the (surjective) reduction map $\mathrm{GL}(R) \longrightarrow \mathrm{GL}(R/I)$, and let $\bar{H}_*$ be reduced homology (i.e. removing the copy of $\mathbb{Z}$ in degree zero). By definition, K-theory and homology of $\mathrm{GL}(R)$ are closely related. For a Henselian pair, this takes the following form:

Proposition 3.1. *a)* [5] *Let (R, I) be a Henselian pair and m invertible in R. Then the following statements are equivalent:*

1. $K_*(R, \mathbb{Z}/m) = K_*(R/I, \mathbb{Z}/m)$.
2. $H_*(\mathrm{GL}(R), \mathbb{Z}/m) = H_*(\mathrm{GL}(R/I), \mathbb{Z}/m)$.
3. $\bar{H}_*(\mathrm{GL}(R, I), \mathbb{Z}/m) = 0$.

b) [4] *If m is not invertible in R, then we still have an equivalence:*

1. *The pro-system $\{K_*(R/I^j, \mathbb{Z}/m)\}_j$ is isomorphic to the constant pro-systen $\{K_*(R, \mathbb{Z}/m)\}_j$.*
2. *The pro-system $\{H_*(\mathrm{GL}(R/I^j), \mathbb{Z}/m)\}_j$ is isomorphic to the constant pro-systen $\{H_*(\mathrm{GL}(R), \mathbb{Z}/m)\}_j$.*

Recall that a map $(\phi_i) : (X_i) \longrightarrow (Y_i)$ of pro-systems is an isomorphism if and only if for all i there is a $j > i$ and a map $s : Y_j \longrightarrow X_i$ making the obvious diagram commutative.

It is a deep theorem in algebraic K-theory that the conditions of (a) are satisfied:

Theorem 3.2. *(Gabber, Suslin) Let (R, I) be a Henselian pair, and m invertible in R. Then*

$$K_i(R, \mathbb{Z}/m) \cong K_i(R/I, \mathbb{Z}/m).$$

The proof consists of three equally difficult steps:

1. The special case R is the Henselian local ring of a smooth variety over a field F in an F-rational point [2][3].
2. Apply this to the Henselization of $\mathrm{GL}_n^{\times i}$ in the unit section to prove the theorem for R containing a field [5].
3. Reduce the general case to the case R containing a field F [2].

Here is the main idea of the second step:

Let (R, I) be a Henselian pair containing a field F. The group homology $H_*(G, \mathbb{Z}/m)$ can be calculated using the complex $C_*(G, \mathbb{Z}/m)$ which in degree i is the free abelian group generated by i-tuples of elements of G, $[g_1, \dots, g_i]$. Since every element of $\mathrm{GL}(R, I)$ is contained in $\mathrm{GL}_n(R, I)$ for some n, it suffices to show according to Proposition 3.1 that the inclusion

$$C_*(\mathrm{GL}_n(R, I), \mathbb{Z}/m) \xrightarrow{\iota} C_*(\mathrm{GL}(R, I), \mathbb{Z}/m)$$

induces the zero map on homology for all n. This holds if the map is null-homotopic, i.e. if we can construct maps

$$C_i(\mathrm{GL}_n(R, I), \mathbb{Z}/m) \xrightarrow{s} C_{i+1}(\mathrm{GL}(R, I), \mathbb{Z}/m)$$

such that $d \circ s + s \circ d = \iota$.

Consider the algebraic variety $\mathrm{GL}_n(F)^{\times i}$. It is smooth over F and has a distinuished F-rational point e, the unit section. Let $\mathcal{O}^h_{n,i}$ be the Henselian local ring at e, and $\mathfrak{m}$ the corresponding maximal ideal.

Let $\beta = [\beta_1, \dots .\beta_i] \in C_i(\mathrm{GL}_n(R, I), \mathbb{Z}/m)$. Each element β_j defines a morphism $\operatorname{Spec} R \longrightarrow \mathrm{GL}_n$ sending the subvariety defined by I to the unit section. Hence β defines a morphism $\operatorname{Spec} R \longrightarrow \mathrm{GL}_n^{\times i}$ with the same property. Since (R, I) is Henselian, this induces a homomorphism $\mathcal{O}^h_{n,i} \longrightarrow R$ sending $\mathfrak{m}$ to I. In particular, we get a map

$$\beta^* : C_*(\mathrm{GL}(\mathcal{O}^h_{n,i}, \mathfrak{m}), \mathbb{Z}/m) \longrightarrow C_*(\mathrm{GL}(R, I), \mathbb{Z}/m).$$

Since by step (1) the theorem is known for $\mathcal{O}^h_{n,i}$, we have

$$K_*(\mathcal{O}^h_{n,i}, \mathbb{Z}/m) \cong K_*(\mathcal{O}^h_{n,i}/\mathfrak{m}, \mathbb{Z}/m)$$

and hence by Proposition 3.1

$$\bar{H}_*(\mathrm{GL}(\mathcal{O}^h_{n,i}, \mathfrak{m}), \mathbb{Z}/m) = 0.$$

This property can be used to construct inductively elements $c_{n,i} \in C_{i+1}(\mathrm{GL}(\mathcal{O}^h_{n,i}, \mathfrak{m}), \mathbb{Z}/m)$ independent of R such that $s(\beta) := \beta^*(c_{n,i})$ is the desired null-homotopy.

4. The general case

If m is not invertible in R, then in general $K_*(R, \mathbb{Z}/m) \neq K_*(R/I, \mathbb{Z}/m)$. One can ask if at least $K_*(R, \mathbb{Z}/m)$ is determined by the pro-system $\{K_*(R/I^j, \mathbb{Z}/m)\}_j$. But this is also wrong in general. For example, let $R = \mathbb{F}_p[X]^h$ be the Henselization of the affine line over a finite field in the origin and $\hat{R} = \mathbb{F}_p[[X]]$ its completion. Then $R/I^j = \hat{R}/I^j$ for all j, but

$$K_1(R, \mathbb{Z}/p) = R^\times/p \neq \hat{R}^\times/p = K_1(\hat{R}, \mathbb{Z}/p)$$

because the former is countable and the latter is uncountable.

However, the idea of the proof of Theorem 3.2 can be used to prove the following:

Theorem 4.1. *Let R be a noetherian ring such that p is not a zero divisor, such that the map from R to the p-completion $\hat{R}$ is regular, and such that (R, p) is a Henselian pair. Then*

$$K_i(R, \mathbb{Z}/p) \cong \varprojlim K_i(R/p^j, \mathbb{Z}/p).$$

An integral domain of characteristic 0 which is complete for the p-adic topology, or the Henselization at a point of the closed fiber of a reduced variety of finite type over a discrete valuation ring satisfies the hypothesis of this theorem. The theorem is a generalization of the special case R a Henselian valuation ring of mixed characteristic. Except the following essential new ingredient, the proof goes back to [5].

Let $S = R[\frac{1}{p}]$, equipped with the p-adic topology. The hypothesis implies that S contains $\mathbb{Q}$.

Proposition 4.2. *Let $e \in X$ a pointed toplogical space and $\mathcal{F}$ be the ring of germs of continuous functions from X to S. Let $\mathcal{I} \subset \mathcal{F}$ be the ideal of germs of functions vanishing at e. Then $(\mathcal{F}, \mathcal{I})$ is a Henselian pair.*

The proof of the proposition will be published in a forthcoming paper. To continue the proof of the theorem, consider $\mathrm{GL}_n(S)^{\times i}$ as a topological space with the p-adic topology. Let $\mathcal{F}_{n,i}$ be the ring of germs of continuous S-valued functions defined in a neighborhood of the unit element e, and let $\mathcal{I}_{n,i}$ be the ideal of germs of functions vanishing at e. We are going to construct a homotopy as above.

Every chain $c \in C_{i+1}(\mathrm{GL}_r(\mathcal{F}_{n,i}, \mathcal{I}_{n,i}), \mathbb{Z}/p)$ defines a map of some neighborhood of $e \in \mathrm{GL}_n(S)^{\times i}$ to $C_{i+1}(\mathrm{GL}_r(S), \mathbb{Z}/p)$ which is continuous, i.e. for each t there is an s such that c is defined on $\mathrm{GL}_n(R, p^s)^{\times i}$ and maps it to $C_{i+1}(\mathrm{GL}_r(R, p^t), \mathbb{Z}/p)$. Let $\bar{c}$ be the $\mathbb{Z}/p$-linear extension

$$C_i(\mathrm{GL}_n(R, p^s), \mathbb{Z}/p) \longrightarrow C_{i+1}(\mathrm{GL}_r(R, p^t), \mathbb{Z}/p).$$

Consider the algebraic variety $X_{n,i} = \mathrm{GL}_n^{\times i}/S$ over S, with affine coordinate ring $S[X_{n,i}]$. Let $\mathcal{J}_{n,i} \subset S[X_{n,i}]$ be the ideal of functions vanishing at the unit section. Then there is a map $S[X_{n,i}] \longrightarrow \mathcal{F}_{n,i}$, sending a polynomial on $X_{n,i}$ to its associated function, sending $\mathcal{J}_{n,i}$ to $\mathcal{I}_{n,i}$, and which induces an isomorphism

$$S[X_{n,i}]/\mathcal{J}_{n,i} \xrightarrow{\sim} \mathcal{F}_{n,i}/\mathcal{I}_{n,i} \cong S.$$

Since the pair $(\mathcal{F}_{n,i}, \mathcal{I}_{n,i})$ is Henselian, this induces a map of the Henselization $(\mathcal{O}^h_{n,i}, \mathcal{L}_{n,i})$ of $S[X_{n,i}]$ at $\mathcal{J}_{n,i}$ to $(\mathcal{F}_{n,i}, \mathcal{I}_{n,i})$.

Because S contains the field $\mathbb{Q}$, there are the elements from above

$$c_{n,i} \in C_{i+1}(\mathrm{GL}(\mathcal{O}^h_{n,i}, \mathcal{L}_{n,i}), \mathbb{Z}/p).$$

Let $c'_{n,i}$ be their image in $C_{i+1}(\mathrm{GL}(\mathcal{F}_{n,i}, \mathcal{I}_{n,i}), \mathbb{Z}/p)$. For fixed N, we can find $r \geq n$ such that all chains $c'_{n,i}$ for $i \leq N$ lie in $C_{i+1}(\mathrm{GL}_r(\mathcal{F}_{n,i}, \mathcal{I}_{n,i}), \mathbb{Z}/p)$, and then we can find $s \geq t$ such that $c'_{n,i}$ are defined on $\mathrm{GL}_n(R, p^s)^{\times i}$ and map it to $C_{i+1}(\mathrm{GL}_r(R, p^t), \mathbb{Z}/p)$. Using the universal construction from above, we get a null-homotopy

$$C_i(\mathrm{GL}_n(R, p^s), \mathbb{Z}/p) \longrightarrow C_{i+1}(\mathrm{GL}_r(R, p^t), \mathbb{Z}/p).$$

This proves

Proposition 4.3. *Let $N, n, t \in \mathbb{N}$. Then there exist $r \geq n$ and $s \geq t$ such that the embedding $\mathrm{GL}_n(R, p^s) \longrightarrow \mathrm{GL}_r(R, p^t)$ induces the zero homomorphism on reduced homology $\bar{H}_i(-, \mathbb{Z}/p)$ for $i \leq N$.*

Finally, one analyzes spectral sequences of the form

$$H_a(\mathrm{GL}_n(R/p^t), H_b(\mathrm{GL}_n(R, p^t), \mathbb{Z}/p)) \Rightarrow H_{a+b}(\mathrm{GL}_n(R), \mathbb{Z}/p)$$

for n and t going to infinity to show that the the constant pro-system $\{H_*(\mathrm{GL}(R), \mathbb{Z}/p)\}_j$ and the pro-system $\{H_*(\mathrm{GL}(R/I^j), \mathbb{Z}/p)\}_j$ are isomorphic.

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