数理解析研究所講究録
1155 巻 2000 年 194-207

A CONDITIONAL STABILITY ESTIMATE FOR AN INVERSE NEUMANN BOUNDARY PROBLEM

J. CHENG, Y.C. HON, AND M. YAMAMOTO
(程晋, 韓耀宗, 山本昌宏)

ABSTRACT. In this paper, we consider an inverse problem of determining an unknown boundary in $\mathcal{R}^2$ where a Neumann condition is imposed. A stability estimate under some a-priori assumptions on the unknown boundary and the solution of the problem is obtained. The proofs are based on using the complex extension method, an estimation of harmonic measure, and our recent stability estimation results for an inverse boundary problem of Laplace's equation on non-smooth domain.

1. INTRODUCTION

In the last decade, the technique of non-destructive testing has been developed and applied to determine the shape of a being corroded part of an unknown boundary by a suitable observation on the other part of the boundary which is accessible. This is a well known inverse boundary determination problem which arises from the engineering industry. In this paper, we consider the uniqueness and stability of an inverse problem in determining an inaccessible boundary, where a Neumann condition is imposed, from an accessible boundary, where Cauchy data in terms of electrostatic measurements can be obtained.

Let Ω be a simply connected bounded domain in $\mathcal{R}^2$ with Lipschitz continuous boundary $\partial\Omega$. Assume that Γ and L are two distinct parts of $\partial\Omega$ which satisfy

---

Date: March 17, 2000.

1991 *Mathematics Subject Classification.* 35R.

Key words and phrases. determining unknown boundary, Neumann boundary condition, conditional stability estimate, non-destructive testing.

The first named author is partially supported by NSF of China. The second named author is supported by the Research Grant Council of the Hong Kong Special Administrative Region, China of project No. 9040428. The third named author is supported partially by the Sanwa Systems Development Co. Ltd. (Tokyo, Japan).

$\Gamma \cap L = \emptyset$ (it is not necessary that $\Gamma \cup L = \partial\Omega$). Suppose that L is an accessible boundary on which we can access the boundary measurements and Γ is an inaccessible boundary to be determined. It is then natural to assume a Neumann condition on the unknown boundary Γ if it has been corroded. From [3], the model can be posed by assuming that the charge potential function $u = u(x)$ satisfies the following Laplace's equation in Ω:

$$\Delta u = 0, \qquad x \in \Omega. \tag{1.1}$$

On the accessible boundary L, we have

$$u = f, \qquad x \in L, \tag{1.2}$$

$$\frac{\partial u}{\partial n} = g, \qquad x \in L, \tag{1.3}$$

where n is the outer unit normal on $\partial\Omega$. On the inaccessible boundary Γ, we assume

$$\frac{\partial u}{\partial n} = 0, \qquad x \in \Gamma. \tag{1.4}$$

The inverse boundary problem is to determine Γ from f and g.

In this paper, we will discuss the uniqueness and stability of this inverse problem. This paper is motivated by a number of recent results in the applications of non-destructive testing technique (we refer to [3], [4], [16], [21] and [22]). Particularly, an inverse problem in determining an unknown boundary was proposed in our recent work [6] where a log-type conditional stability estimate was given. The major improvement is that a zero-Dirichlet condition on the unknown boundary was imposed in [6] whereas a zero-Neumann condition was imposed in this paper. Based on the zero-Dirichlet condition, the proofs on uniqueness and conditional stability were comparatively easier to obtain by using the maximum principle for Laplace's equation without a regularity assumption placed on the domain. The zero-Neumann condition on the unknown boundary, however, is more reasonable from the view point of practical applications. This kind of inverse Neumann boundary problem was also discussed in [3] under some special assumptions. In this paper, we will discuss this inverse problem on non-smooth domain under a more general assumption. The results of uniqueness and conditional stability estimate will first be stated in the following Section 2. The detail proofs are then given in Section

3. Section 4 includes the conclusion and some remarks. It is noted here that the proofs are also extendable to three dimensional problems.

2. Main Results

Let Ω_1, Ω_2 be two simply connected bounded domains in $\mathcal{R}^2$. Assume that $0 < a < b < 1$. For arbitrarily fixed $\alpha > 0$, $\beta > 0$, $m_0 > 0$ and $m_1 > 0$, define

$$
\begin{aligned}
(2.1)\ \mathcal{F} &= \mathcal{F}(\alpha, \beta, m_0, m_1) \\
&= \{F \in C[0,1] |\ \ F(x) = \alpha \quad, 0 < x < a; \quad F(x) = \beta \quad, b < x < 1; \\
&\qquad |F(x) - F(y)| \leq m_1 |x - y|, \quad F(x) \geq m_0, \quad x, y \in [0,1]\},
\end{aligned}
$$

$$
\Omega_1 = \{(x,y) |\ \ 0 < x < 1, \quad 0 < y < F_1(x)\}, \tag{2.2}
$$

$$
\Omega_2 = \{(x,y) |\ \ 0 < x < 1, \quad 0 < y < F_2(x)\}, \tag{2.3}
$$

where $F_1, F_2 \in \mathcal{F}$. In other words, $\Gamma_j = \{(x,y) |\ \ y = F_j(x), \quad a < x < b\}, j = 1,2$ are two inaccessible boundaries given by the Lipschitz continuous functions. The a-priori information $F_1, F_2 \in \mathcal{F}$ means that the shapes of the unknown boundaries are not too complicated.

For the accessible boundary, we let

$$
L = \{(x,0) |\ \ a < x < b\}. \tag{2.4}
$$

Assume that u_j, $j = 1,2$ satisfy

$$
\Delta u_j(x,y) = 0, \qquad (x,y) \in \Omega_j, \tag{2.5}
$$

$$
\frac{\partial u_j}{\partial n}(x,y) = 0, \qquad (x,y) \in \Gamma_j, \tag{2.6}
$$

and

$$
u_j(x,0) = f_j(x), \quad \frac{\partial u_j}{\partial y}(x,0) = g_j(x), \quad c < x < d. \tag{2.7}
$$

The main results of this paper are stated in the following:

Theorem 2.1. *Suppose that $g_j \neq 0$, $j = 1,2$ and*

$$
f_1 = f_2, \qquad g_1 = g_2, \qquad on \quad L, \tag{2.8}
$$

we then have $\Gamma_1 = \Gamma_2$.

Remark 2.1. *In* [3], *the assumption* $g_j \neq 0$ *was missed. However, under the zero-Neumann condition on the unknown boundary, if this assumption is not true, then the uniqueness cannot be obtained simply because* u_j *can be any constant.*

Theorem 2.2. *Let* $m_2, m_3 > 0$, $0 < \alpha < 1$ *and* $M > 0$ *be arbitrarily fixed and assume that*

$$|\frac{\partial f_j}{\partial x}(x_0)| + |g(x_0)| \geq m_2, \qquad j = 1, 2, \tag{2.9}$$

where $x_0 \in [a, b]$ *is a fixed constant. Furthermore, asssume that*

$$u_j \in C^2(\Omega_j) \bigcap C^1(\overline{\Omega}_j), \quad \|u_j\|_{C^1(\overline{\Omega}_j)} \leq M, \quad j = 1, 2, \tag{2.10}$$

and

$$\|u_j\|_{C^{1+\alpha}((a,b)\times(0,m_0))} \leq m_3, \qquad j = 1, 2. \tag{2.11}$$

Then there exists constants $C = C(m_0, m_1, m_2, m_3, \alpha, M) > 0$ *and* $0 < \tau < 1$ *such that*

$$\|F_1 - F_2\|_{C[a,b]} \leq C[\frac{1}{\log(\log \frac{1}{\epsilon})}]^\tau, \tag{2.12}$$

where $\epsilon = \|f_1 - f_2\|_{H^1(a,b)} + \|g_1 - g_2\|_{L^2(a,b)}$.

Before going to the next section on the detail proofs, we would like to give two examples to indicate that the assumptions given in Theorem 2.2 are necessary for obtaining a conditional stability estimation (2.12).

Example 2.1. *Let*

$$\Omega_1 = (0, \pi) \times (0, 1),$$
$$u_1(x, y) = \frac{1}{n^2}[e^{-ny} + e^{-2n}e^{ny}] \cos nx,$$

and

$$\Omega_2 = (0, \pi) \times (0, 1 + \delta y),$$
$$u_2(x, y) = \frac{1}{n^2}[e^{-ny} + e^{-2n(1+\delta y)}e^{ny}] \cos nx.$$

It is easy to verify that $u_i(x, y$ are harmonic functions in $\Omega_i, i = 1, 2$ and

$$\frac{\partial u_1}{\partial x}(0, y) = \frac{\partial u_1}{\partial x}(\pi, y) = \frac{\partial u_1}{\partial y}(x, 1) = 0,$$
$$\frac{\partial u_1}{\partial x}(0, y) = \frac{\partial u_1}{\partial x}(\pi, y) = \frac{\partial u_1}{\partial y}(x, 1 + \delta y) = 0.$$

Here the fixed boundary is $\{(x,0)|0 < x < \pi\}$ and the remained parts are the unknown boundaries.

The distance between the two unknown boundaries is

$$d(\Gamma_1, \Gamma_2) = \delta y,$$

where Γ_i is the unknown boundary of $\partial\Omega_i$.

The Cauchy data are

$$\begin{aligned}\epsilon &= \max_{0<x<\pi}\{|u_1(x,0) - u_2(x,0)| + \max_{0<x<\pi}\{|\frac{\partial u_1}{\partial y}(x,0) - \frac{\partial u_2}{\partial y}(x,0)|, \\ &= \frac{1}{n^2}(e^{-2n} - e^{-2n(1+\delta y)}) + \frac{1}{n}(e^{-2n} - e^{-2n(1+\delta y)}).\end{aligned}$$

It can be shown that u_i tends to 0 when n tends to infinity. This means that u_i does not satisfy the assumption (2.9). Furthermore, if $\delta y = \frac{1}{\ln n}$, we can obtain a double logarithmic estimation. If, however, we choose $\delta y = \frac{1}{\ln \ln n}$, we obtain a weaker estimation.

This example indicates that the estimation can be extremely weak if the assumption in Theorem 2.2 is not imposed.

Example 2.2. *Let Ω_1 and Ω_2 are the same as in Example 1. Consider the following harmonic functions*

$$u_1(x,y) = [e^{-ny} - e^{-2n}e^{ny}]\cos nx,$$

and

$$u_2(x,y) = [e^{-ny} - e^{-2n(1+\delta y}e^{ny}]\cos nx.$$

The distance δy between the two unknown boundaries Γ_1 and Γ_2 is the same as in Example 1.

The Cauchy data are

$$\begin{aligned}\epsilon &= \max_{0<x<\pi}\{|u_1(x,0) - u_2(x,0)| + \max_{0<x<\pi}\{|\frac{\partial u_1}{\partial y}(x,0) - \frac{\partial u_2}{\partial y}(x,0)|, \\ &= (e^{-2n} - e^{-2n(1+\delta y)}) + n(e^{-2n} - e^{-2n(1+\delta y)}).\end{aligned}$$

It is easy to check that $||u_i||_{C^2}$ is unbounded when n tends to infinity. Again, if $\delta y = \frac{1}{\ln n}$, we have a double logarithmic estimation. If $\delta y = \frac{1}{\ln \ln n}$, a weaker estimation can be obtained.

3. Proofs of the Main Results

3.1. **Some Lemmata.** We need the following lemmata:

Lemma 3.1. *Suppose that $(x_0, F_1(x_0)) \in \overline{\Gamma}_1$. Then there exists positive constant C_3 and $0 < \beta < 1$, which are independent of y, satisfying*

$$|u_2(x_0, y) - u_1(x_0, y)| \leq C\epsilon^\beta, \qquad y \in [0, \frac{m_0}{2}]. \tag{3.1}$$

Proof. The proof follows from the results given in Payne [25]. □

Now, define

$$D = \{z = x + iy \in \mathbb{C} | \quad |z| < R, |\arg z| < \theta\},$$

and

$$l = \{z = x_1 | x_1 \in (\rho_1, \rho_2)\} \subset D$$

.

Definition 3.1. *$\psi(z)$ is called a harmonic measure for D and l if it satisfies*

$$\begin{aligned} \Delta\psi &= 0, & z &\in D \setminus l, \\ \psi &= 0, & z &\in \partial D, \\ \psi &= 1, & z &\in l. \end{aligned}$$

For the existence and uniqueness of this harmonic measure, we refer to Kellog's book [17]. By using the same method in [11], we can prove that $\psi \in C^\nu(\overline{D})$ $(0 < \nu < 1)$.

Lemma 3.2. *For the harmonic measure ψ for D and l, we have the following estimation*

$$\psi(x) \geq C((\frac{\rho_2}{x})^{\frac{\pi}{2\theta}} - (\frac{\rho_2}{R})^{\frac{\pi}{2\theta}}), \qquad x \in (\rho_2, R), \tag{3.2}$$

where C is a constant which is independent of x.

Moreover, if ρ_1 is sufficiently small, the constant C can be independent of ρ_1 and ρ_2.

Proof. The proof can be found in [9]. □

By using the estimation for the harmonic measure, we can have the following conditional stability estimation for a holomorphic function in D.

Lemma 3.3. *Suppose that* $v = v(z)$ *is a holomorphic function in* D *and let* $\varepsilon = \max_{x\in[\rho_1,\rho_2]} |v(x)|$. *If* $|v(z)| \leq M_1$, $z \in D$, *then we have*

$$|v(x)| \leq M_1(\frac{\varepsilon}{M_1})^{C((\frac{\rho_2}{x})^{\frac{\pi}{2\theta}}-(\frac{\rho_2}{R})^{\frac{\pi}{2\theta}})}, \qquad x \in [\rho_2, R].$$

Proof. The proof can be found in [9]. □

We now state some results concerning a Neumann problem for Laplace's equation on a Lipschitz domain.

Let D be a Lipschitz domain in $\mathcal{R}^2$ with boundary ∂D. Consider the following Neumann problem

$$\Delta w(x,y) = 0, \quad in \quad D, \tag{3.3}$$

$$\frac{\partial w}{\partial n} = h, \quad on \quad \partial D, \tag{3.4}$$

where $\int_{\partial D} h d\sigma = 0$ and (3.4) is satisfied in a generalized sense (refer to [15]).

Lemma 3.4. *Suppose that* $h \in L^2(\partial D)$ *and* $\int_{\partial D} h d\sigma = 0$. *There exists a unique harmonic function* w *such that*

(i) In a generalized sense, w *satisfies*

$$\frac{\partial w}{\partial n} = h, \quad on \quad \partial D. \tag{3.5}$$

(ii) w *can be expressed as*

$$w(x,y) = \int_{\partial D} \ln[(x-\xi_1)^2 + (y-\xi_2)^2] q(\xi_1,\xi_2) d\sigma(\xi), \tag{3.6}$$

where q *satisfies*

$$||q||_{L^2(\partial D)} \leq C||h||_{L^2(\partial D)}. \tag{3.7}$$

Here, the constant $C > 0$ *depends only on the Lipschitz character of* D.

(iii) Every solution can be expressed as

$$w(x,y) = \int_{\partial D} \ln[(x-\xi_1)^2 + (y-\xi_2)^2] q(\xi_1,\xi_2) d\sigma(\xi) + c, \tag{3.8}$$

where c *is a constant.*

Proof. The proof of this lemma can be found in [15]. □

3.2. **Proof of Theorem 2.1.**

Proof. Suppose that $\Gamma_1 \neq \Gamma_2$. Without loss of generality, there is an interval $(a_1, b_1) \subset (a, b)$ such that

$$F_2(x) > F_1(x), \qquad x \in (a_1, b_1),$$

and

$$F_2(a_1) = F_1(a_1), \qquad F_1(b_1) = F_2(b_1).$$

Here,

$$\mathcal{D} = \{(x, y) \in \mathcal{R}^2 | \quad F_1(x) < y < F_2(x); \quad x \in (a_1, b_1)\}.$$

Since $f_1 = f_2$ and $g_1 = g_2$, by the uniqueness of the Cauchy problem for Laplace's equation, we have

$$u_1(x, y) = u_2(x, y), \qquad (x, y) \in \Omega_1 \cap \Omega_2. \tag{3.9}$$

Therefore, for $x \in (a_1, b_1)$ and $0 < y < F_1(x)$, we obtain

$$\nabla u_1(x, y) = \nabla u_2(x, y). \tag{3.10}$$

Using the boundary condition for u_1 on Γ_1, we further have

$$\frac{\partial u_2}{\partial \nu} = 0, \qquad on \quad \{(x, y) \in \mathcal{R}^2 | \quad y = F_1(x), \quad x \in (a_1, b_1)\}, \tag{3.11}$$

where ν is the unit outer normal on $\mathcal{D}$.

By considering u_2 for $\mathcal{D}$, it follows that u_2 is harmonic in $\mathcal{D}$ and $\frac{\partial u_2}{\partial \nu} = 0$ for $(x, y) \in \partial\mathcal{D} \setminus \{(a_1, F_1(a_1)), (b_1, F_2(b_1))\}$.

It can be verified that the Green's formula is also true for the non-smooth domain $\mathcal{D}$, i.e.,

$$\int_{\mathcal{D}} \Delta u_2 u_2 dxdy = -\int_{\mathcal{D}} |\nabla u_2|^2 dxdy + \int_{\partial\mathcal{D}} \frac{\partial u_2}{\partial \nu} u_2 d\sigma. \tag{3.12}$$

Therefore, we have

$$\nabla u_2(x, y) = 0, \qquad (x, y) \in \mathcal{D}. \tag{3.13}$$

By the unique continuation for the Laplace's equation, we have

$$\nabla u_2(x, y) = 0, \qquad (x, y) \in \Omega_2, \tag{3.14}$$

and

$$g_2(x,y) = 0, \qquad (x,y) \in L. \tag{3.15}$$

This is a contradiction to the assumption $g_j \neq 0$. The proof is then complete. □

3.3. **Proof of Theorem 2.2.**

Proof. Without loss of generality, we assume that $|F_2(x) - F_1(x)|$ attends its maximum at $x = x^* \in (a,b)$ and $d = F_2(x^*) - F_1(x^*) > 0$.

From the assumption $F_j \in \mathcal{F}$, there is an interval $(a_2, b_2) \subset (a,b)$ such that

$$F_2(x) - F_1(x) > 0, \qquad x \in (a_2, b_2), \tag{3.16}$$

and

$$F_2(a_2) = F_1(a_2), \qquad F_2(b_2) = F_1(b_2). \tag{3.17}$$

Let η be a small positive constant. Define

$$\mathcal{U} = \{(x,y)| \ \ F_1(x) < y < F_2(x), \quad a_2 < x < b_2\}, \tag{3.18}$$

and

$$\mathcal{U}_\eta = \{(x,y)| \ \ F_1(x) < y < F_2(x), \quad a_2 + \eta < x < b_2 - \eta\}. \tag{3.19}$$

For $j = 1, 2$, denote

$$\gamma_j = \mathcal{U} \cap \Gamma_j, \tag{3.20}$$

and

$$\gamma_j^\eta = \mathcal{U}_\eta \cap \Gamma_j. \tag{3.21}$$

Proposition 3.1. *The domain $\mathcal{U}_\eta$ is a Lipschitz domain in $\mathcal{R}^2$ and the Lipschitz constant is less than* $\max\{1, m_1\}$.

This proposition can be obtained directly from $\partial\mathcal{U}_\eta$ which can be expressed locally by some Lipschitz functions whose Lipschitz constants are all less than $\max\{1, m_1\}$.

Remark 3.1. *The domain $\mathcal{U}$ may not be a Lipschitz domain because the points $(a_2, F_1(a_2))$ and $(b_2, F_1(b_2))$ can be cusp points.*

We proceed our proof in the following three steps:

Step 1: Boundary value estimation for $\frac{\partial u_2}{\partial \nu}$ on γ_1. (Here ν is the unit outer normal for the domain $\mathcal{U}$ or $\mathcal{U}_\eta$).

In [10], we had proven the following results:

Lemma 3.5. *Under the assumptions given in Theorem 2.2, there exist constants $C > 0$ and $0 < \tau_1 < 1$ which depend on m_j, $j = 0, 1, 2, 3$ and M such that*

$$|\nabla u_2 - \nabla u_1| \leq C(\frac{1}{\ln \frac{1}{\epsilon}})^{\tau_1}. \tag{3.22}$$

Since $\frac{\partial u_1}{\partial n} = 0$ on Γ_1, by Lemma 3.5, we have

Lemma 3.6. *Under the assumptions given in Theorem 2.2, there exist constants $C > 0$ and $0 < \tau_1 < 1$ which depend on m_j, $j = 0, 1, 2, 3$ and M such that*

$$|\frac{\partial u_2}{\partial \nu}| \leq C(\frac{1}{\ln \frac{1}{\epsilon}})^{\tau_1}. \tag{3.23}$$

Step 2: Estimation of d.

Let $\delta = C(\frac{1}{\ln \frac{1}{\epsilon}})^{\tau_1}$.

We consider the following Neumann problem for Laplace's equation in $\mathcal{U}_\eta$:

$$\Delta w = 0, \quad in \quad \mathcal{U}_\eta, \tag{3.24}$$

$$\frac{\partial w}{\partial w} = \frac{\partial u_2}{\partial \nu}, \quad on \quad \partial \mathcal{U}_\eta. \tag{3.25}$$

It is obvious that u_2 is one the solutions for this Neumann problem.

Let ϱ be a small positive parameter. Define

$$\mathcal{U}_\eta^\varrho = \{(x, y) \in \mathcal{U}_\eta | dist((x, y), \partial \mathcal{U}_\eta) > \varrho\}.$$

Clearly, we have

$$|\nabla u_2(x, y)| \leq \frac{C}{\varrho} ||u_2||_{L^2(\partial \mathcal{U}_\eta)}, \qquad (x, y) \in \mathcal{U}_\eta^\varrho, \tag{3.26}$$

where $C > 0$ is a constant which depends on m_1.

Calculating $||u_2||_{L^2(\partial \mathcal{U}_\eta)}$, we have

$$||u_2||_{L^2(\partial \mathcal{U}_\eta)} \leq C_1 \delta + C_2 \eta, \tag{3.27}$$

where $C_1 > 0$ and $C_2 > 0$ are constants which depend on m_1 and M.

From the assumptions (2.9) and (2.11), there is a positive constant ρ which depends on M and m_2 such that

$$|\frac{\partial u_2}{\partial x}(x_0,\rho)|+|\frac{\partial u_2}{\partial y}(x_0,\rho)|\geq\frac{m_2}{2}. \tag{3.28}$$

We choose $\delta_1 = C_1\delta + C_2\eta$ and let $v(z) = \frac{1}{2}(\frac{\partial}{\partial x}+i\frac{\partial}{\partial y})u_2 = \partial_{\bar z}u_2$. It is easy to verify that $v(z)$ is a holomorphic function in Ω_2 and $|v(z)|\leq M, \quad z\in\Omega_2$.

From the assumptions on Ω_2, there is a positive constant θ which depends on M such that

$$\mathcal{V}=\{(x,y)|0<y<r(x),a<x<b\}\subset\Omega_2, \tag{3.29}$$

and

$$\mathcal{V}_1=\{(x,y)|r_1(x)<y<r(x),a<x<b\}\subset\mathcal{U}_\eta. \tag{3.30}$$

Here

$$r(x)=\begin{cases}\tan\theta(x-x^*)+F_2(x^*)-\varrho, & x<x^*,\\ -\tan\theta(x-x^*)+F_2(x^*)-\varrho, & x\geq x^*,\end{cases} \tag{3.31}$$

and

$$r_1(x)=\begin{cases}-\tan\theta(x-x^*)+F_2(x^*)-d+\varrho, & x<x^*,\\ \tan\theta(x-x^*)+F_2(x^*)-d+\varrho, & x\geq x^*.\end{cases} \tag{3.32}$$

From Lemma 3.3, there exists a constant κ which depends on m_j and M such that

$$|v(x,\rho)|\leq C_1\delta_1^{C_2(d-2\varrho)^\alpha}\equiv\delta_2, \qquad x\in[x^*-\kappa,x^*+\kappa] \tag{3.33}$$

where $0<\alpha<1$, $C_1>0$ and $C_2>0$ are constants which depend on M and m_j.

From the results given in [10], there exist positive constants $C_3>0$ and $0<\beta<1$ which depend on M, κ, ρ and x_0 such that

$$|v(x_0,\rho)|\leq C_3\delta_2^{\beta(d-2\varrho)^\alpha}. \tag{3.34}$$

From equation (3.28), we have

$$d\leq 2\varrho+C_4(\frac{1}{\ln\frac{1}{\delta_1}})^{\tau_2}, \tag{3.35}$$

where $C_4>0$ and $0<\tau_2<1$ are constants which depend on m_j and M.

Let $\eta \to 0$. It is now clear that

$$d \le 2\varrho + C_5\left(\frac{1}{\ln[\varrho \ln \frac{1}{\epsilon}]}\right)^{\tau_2}. \tag{3.36}$$

Finally, by selecting a value for ϱ which minimizes $2\varrho + C_5(\frac{1}{\ln[\varrho \ln \frac{1}{\epsilon}]})^{\tau_2}$, we complete the proof for the theorem. $\square$

4. Conclusions

In this paper, we discuss an inverse problem in determining an unknown inaccessible boundary from given Cauchy data on the other part of the boundary. A double logarithmic conditional stability estimate is obtained. This kind of weak stability estimate is common in the studies of the ill-posedness of Cauchy problem for Laplace's equation and in particular, the highly ill-posedness in determining an unknown boundary from an incomplete boundary information of the solution. The results obtained in the paper are compatible with the results given in [1], [2], [6] and [13] except that this paper gives a most likely optimal estimate under a more general assumption.

References

1. G. Alessandrini, Stable determination of a crack from boundary measurements. Proc. R. Soc. Edinburgh A V.127 (1993), 497–516.
2. G. Alessandrini, Examples of instability in inverse boundary-value problems. Inverse Problems V.13 (1997), 887–897.
3. S. Andrieux, A.B. Abda & M. Jaoua, Identifiabilité de frontière inaccessible par des mesures de surface. C. R. Acad. Sci. Paris Sér. I Math. V.316 (1993), 429–434.
4. N.D. Aparicio & M.K. Pidcock, The boundary inverse problem for the Laplace equation in two dimensions. Inverse Problems V.12 (1996), 565–577.
5. A.L. Bukhgeim, Extension of solutions of elliptic equations from discrete sets. J. Inverse and Ill-posed Problems V.1 (1993), 17–32.
6. A.L. Bukhgeim, J. Cheng & M. Yamamoto, Stability for an inverse boundary problem of determining a part of boundary. Inverse Problems V.15 (1999), 1021–1032.
7. J.R. Cannon The One-dimensional Heat Equation. Addison-Wesley Reading, MA (1984).
8. T. Carleman, Sur un problème d'unicité pour les systèmes d'équations aux dérivées partielles à deux variables indépendentes. Ark. Mat. Astr. Fys. V.26B (1939), 1–9.
9. J. Cheng, Y.C. Hon & M. Yamamoto, Conditional stability estimation for an inverse boundary problem with non-Smooth boundary data in R^3. Trans. Amer. Math. Soc. to appear.

10. J. Cheng & M. Yamamoto, Unique continuation on a line for harmonic functions. Inverse Problems V.14 (1998), 869–882.

11. A. Fiedman & M. Vogelius, Determining cracks by boundary measurements. Indiana Math. V.38 (1989), 527–556.

12. D. Gilbarg & N. Trudinger, Elliptic partial differential equations of second order, 2nd ed. Berlin New York Springer-Verlag, 1983.

13. V. Isakov, Stability estimates for obstacles in inverse scattering. J. of Computational and Applied Math. V.42 (1992), 79–88.

14. V. Isakov, New stability results for soft obstacles in inverse scattering. Inverse Problems V.9 (1993), 535–543.

15. D.S. Jerison & C.E. Kenig, The Neumann problem on Lipschschitz domains. Bulletin of Amer. Math. Soc. V.4 (1981), 203–207.

16. P.G. Kaup, F. Santosa & M. Vogelius, Method for imaging corrosion damage in thin plates from electrostatic data. Inverse Problems V.12 (1996), 279–293.

17. O.D. Kellogg, Foundations of Potential Theory. Dover Publications, Inc., New York (1953).

18. E. Landis, Some problems of the qualitative theory of second order elliptic equations. Russian Math. Surveys V.18 (1963), 1–62.

19. M.M. Lavrentiev, Some Improperly Posed Problems of Mathematical Physics. (English translation) Springer-Verlag, Berlin (1967).

20. J.-L. Lions & E. Magenes, Non-homogeneous Boundary Value Problems and Applications. Springer-Verlag, Berlin (1972).

21. M. McIver, Characterization of surface-breaking cracks in metal sheets by using AC electric fields. Proc. R. Soc. London A V.421 (1989), 179–194.

22. M. McIver, An inverse problem in electromagnetic crack detection. IMA Journal of Applied Mathematics V.47 (1991), 127–145.

23. D.H. Micheal, R.T. Waechter & R. Collins, The measurement of surface cracks in metals by using a.c. electric fields. Proc. R. Soc. London A V.381 (1982), 139–157.

24. S. Mizohata, The Theory of Partial Differential Equations. Cambridge University Press, London (1973).

25. L.E. Payne, Bounds in the Cauchy problem for Laplace's equation. Arch. Rational Mech. Anal. V.5 (1960), 35–45.

26. M.H. Protter & H.F. Weinberger, Maximum Principles in Differential Equations. Englewood Cliffe, New Jersey (1967).

27. A.G. Ramm, Stability of the solution to inverse obstacle scattering problem. J. Inverse and Ill-posed Problems V.2 (1994), 269–275.

28. J. Rauch Partial Differential Equations. Springer-Verlag, Berlin (1991).

29. M.E. Taylor, Partial Differential Equations:Basic Theory. Springer-Verlag, Berlin (1996).

30. A.N. Tikhonov & V.Y. Arsenin, Solutions of Ill-posed Problems. English Translation. Winston & Sons, Washington (1977).

Department of Mathematics, Fudan University, Shanghai 200433, China & Department of Mathematics, Faculty of Engineering, Gunma University, Kiryu 376-8515, Japan

E-mail address: jcheng@math.sci.gunma-u.ac.jp & jcheng@fudan.edu.cn

Department of Mathematics, City University of Hong Kong, Tat Chee Avenue, Kowloon, Hong Kong

E-mail address: maychon@cityu.edu.hk

Department of Mathematical Sciences, University of Tokyo, 3-8-1 Komaba, Meguro, Tokyo 153-8914, Japan

E-mail address: myama@ms.u-tokyo.ac.jp