Continuation of Holomorphic Functions from Subvarieties to Pseudoconvex Domains

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1. Introduction

Let $D$ be a bounded pseudoconvex domain in $\mathbb{C}^n$ and $V$ a subvariety of $D$. In the present paper, we give some recent results concerning holomorphic extensions from $V$ to $D$ in some function spaces. In 1965, Hörmander obtained $L^2$ estimates for the $\bar{\partial}$ problem in bounded pseudoconvex domains in $\mathbb{C}^n$. In 1970, Henkin, Grauert-Lieb and Lieb obtained the uniform estimates for the $\bar{\partial}$ problem in strictly pseudoconvex domains in $\mathbb{C}^n$ with smooth boundary. Corresponding to these results, extension problems were studied by two different methods. The one is the extension using the integral formula in the case where $D$ is a bounded pseudoconvex domain with a support function (for example, bounded strictly pseudoconvex domains or bounded convex domains with smooth boundary). The other is the $L^2$ extension using the Hilbert space theory in the case where $D$ is a general bounded pseudoconvex domain. The main purpose of the present paper is to introduce Berndtsson’s another proof of the $L^2$ extension theorem of Ohsawa-Takegoshi in bounded pseudoconvex domains.

2. Some recent results

Definition. Let $D$ be an open set in $\mathbb{C}^n$ and $\varphi \in C^\infty(D)$ a real function. We denote by $L^2(D, \varphi)$ the space of square-integrable functions in $D$ with respect to the measure $e^{-\varphi}d\mu$, where $d\mu$ is the Lebesgue measure in $\mathbb{C}^n$. We denote by $L^2_{(p,q)}(D, \varphi)$ the space of $(p, q)$-forms with coefficients in $L^2(D, \varphi)$,

$$f = \sum' f_{I,J}dz^I \wedge d\overline{z}^J,$$

where $\sum'$ means that the summation is performed only over strictly increasing multi-indices. We set

$$|f|^2 = \sum' |f_{I,J}|^2, \quad \|f\| = \left(\int_D |f|^2e^{-\varphi}d\mu\right)^{\frac{1}{2}}.$$
For $f, g \in L^2_{(p,q)}(D, \varphi)$ with $f = \sum_{I,J} f_{I,J} d_{\mathcal{Z}^{I}} \overline{z}^{J}$, $g = \sum_{I,J} g_{I,J} d_{\mathcal{Z}^{I}} \overline{z}^{J}$, we define the inner product in $L^2_{(p,q)}(D, \varphi)$ by 

$$(f, g) = \sum_{I,J} \int_D f_{I,J} \overline{g_{I,J}} e^{-\varphi} d\mu.$$ 

Then $L^2_{(p,q)}(D, \varphi)$ is a Hilbert space with this inner product.

**Theorem 1.** (Hörmander[14]) Let $D$ be a bounded pseudoconvex open set in $C^n$, let $\delta$ be the diameter of $D$, and let $\psi$ be a plurisubharmonic function in $D$. For every $f \in L^2_{p,q}(D, \varphi)$, $q > 0$, with $\bar{\partial}f = 0$, one can then find $u \in L^2_{(p,q-1)}(D, \varphi)$ such that $ar{\partial}u = f$ and

$$q \int_D |u|^2 e^{-\varphi} dV \leq e\delta^2 \int_D |f|^2 e^{-\varphi} dV.$$ 

**Theorem 2.** (Henkin[10], Ramirez[17]) Let $D$ be a bounded strictly pseudoconvex domain in $C^n$ with smooth boundary. Then there exist a pseudoconvex domain $\tilde{D} \supset \overline{D}$ and functions $K(\zeta, z)$ and $\Phi(\zeta, z)$ defined for $\zeta \in \partial D$ and $z \in \tilde{D}$ such that

1. $K(\zeta, z)$ and $\Phi(\zeta, z)$ are holomorphic in $z \in \tilde{D}$ and continuous in $\zeta \in \partial D$
2. For every $\zeta \in \partial D$ the function $\Phi(\zeta, z)$ vanishes on the closure $\overline{D}$ only at the point $z = \zeta$.
3. For any holomorphic function $f$ in $D$ that is continuous on $\overline{D}$ and any $z \in D$, the integral formula

$$f(z) = \int_{\partial D} f(\zeta) \frac{K(\zeta, z)}{\Phi(\zeta, z)} d\sigma(\zeta)$$

holds, where $d\sigma$ is the $(2n-1)$ dimensional Lebesgue measure on $\partial D$.

**Definition.** Let $f(x)$ be a function on $D$. Then we define

$$|f|_0 = \sup_{x \in D} |f(x)|.$$ 

Let $f$ be a $(0,q)$-form with the coefficients $f_{i_1, \ldots, i_q}$. Then we define

$$|f|_0 = \max_{i_1, \ldots, i_q} |f_{i_1, \ldots, i_q}|_0.$$ 

**Theorem 3.** (Henkin[11], Grauert-Lieb[8], Lieb[15]) Let $D$ be a bounded strictly pseudoconvex domain in $C^n$ with smooth boundary. Then there exists a constant $K$ such that if $f$ is a $\bar{\partial}$ closed $C^\infty(0,q+1)$-form on $D$, then there exists a $C^\infty(0,q)$-form $u$ on $D$ with

$$\bar{\partial}u = f \quad \text{and} \quad |u|_0 \leq K|f|_0.$$
Let $D$ be a strictly pseudoconvex domain in $\mathbb{C}^n$ with smooth boundary and let $\bar{M}$ be a submanifold in a neighborhood $\hat{D}$ of $\overline{D}$ which meets $\partial D$ transversally. We set $M = \bar{M} \cap D$. Let $\Omega$ be a domain in some complex manifold. We denote by $H^\infty(\Omega)$ the space of all bounded holomorphic functions in $\Omega$. We also denote by $A^\infty(\Omega)$ the space of all holomorphic functions in $\Omega$ that are $C^\infty$ on $\overline{\Omega}$. In this setting, we have theorem 4 and 5.

**Theorem 4.** (Henkin[12]) There exists a linear extension operator $E : H^\infty(M) \to H^\infty(D)$. Moreover, $Ef$ is continuous on $\overline{D}$ if $f$ is continuous on $\overline{M}$.

**Theorem 5.** (Adachi[1], Elgueta[7]) There exists a linear extension operator $E : A^\infty(M) \to A^\infty(D)$.


Let $D$ be a bounded pseudoconvex domain in $\mathbb{C}^n$ with smooth boundary. Let $\gamma : \partial D \times D \to \mathbb{C}^n$ be a smooth mapping such that

$$(\zeta - z, \gamma) = \sum_{j=1}^n (\zeta_j - z_j) \gamma_j(\zeta, z) \neq 0 \quad \text{on} \quad \partial D \times D.$$

Let $h_1, \ldots, h_m (m < n)$ be holomorphic functions in a neighborhood $\hat{D}$ of $\overline{D}. Define$

$$\tilde{V} = \{ z \in \hat{D} | h_1(z) = \cdots = h_m(z) = 0 \}, \quad V = \tilde{V} \cap D.$$  

We say $V$ intersects $\partial D$ transversally if

$$d\rho \wedge \partial h_1 \wedge \cdots \wedge \partial h_m \neq 0 \quad \text{on} \quad \partial V.$$

In the above setting, we have the following:

**Theorem 6.** (Stout[19], Hatziafratis[9]) There is a smooth form $K_V(\zeta, z)$ on $\partial V \times \overline{V}$ which is of type $(0,0)$ in $z$ and $(n-m-1, n-m)$ in $\zeta$ such that if $f$ is holomorphic in $V$ and continuous on $\overline{V}$, then for $z \in V$

$$f(z) = \int_{\zeta \in \partial V} f(\zeta) \frac{K_V(\zeta, z)}{(\zeta - z, \gamma(\zeta, z))^{n-m}}.$$

Moreover, $K_V(\zeta, z)$ is holomorphic in $z \in D$ provided that $\gamma(\zeta, z)$ is holomorphic in $z \in D$.

Let $D$ be a bounded convex domain with a defining function $\rho$. Then we can choose

$$\gamma_i(\zeta, z) = \frac{\partial \rho}{\partial \zeta_i}(\zeta).$$
Let $E(f)(z)$ be the right hand side of (1). Then we have

**Theorem 7.** (Adachi-Cho[3]) Let $D$ be a bounded convex domain in $\mathbb{C}^n$ with real analytic boundary and let $V$ be a one dimensional subvariety of $D$ defined above. Then we have

1. Let $1 \leq p < \infty$. If $f \in H^p(V)$, then $E(f) \in H^p(D)$.
2. Suppose that $V$ has no singular points and $1 \leq p < \infty$. If $f \in \mathcal{O}(V) \cap L^p(V)$, then $E(f) \in \mathcal{O}(D) \cap L^p(D)$.

where $\mathcal{O}(V)$ (resp. $\mathcal{O}(D)$) denotes the space of all holomorphic functions in $V$ (resp. $D$).

A bounded domain $\Omega \subset \mathbb{C}^n$ is an analytic polyhedron with defining functions $\phi_j$ if

$$\Omega = \{z \in \mathbb{C}^n | |\phi_j(z)| < 1, j = 1, \ldots, N\},$$

where the defining functions $\phi_j$ are holomorphic in some neighborhood $\tilde{\Omega}$ of $\overline{\Omega}$. We set $\sigma_I = \{z \in \overline{\Omega} | |\phi_j(z)| = 1, j \in I\}$. We say that $\Omega$ is non-degenerate if $\partial \phi_{i_1} \wedge \cdots \wedge \partial \phi_{i_k} \neq 0$ on $\sigma_I$ for every multiindex $I = \{i_1, \ldots, i_k\}$ such that $|I| = k \leq n$. We say that $\Omega$ is strongly non-degenerate if $\partial \phi_{i_1} \wedge \cdots \wedge \partial \phi_{i_k} \neq 0$ on $\sigma_I$ for all multiindices $I$. Let $\tilde{V}$ be a regular subvariety of $\tilde{\Omega}$ of codimension $m$ given by

$$\tilde{V} = \{z \in \tilde{\Omega} | h_1(z) = \cdots = h_m(z) = 0\},$$

where $h_j \in \mathcal{O}(\tilde{\Omega})$, and $\partial h_1 \wedge \cdots \wedge h_m \neq 0$ on $\tilde{V}$. We set $V = \tilde{V} \cap \Omega$. We impose the transversal assumption that

$$\partial h_1 \wedge \cdots \wedge \partial h_m \wedge \partial \phi_{i_1} \wedge \cdots \wedge \partial \phi_{i_k} \neq 0 \quad \text{on} \quad \overline{V} \cap \sigma_I,$$

for every multiindex $I$ such that $|I| = k \leq n - m$. For a strongly non-degenerate polyhedron $\Omega$ we can define the Hardy spaces

$$H^p(\Omega) = \left\{ f \in \mathcal{O}(\Omega) | \sup_{\epsilon > 0} \|f\|_{L^p(\sigma_\epsilon)} < \infty \right\}.$$

In the above setting, we have by applying the integral formula obtained by Berndtsson[5]:

**Theorem 8.** (Adachi-Andersson-Cho[3])

1. Let $\Omega$ be a non-degenerate analytic polyhedron. For each $f \in \mathcal{O}(V) \cap L^p(V), 1 \leq p < \infty$, there exists $F \in \mathcal{O}(\Omega) \cap L^p(\Omega)$ such that $F(z) = f(z)$ for $z \in V$ and $\|F\|_{L^p(\Omega)} \leq C \|f\|_{L^p(V)}$. 
Let $\Omega$ be a strongly non-degenerate analytic polyhedron. Then for all $f \in H^p(V)$, $1 < p \leq \infty$, there exists $F \in H^p(\Omega)$ such that $F(z) = f(z)$ for $z \in V$ and $\|F\|_{H^p(\Omega)} \leq c\|f\|_{H^p(V)}$.

### 3. Outline of the proof of the theorem of Ohsawa-Takegoshi due to Berndtsson

In this section, we shall prove the extension theorem of Ohsawa-Takegoshi by following the Berndtsson's proof[6]. Using $L^2$ space techniques, Ohsawa and Takegoshi obtained the following:

**Theorem 9.** (Ohsawa-Takegoshi[16]) Let $D$ be a bounded pseudoconvex domain in $\mathbb{C}^n$. We set $H = \{z \in \mathbb{C}^n | z_1 = 0\}$. Then there exists a constant $C$ which depends only on the diameter of $D$ such that, for any plurisubharmonic function $\varphi$ on $D$, and for any holomorphic function $f$ on $H \cap D$, there exists a holomorphic function $F$ in $D$ such that

$$F|_{H \cap D} = f, \quad \int_D |F|^2 e^{-\varphi} \, d\mu \leq C \int_{H \cap D} |f|^2 e^{-\varphi} \, d\mu,$$

where $d\mu$ and $d\mu_1$ are Lebesgue measures in $\mathbb{C}^n$ and $\mathbb{C}^{n-1}$, respectively.

**Lemma 1.** (Hörmander[14]) Let $D$ be a bounded open set in $\mathbb{C}^n$ with smooth boundary $\partial D$ and let $\rho$ be a smooth defining function for $D$. For $f = \sum f_J d\overline{z}^J \in C_{(q)}^1(0, \overline{D})$ and $u = \sum u_K d\overline{z}^K \in C_{(0,q-1)}^1(\overline{D})$, the following equality is valid

$$(\overline{\partial} u, f) = - \int_D \sum_J \sum_k u_K \overline{\delta_j f_{jK}} e^{-\varphi} \, d\mu + \int_{\partial D} \sum_k u_K \sum_{j=1}^n f_{jK} \overline{\frac{\partial \rho}{\partial z_j}} e^{-\varphi} \, dS.$$

**Definition.** For $u \in C^1(D)$, define

$$\delta_j u = e^{-\varphi} \frac{\partial}{\partial z_j} (ue^{-\varphi}) = \frac{\partial u}{\partial z_j} - \frac{\partial \varphi}{\partial z_j} u, \quad \partial_k u = \frac{\partial u}{\partial z_k} - \frac{\partial \varphi}{\partial z_k}, \quad \overline{\partial}_k u = \frac{\partial u}{\partial \overline{z}_k}.$$

For $C^1(0,q)$-form $f = \sum f_J d\overline{z}^J$, define $\overline{\partial} f = -\sum_{|J|=q}^n \delta_j f_{jK} d\overline{z}^K$. We define

$$f \in \text{Def}(\overline{\partial}^* ) \iff \sum_{|J|=q}^n f_{jK} \frac{\partial \rho}{\partial z_j} = 0 \quad \text{on} \quad \partial D \quad \text{for all} \quad K.$$
We say $f$ satisfies the boundary condition if $f \in \text{Def}(\bar{\partial}^*)$. When $f$ satisfies the boundary condition, we have from lemma 1
\[(\bar{\partial}u, f) = (u, \bar{\partial}^* f)\].

**Lemma 2.** (Hörmander[14]) Let $\alpha = \sum'_{|J|=q} \alpha_Jd\bar{z}^J$ be a smooth $(0,q)$-form in $\overline{D}$ and $\alpha \in \text{Def}(\bar{\partial}^*)$. For $\varphi \in C^\infty(\overline{D})$ we have
\[
\|\bar{\partial}^* \alpha\|^2 + \|\bar{\partial} \alpha\|^2 = \sum_K' \sum_{j,k=1}^n \int_D \alpha_{jK} \overline{\alpha_{kK}} \frac{\partial^2 \varphi}{\partial z_j \partial \bar{z}_k} e^{-\varphi} d\mu + \sum_K' \sum_{j=1}^n \int_D \left| \frac{\partial \alpha_j}{\partial z_j} \right|^2 e^{-\varphi} d\mu + \sum_K' \sum_{j,k=1}^n \int_{\partial D} \alpha_{jK} \overline{\alpha_{kK}} \frac{\partial \rho}{\partial z_j \partial \bar{z}_k} e^{-\varphi} dS |\partial \rho|.
\]

We assume that $\varphi$ is a smooth function in $\overline{D}$ from lemma 3 to lemma 7. Thus $f \in L^2(D, \varphi)$ means $f \in L^2(D)$. We omit the proof of lemma 3, since the detailed proof of lemma 3 is given in [6].

**Lemma 3.** Let $w$ be a real valued smooth function in $\overline{D}$. $\alpha = \sum_{j=1}^n \alpha_jd\bar{z}_j$ is a smooth $(0,1)$-form in $\overline{D}$ satisfying the boundary condition. Then we have
\[
\int_D w \sum_{j,k=1}^n \varphi_{jk} \alpha_j \overline{\alpha_k} e^{-\varphi} d\mu - \int_D w_{jK} \alpha_j \overline{\alpha_k} e^{-\varphi} d\mu + \int_D w |\bar{\partial}^* \alpha|^2 e^{-\varphi} d\mu + \int_D w \sum_{j,k=1}^n \left| \frac{\partial \alpha_k}{\partial \bar{z}_j} \right|^2 e^{-\varphi} d\mu + \int_{\partial D} \sum_{j,k=1}^n \rho_{jk} \alpha_j \overline{\alpha_k} e^{-\varphi} dS |\partial \rho| = 2\text{Re} \int_D w \bar{\partial} \bar{\partial}^* \alpha \cdot \bar{\alpha} e^{-\varphi} d\mu + \int_D w |\bar{\partial} \alpha|^2 e^{-\varphi} d\mu.
\]

**Definition.** Let $\psi \in C^\infty(\overline{D})$ and $\alpha = \sum_{j=1}^n \alpha_jd\bar{z}_j \in C^\infty_{(0,1)}(\overline{D})$. We define the inner product of $g = \psi \bar{\partial} \left( \frac{1}{z_1} \right)$ and $\alpha$ by
\[
<g, \alpha> = \sum_{j=1}^n \psi \frac{\partial}{\partial \bar{z}_j} \left( \frac{1}{z_1} \right) \alpha_j = \lim_{\varepsilon \to 0} \int_D \psi(z) \frac{\partial}{\partial \bar{z}_1} \left( \frac{z_1}{|z_1|^2 + \varepsilon} \right) \overline{\alpha_1(z)} e^{-\varphi(z)} d\mu(z).
\]
Moreover, if we define

$$h_\varepsilon(z) = \psi(z) \overline{\bar{\partial}} \left( \frac{z}{|z|^2 + \varepsilon} \right),$$

then we obtain

(2) $$< g, \alpha > = \lim_{\varepsilon \to 0} < h_\varepsilon, \alpha >.$$

In view of lemma 6, the right hand side of (2) exists. For \( u \in L^1(D) \) and a \((0,1)\)-form \( \alpha \) in \( D \) with compact support, we define

$$< \partial u, \alpha > = (u, \bar{\partial}^* \alpha).$$

Then we have the following:

**Lemma 4.** Let \( D \) be a bounded strictly pseudoconvex domain in \( \mathbb{C}^n \) with smooth boundary. Let \( f \) be a holomorphic function in \( \bar{D} \) and \( g = f \overline{\partial} \left( \frac{1}{z_1} \right) \). Let \( u \in L^1(D) \). If the equality

$$< g, \alpha > = \int_D u \overline{\partial}^* \alpha e^{-\varphi} d\mu$$

holds for any \( \partial \) closed \( \alpha \in C^\infty_{(0,1)}(\bar{D}) \) which satisfies the boundary condition, then \( g = \partial u \) in the sense of distribution.

**Proof.** Let \( \alpha \) be a \( C^\infty(0,1) \)-form in \( D \) with compact support. We define

$$\text{Def}(\partial) = \{ g \in L^2_{(0,q)}(D, \varphi) | \partial g \in L^2_{(0,q+1)}(D, \varphi) \}.$$

For Laplace-Beltrami operator \( \Box = \overline{\partial} \overline{\partial}^* + \overline{\partial}^* \overline{\partial} : L^2_{(0,1)}(D, \varphi) \to L^2_{(0,1)}(D, \varphi) \), define

$$\text{Def}(\Box) = \{ \alpha \in L^2_{(0,1)}(D, \varphi) | \alpha \in \text{Def}(\partial), \overline{\partial} \alpha \in \text{Def}(\overline{\partial}^*), \partial^* \alpha \in \text{Def}(\overline{\partial}) \},$$

$$\mathcal{H} = \{ \alpha \in \text{Def}(\Box) | \Box \alpha = 0 \}.$$

Then \( \mathcal{H} \) is a closed subspace of the Hilbert space \( L^2_{(0,1)}(D, \varphi) \). Let \( H : L^2_{(0,1)}(D, \varphi) \to \mathcal{H} \) be the orthogonal projection. From the theory of the \( \partial \) Neumann problem, there exists Neumann operator \( N : L^2_{(0,1)}(D, \varphi) \to \text{Def}(\Box) \) such that

$$\alpha = \overline{\partial} \overline{\partial}^* N \alpha + \overline{\partial}^* \overline{\partial} N \alpha + H \alpha.$$

For \( \beta \in \mathcal{H} \), we have

$$0 = (\Box \beta, \beta) = (\overline{\partial} \overline{\partial}^* \beta, \beta) + (\overline{\partial}^* \overline{\partial} \beta, \beta) = \| \overline{\partial}^* \beta \|^2 + \| \overline{\partial} \beta \|^2.$$
Hence we obtain $\bar{\partial}\beta = \bar{\partial}^*\beta = 0$. From lemma 2, it holds that

$$0 = \|\bar{\partial}\beta\|^2 + \|\bar{\partial}^*\beta\|^2 \geq \sum_{j,k=1}^n \left\| \frac{\partial \beta_j}{\partial \overline{z}_k} \right\|^2 + \int_{\partial D} \sum_{j,k=1}^n \frac{\partial^2 \rho}{\partial z_j \partial \overline{z}_k} \beta_j \beta_k \frac{dS}{|\partial \rho|}$$

Thus $\beta_j$ is holomorphic in $D$ and $0$ in $\partial D$ so that $\beta = 0$. Therefore $\mathcal{H} = 0$. We set

$$\alpha_1 = \bar{\partial} \bar{\partial}^* \iota^{r}, \alpha_2 = \bar{\partial}^* \bar{\partial} \iota_{\alpha}'$$

Since Neumann operator maps smooth $(0,1)$-forms to smooth $(0,1)$-forms in the strictly pseudoconvex domain $D$, $\alpha_1$ and $\alpha_2$ are both smooth $(0,1)$-forms in $\overline{D}$. Obviously, $\delta_1 = 0$. If $\bar{\partial}\beta = 0$, then by lemma 1 $(\beta, \alpha_2) = (\bar{\partial}\beta, \bar{\partial}N\alpha) = 0$. Hence $\alpha_2 \perp \text{Ker}(\bar{\partial})$. On the other hand, from lemma 1, for any smooth function $\beta$ on $\overline{D}$, we have

$$0 = (\bar{\partial}\beta, \alpha_2) = (\beta, \bar{\partial}^* \alpha_2) + \int_{\partial D} \beta \alpha_2 \cdot \partial p e^{-\varphi} \frac{dS}{|\partial \rho|}.$$

Thus $\bar{\partial}^* \alpha_2 = 0$. Therefore $\alpha_2$ satisfies the boundary condition. Hence, $\alpha_1$ satisfies the boundary condition. If we set

$$h_\varepsilon(z) = f(z) \bar{\partial} \left( \frac{z_1}{|z_1|^2 + \varepsilon} \right),$$

then we have

$$< g, \alpha_2 > = \lim_{\varepsilon \to 0} (h_\varepsilon, \alpha_2) = 0.$$

Thus we have

$$< g, \alpha > = < g, \alpha_1 > = \int_D u \bar{\partial}^* \alpha_1 e^{-\varphi} d\mu = \int_D u \bar{\partial}^* \alpha e^{-\varphi} d\mu = (u, \bar{\partial}^* \alpha) = < \bar{\partial} u, \alpha >,$$

which means $g = \bar{\partial} u$.

**Lemma 5.** Let $g$ be the same as in lemma 4. Let $\lambda$ be a non-negative real valued function in $D$ with the property that $\frac{1}{\lambda}$ is integrable. If the inequality

$$| < g, \alpha > |^2 \leq C \int_D |\bar{\partial}^* \alpha|^2 \frac{e^{-\varphi}}{\lambda} d\mu$$

holds for any $\bar{\partial}$ closed $\alpha \in C^\infty_{(0,1)}(\overline{D})$ which satisfies the boundary condition, then there exists $u \in L^1(D, \varphi)$ such that

$$\bar{\partial} u = g, \quad \int_D |u|^2 \lambda e^{-\varphi} d\mu \leq C.$$
Proof. Let $C_b^\infty(\overline{D})$ be the space of all $\overline{\partial}$ closed $C^\infty(0,1)$-forms in $\overline{D}$ which satisfies the boundary condition. We set

$$F = \{ \overline{\partial}^* \alpha | \alpha \in C_b^\infty(\overline{D}) \}, \quad \varphi_1 = \frac{e^{-\varphi}}{\lambda}.$$ 

Then, $F$ is a vector subspace of $L^2(D, \varphi_1)$. For $w \in F$, there exists $\alpha \in C_b^\infty(\overline{D})$ such that $w = \overline{\partial}^* \alpha$. We define

$$\Phi(w) = \langle g, \alpha \rangle.$$ 

Then $\Phi(w)$ is independent of the choice of $\alpha$. Also, we have

$$|\Phi(w)|^2 \leq C||w||_{\varphi_1}^2, \quad ||\Phi|| \leq \sqrt{C}.$$ 

Thus $\Phi$ is a bounded anti-linear operator on $F$. From the Hahn-Banach theorem, $\Phi$ is extended to a bounded anti-linear operator on $L^2(D, \varphi_1)$. From the Riesz representation theorem, there exists $v \in L^2(D, \varphi_1)$ such that

$$\Phi(w) = (v, w)_{\varphi_1}, \quad ||v||_{\varphi_1} = ||\Phi|| \leq \sqrt{C}.$$ 

Therefore we have

$$\langle g, \alpha \rangle = \int_D \overline{\partial}^* \alpha \frac{e^{-\varphi}}{\lambda} d\mu = \int_D |v|^2 \frac{e^{-\varphi}}{\lambda} d\mu = ||v||_{\varphi_1}^2 \leq C.$$ 

If we set $u = \frac{v}{\lambda}$, then

$$\int_D |u|^2 \lambda e^{-\varphi} d\mu \leq C, \quad \langle g, \alpha \rangle = \int_D \overline{\partial}^* \alpha e^{-\varphi} d\mu.$$ 

On the other hand, we have

$$\int_D |u| e^{-\varphi} d\mu \leq \int_D |v|^2 \frac{e^{-\varphi}}{\lambda} d\mu \int_D \frac{e^{-\varphi}}{\lambda} d\mu \leq C \int_D \frac{e^{-\varphi}}{\lambda} d\mu < \infty.$$ 

Thus, $u \in L^1(D, \varphi)$. From lemma 4, we obtain $\overline{\partial} u = g$.

Lemma 6. For $\varphi \in C^\infty(\overline{D})$, it holds that

$$\lim_{\varepsilon \to 0} \int_D \frac{\varepsilon}{(|z_1|^2 + \varepsilon)^2} \varphi(z) d\mu(z) = \pi \int_{\{z_1 = 0\} \cap D} \varphi(z) d\mu_1(z),$$

where $d\mu$ and $d\mu_1$ are Lebesgue measures in $\mathbb{C}^n$ and $\mathbb{C}^{n-1}$, respectively.
Lemma 7. Let $D$ be a bounded strictly pseudoconvex domain in $\mathbb{C}^n$ with smooth boundary and $D \subset \{z||z_1| \leq 1\}$. Let $\varphi$ be a smooth plurisubharmonic function in $\overline{D}$ and let $\alpha$ be a $\overline{\partial}$ closed smooth $(0,1)$-form in $\overline{D}$ which satisfies the boundary condition. Then, for $0 < \delta < 1$, we have

$$\int_{\{z_1=0\} \cap D} |\alpha_1|^2 e^{-\varphi} d\mu_1 \leq \frac{2}{\pi} \left( 1 + \frac{1}{\delta^2} \right) \int_D |\overline{\partial}\alpha|^2 e^{-\varphi} d\mu.$$

Proof. For $0 < \delta < 1$, we set

$$w^\delta = 1 - |z_1|^{2\delta} = 1 - (z_1 \overline{z_1})^\delta.$$

From lemma 3, we have

$$\int_D w^\delta \sum_{j,k=1}^n \varphi_{j,k} \alpha_j \overline{\alpha_k} e^{-\varphi} d\mu + \delta^2 \int_D |z_1|^{2\delta-2} |\alpha_1|^2 e^{-\varphi} d\mu + \int_D w^\delta |\overline{\partial}\alpha|^2 e^{-\varphi} d\mu$$

$$+ \int_D w^\delta \sum_{j,k=1}^n \left| \frac{\partial \alpha_j}{\partial z_k} \right|^2 e^{-\varphi} d\mu + \int_{\partial D} w^\delta \sum_{j,k=1}^n \rho_{j,k} \alpha_j \overline{\alpha_k} e^{-\varphi} dS$$

$$= 2\text{Re} \int_D w^\delta \overline{\partial} \overline{\partial} \alpha \cdot \overline{\alpha} e^{-\varphi} d\mu.$$

Hence we have

$$\delta^2 \int_D |z_1|^{2\delta-2} |\alpha_1|^2 e^{-\varphi} d\mu + \int_D w^\delta |\overline{\partial}\alpha|^2 e^{-\varphi} d\mu \leq 2\text{Re} \int_D w^\delta \overline{\partial} \overline{\partial} \alpha \cdot \overline{\alpha} e^{-\varphi} d\mu$$

$$= 2\text{Re}(\overline{\partial}\alpha, \overline{\partial} (w^\delta \alpha)) = 2\text{Re}(\overline{\partial} \alpha, w^\delta \overline{\partial} \alpha - \sum_{j=1}^n \frac{\partial w^\delta}{\partial z_j} \alpha_j)$$

$$= 2 \int_D w^\delta |\overline{\partial}\alpha|^2 e^{-\varphi} d\mu - 2\text{Re} \int_D \overline{\partial} \alpha \frac{\partial w^\delta}{\partial z_1} \alpha_1 e^{-\varphi} d\mu$$

$$\leq 2 \int_D w^\delta |\overline{\partial}\alpha|^2 e^{-\varphi} d\mu + 2 \int_D |\overline{\partial}\alpha| |z_1|^{2\delta-1} |\alpha_1| e^{-\varphi} d\mu$$

$$\leq 2 \int_D w^\delta |\overline{\partial}\alpha|^2 e^{-\varphi} d\mu + 2 \int_D |\overline{\partial}\alpha|^2 |z_1|^{2\delta} e^{-\varphi} d\mu + \frac{1}{2} \int_D \delta^2 |\alpha_1|^2 |z_1|^{2\delta-2} e^{-\varphi} d\mu.$$

Thus we have

$$\frac{1}{2} \delta^2 \int_D |z_1|^{2\delta-2} |\alpha_1|^2 e^{-\varphi} d\mu \leq \int_D (1 - |z_1|^{2\delta}) |\overline{\partial}\alpha|^2 e^{-\varphi} d\mu$$

$$+ \int_D |\overline{\partial}\alpha|^2 |z_1|^{2\delta} e^{-\varphi} d\mu$$

$$= \int_D |\overline{\partial}\alpha|^2 e^{-\varphi} d\mu + \int_D |\overline{\partial}\alpha|^2 |z_1|^{2\delta} e^{-\varphi} d\mu \leq 2 \int_D |\overline{\partial}\alpha|^2 e^{-\varphi} d\mu.$$
Therefore, for $0 < \delta < 1$, we obtain

$$
\delta^2 \int_D |z_1|^{25-2} |\alpha_1|^2 e^{-\varphi} d\mu \leq 4 \int_D |\bar{\partial}^* \alpha|^2 e^{-\varphi} d\mu.
$$

On the other hand, we set

$$
w_\varepsilon = \frac{1}{\pi} \log \frac{1}{|z_1|^2 + \varepsilon}, \quad w = \frac{1}{\pi} \log \frac{1}{|z_1|^2}.
$$

We apply lemma 3 to $w_\varepsilon$ and let $\varepsilon \to 0$, then by lemma 6

$$
\int_{\{z_1 = 0\} \cap D} |\alpha_1|^2 e^{-\varphi} d\mu_1 + \int_D w^* \bar{\partial}^* \alpha|^2 e^{-\varphi} d\mu \leq 2 \text{Re} \int_D w \bar{\partial} \alpha \cdot \overline{\alpha} e^{-\varphi} d\mu.
$$

By the same calculation as the first part and applying (3) to $0 < \delta < 1$, we have

$$
\int_{\{z_1 = 0\} \cap D} |\alpha_1|^2 e^{-\varphi} d\mu_1 \leq \frac{1}{\pi} \int_D \log \frac{1}{|z_1|^2} |\bar{\partial}^* \alpha|^2 e^{-\varphi} d\mu + \frac{2}{\pi} \int_D |\bar{\partial}^* \alpha| |\alpha_1| e^{-\varphi} d\mu
$$

$$
\leq \frac{1}{\pi} \int_D \log \frac{1}{|z_1|^2} |\bar{\partial}^* \alpha|^2 e^{-\varphi} d\mu + \frac{2}{\pi} \int_D |\bar{\partial}^* \alpha|^2 e^{-\varphi} d\mu + \frac{1}{2\pi} \int_D |z_1|^{26-2} |\alpha_1|^2 e^{-\varphi} d\mu
$$

$$
\leq \frac{1}{\pi} \int_D \log \frac{1}{|z_1|^2} |\bar{\partial}^* \alpha|^2 e^{-\varphi} d\mu + \frac{2}{\pi} \int_D |\bar{\partial}^* \alpha|^2 e^{-\varphi} d\mu + \frac{\delta^2}{2\pi} \int_D |\bar{\partial}^* \alpha|^2 e^{-\varphi} d\mu.
$$

Using the fact that $x \left( \log \frac{1}{x} + 2 \right) \leq 2$ for $0 < x \leq 1$, we have

$$
\int_{\{z_1 = 0\} \cap D} |\alpha_1|^2 e^{-\varphi} d\mu_1 \leq \frac{2}{\pi} \int_D |\bar{\partial}^* \alpha|^2 e^{-\varphi} d\mu + \frac{1}{\pi \delta^2} \int_D |\bar{\partial}^* \alpha|^2 e^{-\varphi} d\mu
$$

$$
= \frac{2}{\pi} \left( 1 + \frac{1}{\delta^2} \right) \int_D |\bar{\partial}^* \alpha|^2 e^{-\varphi} d\mu,
$$

which completes the proof.

**Lemma 8.** Let $D$ be a pseudoconvex domain in $\mathbb{C}^n$ and $X = \{ z \in D | z_1 = 0 \}$. Let $f$ be a holomorphic function in $X$. If $H$ is locally integrable in $D$ and satisfies $\bar{\partial} H = f \bar{\partial} \left( \frac{1}{z_1} \right)$, then there exists a holomorphic function $\hat{H}$ in $D$ such that $\hat{H}(z) = z_1 H(z)$ a.e. and $\hat{H}(z) = f(z)$ for $z \in X$. 
Proof. There exists a neighborhood $\omega$ of $X$ in $D$ such that $f$ can be extended to be holomorphic in $\omega$. Let $\chi \in C^\infty(D)$ be a function such that $\chi = 1$ in a neighborhood of $X$ in $\omega$, $\text{supp}(\chi) \subset \omega$ and $0 \leq \chi \leq 1$ in $D$. We set

$$\omega = \frac{\bar{\partial} \chi}{z_1}.$$ 

Then $\omega$ satisfies that $\omega \in C^\infty_{(0,1)}(D)$, $\bar{\partial} \omega = 0$. Define

$$G = \frac{\chi f}{z_1} - H,$$

then $G$ is locally integrable. Since we have

$$\bar{\partial} G = \bar{\partial}(\chi f) = \frac{\chi f}{z_1} - \bar{\partial} H = \frac{\chi f}{z_1} - \bar{\partial} H = \frac{\chi f}{z_1} = \omega,$$

there exists a smooth function $\tilde{G}$ in $D$ such that $\tilde{G} = G$ a.e. We set

$$\chi(z)f(z) - z_1\tilde{G}(z) = \tilde{H}(z),$$

then we have $z_1 H(z) = \tilde{H}(z)$ a.e. and $\tilde{H}(z) = f(z)$ for $z \in X$. Moreover we have

$$\bar{\partial} \tilde{H}(z) = (\bar{\partial} \chi(z)) f(z) - \bar{\partial} \tilde{G}(z) = (\bar{\partial} \chi(z)) f(z) - z_1 \omega(z) = 0.$$

Hence $\tilde{H}(z)$ is holomorphic in $D$.

Lemma 9. Let $D$ be an open set in $\mathbb{C}^n$ and let $K \subset D$ be a compact set. Then there exists a constant $C$ such that for any holomorphic function $f$ in $D$ and any neighborhood $\omega$ of $K$

$$\sup_K |f| \leq C \|f\|_{L^1(\omega)}.$$

Lemma 10. Let $\{u_k\}$ be a sequence of holomorphic functions in $D$ which are uniformly bounded on any compact subset of $D$. Then there exists a subsequence $\{u_{k_j}\}$ of $\{u_k\}$ such that $\{u_{k_j}\}$ converges uniformly on any compact subset of $D$ to a holomorphic function in $D$.

Theorem 10. (Berndtsson[6]) Let $D$ be a bounded pseudoconvex domain in $\mathbb{C}^n$ and let $\varphi$ be a plurisubharmonic function in $D$. We set $X = \{z \in D|z_1 = 0\}$. Suppose that
$D \subset \{ z \in \mathbb{C}^n \mid |z_1| \leq A \}$. If $f$ is holomorphic in $X$, then there exists a holomorphic function $F$ in $D$ such that

$$F|_X = f, \quad \int_D |F|^2 e^{-\varphi} d\mu \leq 4A^2 \pi \int_X |f|^2 e^{-\varphi} d\mu_1,$$

where $d\mu$ and $d\mu_1$ are Lebesgue measures in $\mathbb{C}^n$ and $\mathbb{C}^{n-1}$, respectively.

**Proof.** Without loss of generality, we may assume that $A = 1$. There exists an increasing sequence of bounded strictly pseudoconvex domains in $\mathbb{C}^n$ with smooth boundary such that $\overline{D_n} \subset\subset D$ and $\bigcup_{n=1}^{\infty} D_n = D$. Let $\{\varphi_n\}$ be a sequence of $C^\infty$ plurisubharmonic functions in $\overline{D_n}$ such that $\varphi_n \downarrow \varphi$. We set $g = f \bar{\partial} \left( \frac{1}{z_1} \right)$. Let $\alpha$ be a $\bar{\partial}$ closed $(0,1)$-form which satisfies the boundary condition on $\partial D_n$. From lemma 7, we have

$$|<g, \alpha>_{\varphi_n}|^2 = \left| \lim_{\epsilon \to 0} \int_{D_n} f \frac{\epsilon}{(|z_1|^2 + \epsilon)^2} \bar{\alpha} e^{-\varphi_n} d\mu \right|^2 = \left| \int_{\{z_1 = 0\} \cap D_n} \pi f \bar{\alpha} e^{-\varphi_n} d\mu_1 \right|^2 \leq \pi^2 \int_{\{z_1 = 0\} \cap D_n} |f|^2 e^{-\varphi_n} d\mu_1 \leq 2\pi \left( 1 + \frac{1}{\delta^2} \right) \int_{\{z_1 = 0\} \cap D_n} |f|^2 e^{-\varphi_n} d\mu_1.$$

From lemma 5, there exist integrable functions $u^n_\delta$ in $D_n$ such that

$$\bar{\partial} u^n_\delta = g, \quad \int_{D_n} |u^n_\delta|^2 |z_1|^{2\delta} e^{-\varphi_n} d\mu \leq 2\pi \left( 1 + \frac{1}{\delta^2} \right) \int_{\{z_1 = 0\} \cap D_n} |f|^2 e^{-\varphi_n} d\mu_1.$$

We set $F^n_\delta = u^n_\delta z_1$. Then, from lemma 8, $F^n_\delta$ are holomorphic in $D_n$ and satisfy $F^n_\delta|_{\{z_1 = 0\} \cap D_n} = f|_{\{z_1 = 0\} \cap D_n}$. Suppose that

$$\int_X |f|^2 e^{-\varphi} d\mu_1 = C < \infty,$$

then it holds that

$$\int_{D_n} |F^n_\delta|^2 e^{-\varphi_n} d\mu = \int_{D_n} |u^n_\delta|^2 |z_1|^2 e^{-\varphi_n} d\mu \leq \int_{D_n} |u^n_\delta|^2 |z_1|^{2\delta} e^{-\varphi_n} d\mu \leq 2\pi \left( 1 + \frac{1}{\delta^2} \right) \int_{\{z_1 = 0\} \cap D_n} |f|^2 e^{-\varphi_n} d\mu_1 \leq 2\pi \left( 1 + \frac{1}{\delta^2} \right) C.$$
From lemma 9.10, there exists a sequence \( \{\delta_j\} \) with \( \delta_j \to 1 \) such that \( F^n_{\delta_j} \) converges uniformly on any compact subset of \( D_n \) to \( F^n \). Then \( F^n \) are holomorphic in \( D_n \) and satisfy \( F^n|_{\{z=0\}\cap D_n} = f|_{\{z=0\}\cap D_n} \). Moreover, we have

\[
\int_{D_n} |F^n|^2 e^{-\varphi_s} d\mu \leq 4\pi C.
\]

Let \( K \) be a compact subset of \( D \). There exists a natural number \( N \) such that \( K \subset D_n, (n \geq N) \). If we set

\[
M_n = \min_{\overline{D}_n} e^{-\varphi_n},
\]

then, for \( n \geq N \), there exist a constant \( C_2 \) such that

\[
4\pi C \geq \int_{D_n} |F^n|^2 e^{-\varphi_s} d\mu \geq M_N \int_{D_n} |F^n|^2 d\mu \geq C_2 \sup_K |F^n|^2.
\]

Thus \( \{F^n\} \) are uniformly bounded on any compact subset of \( D \). Then we can find a subsequence \( \{F^{k_n}\} \) of \( \{F^n\} \) which converges uniformly on any compact subset of \( D \). We set \( \lim_{n \to \infty} F^{k_n} = F \). Then \( F \) is holomorphic in \( D \) and satisfies \( F|_X = f \). For any compact subset \( K \) of \( D \), we have

\[
\int_K |F|^2 e^{-\varphi} d\mu = \lim_{n \to \infty} \int_K |F^{k_n}|^2 e^{-\varphi_{k_n}} d\mu \leq 4\pi C,
\]

which completes the proof.

**Remark.** Siu[18] also obtained another proof of the theorem of Ohsawa-Takegoshi in which the constant \( C = \frac{64}{3} \pi A^2 \left( 1 + \frac{1}{n} \right)^{1/2} \) provided \( D \subset \{ z ||z| \leq A \} \).

**References**


