Continuation of Holomorphic Functions from Subvarieties to Pseudoconvex Domains

1. Introduction

Let $D$ be a bounded pseudoconvex domain in $\mathbb{C}^n$ and $V$ a subvariety of $D$. In the present paper, we give some recent results concerning holomorphic extensions from $V$ to $D$ in some function spaces. In 1965, Hörmander obtained $L^2$ estimates for the $\bar{\partial}$ problem in bounded pseudoconvex domains in $\mathbb{C}^n$. In 1970, Henkin, Grauert-Lieb and Lieb obtained the uniform estimates for the $\bar{\partial}$ problem in strictly pseudoconvex domains in $\mathbb{C}^n$ with smooth boundary. Corresponding to these results, extension problems were studied by two different methods. The one is the extension using the integral formula in the case where $D$ is a bounded pseudoconvex domain with a support function (for example, bounded strictly pseudoconvex domains or bounded convex domains with smooth boundary). The other is the $L^2$ extension using the Hilbert space theory in the case where $D$ is a general bounded pseudoconvex domain. The main purpose of the present paper is to introduce Berndtsson's another proof of the $L^2$ extension theorem of Ohsawa-Takegoshi in bounded pseudoconvex domains.

2. Some recent results

**Definition.** Let $D$ be an open set in $\mathbb{C}^n$ and $\varphi \in C^\infty(D)$ a real function. We denote by $L^2(D, \varphi)$ the space of square-integrable functions in $D$ with respect to the measure $e^{-\varphi}d\mu$, where $d\mu$ is the Lebesgue measure in $\mathbb{C}^n$. We denote by $L^2_{(p,q)}(D, \varphi)$ the space of $(p, q)$-forms with coefficients in $L^2(D, \varphi)$,

$$ f = \sum' f_{I,J}dz^I \wedge d\overline{z}^J, $$

where $\sum'$ means that the summation is performed only over strictly increasing multi-indices. We set

$$ |f|^2 = \sum' |f_{I,J}|^2; \quad \|f\| = \left( \int_D |f|^2 e^{-\varphi}d\mu \right)^{1/2}. $$
For $f, g \in L_{(p,q)}^2(D, \varphi)$ with $f = \sum_{I,J} f_{i,J} d\overline{z}^I$, $g = \sum_{I,J} g_{i,J} d\overline{z}^I$, we define the inner product in $L_{(p,q)}^2(D, \varphi)$ by

$$(f,g) = \sum_{I,J} f_{i,J} \overline{g_{i,J}} d\varphi \mu.$$ 

Then $L_{(p,q)}^2(D, \varphi)$ is a Hilbert space with this inner product.

**Theorem 1.** (Hörmander[14]) Let $D$ be a bounded pseudoconvex open set in $\mathbb{C}^n$, let $\delta$ be the diameter of $D$, and let $\psi$ be a plurisubharmonic function in $D$. For every $f \in L_{(p,q)}^2(D, \varphi)$, $q > 0$, with $\overline{\partial} f = 0$, one can then find $u \in L_{(p,q-1)}^2(D, \varphi)$ such that $\overline{\partial} u = f$ and

$$q \int_D |u|^2 e^{-\varphi} dV \leq e \delta^2 \int_D |f|^2 e^{-\varphi} dV.$$ 

**Theorem 2.** (Henkin[10], Ramírez[17]) Let $D$ be a bounded strictly pseudoconvex domain in $\mathbb{C}^n$ with smooth boundary. Then there exist a pseudoconvex domain $\tilde{D} \supset \overline{D}$ and functions $K(\zeta, z)$ and $\Phi(\zeta, z)$ defined for $\zeta \in \partial D$ and $z \in \tilde{D}$ such that

(1) $K(\zeta, z)$ and $\Phi(\zeta, z)$ are holomorphic in $z \in \tilde{D}$ and continuous in $\zeta \in \partial D$

(2) For every $\zeta \in \partial D$ the function $\Phi(\zeta, z)$ vanishes on the closure $\overline{D}$ only at the point $z = \zeta$.

(3) For any holomorphic function $f$ in $D$ that is continuous on $\overline{D}$ and any $z \in D$, the integral formula

$$f(z) = \int_{\partial D} f(\zeta) \frac{K(\zeta, z)}{\Phi(\zeta, z)^n} d\sigma(\zeta)$$

holds, where $d\sigma$ is the $(2n-1)$ dimensional Lebesgue measure on $\partial D$.

**Definition.** Let $f(x)$ be a function on $D$. Then we define

$$|f|_0 = \sup_{x \in D} |f(x)|.$$ 

Let $f$ be a $(0,q)$-form with the coefficients $f_{i_1,\ldots,i_q}$. Then we define

$$|f|_0 = \max_{i_1,\ldots,i_q} |f_{i_1,\ldots,i_q}|_0.$$ 

**Theorem 3.** (Henkin[11], Grauert-Lieb[8], Lieb[15]) Let $D$ be a bounded strictly pseudoconvex domain in $\mathbb{C}^n$ with smooth boundary. Then there exists a constant $K$ such that if $f$ is a $\overline{\partial}$ closed $C^\infty(0,q+1)$-form on $D$, then there exists a $C^\infty(0,q)$-form $u$ on $D$ with $\overline{\partial} u = f$ and $|u|_0 \leq K|f|_0$. 
Let $D$ be a strictly pseudoconvex domain in $\mathbb{C}^n$ with smooth boundary and let $\tilde{M}$ be a submanifold in a neighborhood $\tilde{D}$ of $\overline{D}$ which meets $\partial D$ transversally. We set $M = \tilde{M} \cap D$. Let $\Omega$ be a domain in some complex manifold. We denote by $H^\infty(\Omega)$ the space of all bounded holomorphic functions in $\Omega$. We also denote by $A^\infty(\Omega)$ the space of all holomorphic functions in $\Omega$ that are $C^\infty$ on $\overline{\Omega}$. In this setting, we have theorem 4 and 5.

**Theorem 4.** (Henkin[12]) There exists a linear extension operator $E : H^\infty(M) \to H^\infty(D)$. Moreover, $Ef$ is continuous on $\overline{D}$ if $f$ is continuous on $\overline{M}$.

**Theorem 5.** (Adachi[1], Elgueta[7]) There exists a linear extension operator $E : A^\infty(M) \to A^\infty(D)$.


Let $D$ be a bounded pseudoconvex domain in $\mathbb{C}^n$ with smooth boundary. Let $\gamma : \partial D \times D \to \mathbb{C}^n$ be a smooth mapping such that

\[(\zeta - z, \gamma) = \sum_{j=1}^{n}(\zeta_j - z_j)\gamma_j(\zeta, z) \neq 0 \quad \text{on} \quad \partial D \times D.\]

Let $h_1, \ldots, h_m (m < n)$ be holomorphic functions in a neighborhood $\tilde{D}$ of $\overline{D}$. Define

$\tilde{V} = \{z \in \tilde{D} | h_1(z) = \cdots = h_m(z) = 0\}$,  \quad $V = \tilde{V} \cap D$.

We say $V$ intersects $\partial D$ transversally if

$d\rho \wedge \partial h_1 \wedge \cdots \wedge \partial h_m \neq 0 \quad \text{on} \quad \partial V$.

In the above setting, we have the following:

**Theorem 6.** (Stout[19], Hatziafratis[9]) There is a smooth form $K_V(\zeta, z)$ on $\partial V \times V$ which is of type $(0,0)$ in $z$ and $(n-m-1, n-m)$ in $\zeta$ such that if $f$ is holomorphic in $V$ and continuous on $\overline{V}$, then for $z \in V$

\[(1) \quad f(z) = \int_{\zeta \in \partial V} f(\zeta) \frac{K_V(\zeta, z)}{(\zeta - z, \gamma(\zeta, z))^{n-m}}.\]

Moreover, $K_V(\zeta, z)$ is holomorphic in $z \in D$ provided that $\gamma(\zeta, z)$ is holomorphic in $z \in D$.

Let $D$ be a bounded convex domain with a defining function $\rho$. Then we can choose

\[\gamma_i(\zeta, z) = \frac{\partial \rho}{\partial \zeta_i}(\zeta).\]
Let $E(f)(z)$ be the right hand side of (1). Then we have

**Theorem 7.** (Adachi-Cho[3]) Let $D$ be a bounded convex domain in $\mathbb{C}^n$ with real analytic boundary and let $V$ be a one dimensional subvariety of $D$ defined above. Then we have

1. Let $1 \leq p < \infty$. If $f \in H^p(V)$, then $E(f) \in H^p(D)$. 
2. Suppose that $V$ has no singular points and $1 \leq p < \infty$. If $f \in \mathcal{O}(V) \cap L^p(V)$, then $E(f) \in \mathcal{O}(D) \cap L^p(D)$.

where $\mathcal{O}(V)$ (resp. $\mathcal{O}(D)$) denotes the space of all holomorphic functions in $V$ (resp. $D$).

A bounded domain $\Omega \subset \mathbb{C}^n$ is an analytic polyhedron with defining functions $\phi_j$ if

$$\Omega = \{z \in \mathbb{C}^n | |\phi_j(z)| < 1, j = 1, \ldots, N\},$$

where the defining functions $\phi_j$ are holomorphic in some neighborhood $\tilde{\Omega}$ of $\Omega$. We set $\sigma_I = \{z \in \bar{\Omega} | |\phi_j(z)| = 1, j \in I\}$. We say that $\Omega$ is non-degenerate if $\partial \phi_{i_1} \wedge \cdots \wedge \partial \phi_{i_k} \neq 0$ on $\sigma_I$ for every multiindex $I = \{i_1, \ldots, i_k\}$ such that $|I| = k \leq n$. We say that $\Omega$ is strongly non-degenerate if $\partial \phi_{i_1} \wedge \cdots \partial \phi_{i_k} \neq 0$ on $\sigma_I$ for all multiindices $I$. Let $\tilde{V}$ be a regular subvariety of $\tilde{\Omega}$ of codimension $m$ given by

$$\tilde{V} = \{z \in \tilde{\Omega} | h_1(z) = \cdots = h_m(z) = 0\},$$

where $h_j \in \mathcal{O}(\tilde{\Omega})$, and $\partial h_1 \wedge \cdots \wedge \partial h_m \neq 0$ on $\tilde{V}$. We set $V = \tilde{V} \cap \Omega$. We impose the transversal assumption that

$$\partial h_1 \wedge \cdots \wedge \partial h_m \wedge \partial \phi_{i_1} \wedge \cdots \partial \phi_{i_k} \neq 0 \quad \text{on} \quad \overline{V} \cap \sigma_I,$$

for every multiindex $I$ such that $|I| = k \leq n - m$. For a strongly non-degenerate polyhedron $\Omega$ we can define the Hardy spaces

$$H^p(\Omega) = \left\{ f \in \mathcal{O}(\Omega) | \sup_{\varepsilon > 0} \|f\|_{L^p(\sigma_\varepsilon)} < \infty \right\}.$$ 

In the above setting, we have by applying the integral formula obtained by Berndtsson[5]:

**Theorem 8.** (Adachi-Andersson-Cho[3])

1. Let $\Omega$ be a non-degenerate analytic polyhedron. For each $f \in \mathcal{O}(V) \cap L^p(V), 1 \leq p < \infty$, there exists $F \in \mathcal{O}(\Omega) \cap L^p(\Omega)$ such that $F(z) = f(z)$ for $z \in V$ and $\|F\|_{L^p(\Omega)} \leq C\|f\|_{L^p(V)}$. 

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(2) Let $\Omega$ be a strongly non-degenerate analytic polyhedron. Then for all $f \in H^p(V)$, $1 < p \leq \infty$, there exists $F \in H^p(\Omega)$ such that $F(z) = f(z)$ for $z \in V$ and $\|F\|_{H^p(\Omega)} \leq c\|f\|_{H^p(V)}$.

3. Outline of the proof of the theorem of Ohsawa-Takegoshi due to Berndtsson

In this section, we shall prove the extension theorem of Ohsawa-Takegoshi by following the Berndtsson’s proof[6]. Using $L^2$ space techniques, Ohsawa and Takegoshi obtained the following:

**Theorem 9.** (Ohsawa-Takegoshi[16]) Let $D$ be a bounded pseudoconvex domain in $\mathbb{C}^n$. We set $H = \{z \in \mathbb{C}^n|z_1 = 0\}$. Then there exists a constant $C$ which depends only on the diameter of $D$ such that, for any plurisubharmonic function $\varphi$ on $D$, and for any holomorphic function $f$ on $H \cap D$, there exists a holomorphic function $F$ in $D$ such that

$$F|_{H \cap D} = f, \quad \int_D |F|^2 e^{-\varphi} d\mu \leq C \int_{H \cap D} |f|^2 e^{-\varphi} d\mu,$$

where $d\mu$ and $d\mu_1$ are Lebesgue measures in $\mathbb{C}^n$ and $\mathbb{C}^{n-1}$, respectively.

**Lemma 1.** (Hörmander[14]) Let $D$ be a bounded open set in $\mathbb{C}^n$ with smooth boundary $\partial D$ and let $\rho$ be a smooth defining function for $D$. For $f = \sum_{j} f_j dz^j \in C_{(q)}^1(0, \overline{D})$ and $u = \sum_{K} u_K d\overline{z}^K \in C_{(0, q-1)}^1(\overline{D})$, the following equality is valid

$$(\partial u, f) = -\int_D \sum_{j=1}^{n} u_k \delta_j f_{jK} e^{-\varphi} d\mu + \int_{\partial D} \sum_{j=1}^{n} u_k \frac{\partial \rho}{\partial z_j} e^{-\varphi} dS.$$

**Definition.** For $u \in C^1(D)$, define

$$\delta_j u = e^\varphi \frac{\partial}{\partial z_j} (ue^{-\varphi}) = \frac{\partial u}{\partial z_j} - \frac{\partial \varphi}{\partial z_j} u, \quad \partial_k u = \frac{\partial u}{\partial z_k}, \quad \bar{\partial} u = \frac{\partial u}{\partial \overline{z}_k}.$$
We say $f$ satisfies the boundary condition if $f \in \text{Def}(\bar{\partial}^*)$. When $f$ satisfies the boundary condition, we have from lemma 1
\[(\bar{\partial}u, f) = (u, \bar{\partial}^* f).\]

**Lemma 2.** (Hörmander[14]) Let $\alpha = \sum_{|J|=q} \alpha_J \bar{dz}^J$ be a smooth $(0,q)$-form in $\overline{D}$ and $\alpha \in \text{Def}(\bar{\partial}^*)$. For $\varphi \in C^\infty(\overline{D})$ we have
\[
\|\bar{\partial}^{*}\alpha\| + \|\bar{\partial}\alpha\|^2 = \sum_{k, j=1}^{n} \int_{\partial D} \alpha_{j} \bar{\alpha}_{k} \bar{\partial}_{z_j} e^{-\varphi} d\mu + \sum_{j=1}^{n} \int_{D} \left| \frac{\partial \alpha_{j}}{\partial z_j} \right|^2 e^{-\varphi} d\mu
\]
\[
+ \sum_{k, j=1}^{n} \int_{\partial D} \alpha_{j} \bar{\alpha}_{k} \bar{\partial}_{z_j} e^{-\varphi} dS_{|\partial \rho|}.
\]

We assume that $\varphi$ is a smooth function in $\overline{D}$ from lemma 3 to lemma 7. Thus $f \in L^2(D, \varphi)$ means $f \in L^2(D)$. We omit the proof of lemma 3, since the detailed proof of lemma 3 is given in [6].

**Lemma 3.** Let $w$ be a real valued smooth function in $\overline{D}$. $\alpha = \sum_{j=1}^{n} \alpha_j \bar{dz}_j$ is a smooth $(0,1)$-form in $\overline{D}$ satisfying the boundary condition. Then we have
\[
\int_{D} w \sum_{j,k=1}^{n} \varphi_{j} \alpha_{j} \bar{\alpha}_{k} e^{-\varphi} d\mu - \int_{D} w' \sum_{j,k=1}^{n} \alpha_{j} \bar{\alpha}_{k} e^{-\varphi} d\mu
\]
\[
+ \int_{D} w |\bar{\partial}^{*}\alpha|^2 e^{-\varphi} d\mu + \int_{D} w \sum_{j,k=1}^{n} \left| \frac{\partial \alpha_{j}}{\partial z_j} \right|^2 e^{-\varphi} d\mu + \int_{\partial D} w \sum_{j,k=1}^{n} \rho_{j} \alpha_{j} \bar{\alpha}_{k} e^{-\varphi} dS_{|\partial \rho|}
\]
\[
= 2\text{Re} \int_{D} w \bar{\partial} \bar{\partial}^{*}\alpha \cdot \bar{\alpha} e^{-\varphi} d\mu + \int_{\partial D} w |\bar{\partial} \alpha|^2 e^{-\varphi} d\mu.
\]

**Definition.** Let $\psi \in C^\infty(\overline{D})$ and $\alpha = \sum_{j=1}^{n} \alpha_j \bar{dz}_j \in C^\infty_{(0,1)}(\overline{D})$. We define the inner product of $g = \psi \bar{\partial} \left( \frac{1}{z_1} \right)$ and $\alpha$ by
\[
< g, \alpha > = \sum_{j=1}^{n} \left< \psi \frac{\partial}{\partial z_j} \left( \frac{1}{z_1} \right), \alpha_j \right> = \lim_{\varepsilon \to 0} \int_{D} \psi(z) \frac{\partial}{\partial z_1} \left( \frac{\bar{z}_1}{|z_1|^2 + \varepsilon} \right) \overline{\alpha_1(z)} e^{-\varphi(z)} d\mu(z).
Moreover, if we define
\[ h_\varepsilon(z) = \psi(z) \bar{\partial} \left( \frac{z_i}{|z_i|^2 + \varepsilon} \right), \]
then we obtain
\[ (2) \quad < g, \alpha > = \lim_{\varepsilon \to 0} < h_\varepsilon, \alpha >. \]

In view of lemma 6, the right hand side of (2) exists. For \( u \in L^1(D) \) and a \((0,1)\)-form \( \alpha \) in \( D \) with compact support, we define
\[ < \bar{\partial} u, \alpha > = (u, \bar{\partial}^* \alpha). \]

Then we have the following:

**Lemma 4.** Let \( D \) be a bounded strictly pseudoconvex domain in \( \mathbb{C}^n \) with smooth boundary. Let \( f \) be a holomorphic function in \( \overline{D} \) and \( g = f \bar{\partial} \left( \frac{1}{z_i} \right) \). Let \( u \in L^1(D) \). If the equality
\[ < g, \alpha > = \int_D u \bar{\partial}^* \alpha e^{-\varphi} d\mu \]
holds for any \( \bar{\partial} \) closed \( \alpha \in C_{(0.1)}^\infty(\overline{D}) \) which satisfies the boundary condition, then \( g = \bar{\partial} u \) in the sense of distribution.

**Proof.** Let \( \alpha \) be a \( C^\infty(0,1) \)-form in \( D \) with compact support. We define
\[ \text{Def}(\bar{\partial}) = \{ g \in L^2_{(0,0)}(D, \varphi) \mid \bar{\partial} g \in L^2_{(0,q+1)}(D, \varphi) \}. \]
For Laplace-Beltrami operator \( \Box = \bar{\partial}\bar{\partial}^* + \bar{\partial}^*\bar{\partial} : L^2_{(0,1)}(D, \varphi) \to L^2_{(0,1)}(D, \varphi) \), define
\[ \text{Def}(\Box) = \{ \alpha \in L^2_{(0,1)}(D, \varphi) \mid \alpha \in \text{Def}(\bar{\partial}), \bar{\partial}^* \alpha \in \text{Def}(\bar{\partial}^*), \alpha \in \text{Def}(\bar{\partial}^*), \bar{\partial}^* \alpha \in \text{Def}(\bar{\partial}) \}, \]
\[ \mathcal{H} = \{ \alpha \in \text{Def}(\Box) \mid \Box \alpha = 0 \}. \]
Then \( \mathcal{H} \) is a closed subspace of the Hilbert space \( L^2_{(0,1)}(D, \varphi) \). Let \( H : L^2_{(0,1)}(D, \varphi) \to \mathcal{H} \) be the orthogonal projection. From the theory of the \( \bar{\partial} \) Neumann problem, there exists Neumann operator \( \mathcal{N} : L^2_{(0,1)}(D, \varphi) \to \text{Def}(\Box) \) such that
\[ \alpha = \bar{\partial}\bar{\partial}^* \mathcal{N} \alpha + \bar{\partial}^* \bar{\partial} \mathcal{N} \alpha + H \alpha. \]

For \( \beta \in \mathcal{H} \), we have
\[ 0 = (\Box \beta, \beta) = (\bar{\partial}\bar{\partial}^* \beta, \beta) + (\bar{\partial}^* \bar{\partial} \beta, \beta) = \| \bar{\partial}^* \beta \|^2 + \| \bar{\partial} \beta \|^2. \]
Hence we obtain $\bar{\partial}\beta = \bar{\partial}^{*}\beta = 0$. From lemma 2, it holds that

$$0 = \|\bar{\partial}\beta\|^2 + \|\bar{\partial}^{*}\beta\|^2 \geq \sum_{j,k=1}^{n} \left| \frac{\partial \beta_j}{\partial \bar{z}_k} \right|^2 + \int_{\partial D} \sum_{j,k=1}^{n} \frac{\partial^2 \rho}{\partial z_j \partial \bar{z}_k} \beta_j \bar{\beta}_k \frac{dS}{|\partial \rho|}$$

$$\geq \sum_{j,k=1}^{n} \left| \frac{\partial \beta_j}{\partial \bar{z}_k} \right|^2 + c \int_{\partial D} |\beta|^2 \frac{dS}{|\partial \rho|}.$$ 

Thus $\beta_j$ is holomorphic in $D$ and $0$ in $\partial D$ so that $\beta = 0$. Therefore $\mathcal{H} = 0$. We set

$$\alpha_1 = \bar{\partial}\bar{\partial}^{*}\alpha, \quad \alpha_2 = \bar{\partial}^{*}\bar{\partial} \alpha.$$ 

Since Neumann operator maps smooth $(0,1)$-forms to smooth $(0,1)$-forms in the strictly pseudoconvex domain $D$, $\alpha_1$ and $\alpha_2$ are both smooth $(0,1)$-forms in $\overline{D}$. Obviously, $\bar{\partial}\alpha_1 = 0$. If $\bar{\partial}\beta = 0$, then by lemma 1 $(\beta, \alpha_2) = (\bar{\partial}\beta, \bar{\partial} \mathcal{N} \alpha) = 0$. Hence $\alpha_2 \perp \text{Ker}(\bar{\partial})$. On the other hand, from lemma 1, for any smooth function $\beta$ on $\overline{D}$, we have

$$0 = (\bar{\partial}\beta, \alpha_2) = (\beta, \bar{\partial}^{*}\alpha_2) + \int_{\partial D} \beta \alpha_2 \cdot \bar{\partial} e^{-\varphi} \frac{dS}{|\partial \rho|}.$$ 

Thus $\bar{\partial}^{*}\alpha_2 = 0$. Therefore $\alpha_2$ satisfies the boundary condition. Hence, $\alpha_1$ satisfies the boundary condition. If we set

$$h_\varepsilon(z) = f(z) \bar{\partial}\left( \frac{\overline{z_1}}{|z_1|^2 + \varepsilon} \right),$$
then we have

$$< g, \alpha_2 > = \lim_{\varepsilon \to 0} (h_\varepsilon, \alpha_2) = 0.$$ 

Thus we have

$$< g, \alpha > = < g, \alpha_1 > = \int_{D} u \bar{\partial}^{*}\alpha_1 e^{-\varphi} d\mu = \int_{D} u \bar{\partial}^{*}\alpha e^{-\varphi} d\mu = (u, \bar{\partial}^{*}\alpha) = < \bar{\partial}u, \alpha >,$$

which means $g = \bar{\partial}u$.

**Lemma 5.** Let $g$ be the same as in lemma 4. Let $\lambda$ be a non-negative real valued function in $D$ with the property that $\frac{1}{\lambda}$ is integrable. If the inequality

$$|< g, \alpha >|^2 \leq C \int_{D} |\bar{\partial}^{*}\alpha|^2 \frac{e^{-\varphi}}{\lambda} d\mu$$

holds for any $\bar{\partial}$ closed $\alpha \in C_{(0,1)}^\infty(D)$ which satisfies the boundary condition, then there exists $u \in L^1(D, \varphi)$ such that

$$\bar{\partial}u = g, \quad \int_{D} |u|^2 \lambda e^{-\varphi} d\mu \leq C.$$
Proof. Let $C_{b}^{\infty}(\overline{D})$ be the space of all $\bar{\partial}$ closed $C^{\infty}(0,1)$-forms in $\overline{D}$ which satisfies the boundary condition. We set

$$F = \{ \bar{\partial}^{*}\alpha | \alpha \in C_{b}^{\infty}(\overline{D}) \}, \quad \varphi_{1} = \frac{e^{-\varphi}}{\lambda}.$$  

Then, $F$ is a vector subspace of $L^{2}(D, \varphi_{1})$. For $w \in F$, there exists $\alpha \in C_{b}^{\infty}(\overline{D})$ such that $w = \bar{\partial}^{*}\alpha$. We define

$$\Phi(w) = \langle g, \alpha \rangle.$$

Then $\Phi(w)$ is independent of the choice of $\alpha$. Also, we have

$$|\Phi(w)|^{2} \leq C\|w\|_{\varphi_{1}}^{2}, \quad \|\Phi\| \leq \sqrt{C}.$$  

Thus $\Phi$ is a bounded anti-linear operator on $F$. From the Hahn-Banach theorem, $\Phi$ is extended to a bounded anti-linear operator on $L^{2}(D, \varphi_{1})$. From the Riesz representation theorem, there exists $v \in L^{2}(D, \varphi_{1})$ such that

$$\Phi(w) = (v, w)_{\varphi_{1}}, \quad \|v\|_{\varphi_{1}} = \|\Phi\| \leq \sqrt{C}.$$  

Therefore we have

$$\langle g, \alpha \rangle = \Phi(w) = (v, w)_{\varphi_{1}} = \int_{D} \overline{v} \bar{\partial}^{*} e^{\varphi} \frac{e^{-\varphi}}{\lambda} \lambda, \quad \int_{D} |v|^{2} e^{\varphi} \frac{e^{-\varphi}}{\lambda} d\mu = \|v\|_{\varphi_{1}}^{2} \leq C.$$  

If we set $u = \frac{v}{\lambda}$, then

$$\int_{D} |u|^{2} e^{\varphi} \frac{e^{-\varphi}}{\lambda} d\mu \leq C, \quad \langle g, \alpha \rangle = \int_{D} \overline{u} \bar{\partial}^{*} e^{-\varphi} d\mu.$$  

On the other hand, we have

$$\int_{D} |u| e^{-\varphi} d\mu \leq \int_{D} \frac{|v|^{2}}{\lambda} e^{-\varphi} d\mu \int_{D} e^{-\varphi} \frac{e^{-\varphi}}{\lambda} d\mu \leq C \int_{D} e^{-\varphi} \frac{e^{-\varphi}}{\lambda} d\mu < \infty.$$  

Thus, $u \in L^{1}(D, \varphi)$. From lemma 4, we obtain $\bar{\partial} u = g$.

**Lemma 6.** For $\varphi \in C^{\infty}(\overline{D})$, it holds that

$$\lim_{\varepsilon \to 0} \int_{D} \frac{\varepsilon}{|z_{1}|^{2} + \varepsilon^{2}} \varphi(z) d\mu(z) = \pi \int_{\{z_{1} = 0\} \cap D} \varphi(z) d\mu_{1}(z),$$

where $d\mu$ and $d\mu_{1}$ are Lebesgue measures in $\mathbb{C}^{n}$ and $\mathbb{C}^{n-1}$, respectively.
Lemma 7. Let $D$ be a bounded strictly pseudoconvex domain in $\mathbb{C}^n$ with smooth boundary and $D \subset \{ |z| \leq 1 \}$. Let $\varphi$ be a smooth plurisubharmonic function in $D$ and let $\alpha$ be a $\overline{\partial}$ closed smooth $(0,1)$-form in $D$ which satisfies the boundary condition. Then, for $0 < \delta < 1$, we have

$$\int_{\{z_1=0\} \cap D} |\alpha_1|^2 e^{-\varphi} d\mu_1 \leq \frac{2}{\pi} \left(1 + \frac{1}{\delta^2}\right) \int_D |\overline{\partial}^* \alpha|^2 e^{-\varphi} d\mu.$$

Proof. For $0 < \delta < 1$, we set

$$w^\delta = 1 - |z_1|^{2\delta} = 1 - (z_1 \bar{z}_1)^{\delta}.$$

From Lemma 3, we have

$$\int_D w^\delta \sum_{j,k=1}^n \varphi_{j\alpha_k} \alpha_k e^{-\varphi} d\mu + \delta^2 \int_D |z_1|^{2\delta - 2} |\alpha_1|^2 e^{-\varphi} d\mu + \int_D w^\delta |\overline{\partial}^* \alpha|^2 e^{-\varphi} d\mu$$

$$+ \int_D w^\delta \sum_{j,k=1}^n \left| \frac{\partial \alpha_j}{\partial \bar{z}_k} \right|^2 e^{-\varphi} d\mu + \int_{\partial D} w^\delta \sum_{j,k=1}^n p_{j\alpha_k} \alpha_k e^{-\varphi} dS_{\partial \rho} \leq 2 \text{Re} \int_D w^\delta \overline{\partial}^* \alpha \cdot \bar{\alpha} e^{-\varphi} d\mu.$$

Hence we have

$$\delta^2 \int_D |z_1|^{2\delta - 2} |\alpha_1|^2 e^{-\varphi} d\mu + \int_D w^\delta |\overline{\partial}^* \alpha|^2 e^{-\varphi} d\mu \leq 2 \text{Re} \int_D w^\delta \overline{\partial}^* \alpha \cdot \bar{\alpha} e^{-\varphi} d\mu$$

$$= 2 \text{Re} (\overline{\partial}^* \alpha, \overline{\partial}^* (w^\delta \alpha)) = 2 \text{Re} (\overline{\partial}^* \alpha, \overline{\partial}^* \alpha - \sum_{j=1}^n \frac{\partial w^\delta}{\partial z_j} \alpha_j)$$

$$= 2 \int_D w^\delta |\overline{\partial}^* \alpha|^2 e^{-\varphi} d\mu - 2 \text{Re} \int_D \overline{\partial}^* \alpha \frac{\partial w^\delta}{\partial z_1} \alpha \bar{e}^{-\varphi} d\mu$$

$$\leq 2 \int_D w^\delta |\overline{\partial}^* \alpha|^2 e^{-\varphi} d\mu + 2 \int_D \overline{\partial}^* \alpha \delta |z_1|^{2\delta - 1} |\alpha_1| e^{-\varphi} d\mu$$

$$\leq 2 \int_D w^\delta |\overline{\partial}^* \alpha|^2 e^{-\varphi} d\mu + 2 \int_D |\overline{\partial}^* \alpha|^2 |z_1|^{2\delta - 1} e^{-\varphi} d\mu + \frac{1}{2} \int_D \delta^2 |\alpha_1|^2 |z_1|^{2\delta - 2} e^{-\varphi} d\mu.$$

Thus we have

$$\frac{1}{2} \delta^2 \int_D |z_1|^{2\delta - 2} |\alpha_1|^2 e^{-\varphi} d\mu \leq \int_D (1 - |z_1|^{2\delta}) |\overline{\partial}^* \alpha|^2 e^{-\varphi} d\mu + 2 \int_D |\overline{\partial}^* \alpha|^2 |z_1|^{2\delta} e^{-\varphi} d\mu$$

$$= \int_D |\overline{\partial}^* \alpha|^2 e^{-\varphi} d\mu + \int_D |\overline{\partial}^* \alpha|^2 |z_1|^{2\delta} e^{-\varphi} d\mu \leq 2 \int_D |\overline{\partial}^* \alpha|^2 e^{-\varphi} d\mu.$
Therefore, for $0 < \delta < 1$, we obtain

$$\delta^2 \int_D |z_1|^{25-2} |\alpha_1|^2 e^{-\varphi} d\mu \leq 4 \int_D |\bar{\partial}^* \alpha|^2 e^{-\varphi} d\mu.$$  

On the other hand, we set

$$w_\epsilon = \frac{1}{\pi} \log \frac{1}{|z_1|^2 + \epsilon}, \quad w = \frac{1}{\pi} \log \frac{1}{|z_1|^2}.$$  

We apply lemma 3 to $w_\epsilon$ and let $\epsilon \to 0$, then by lemma 6

$$\int_{\{z_1=0\} \cap D} |\alpha_1|^2 e^{-\varphi} d\mu_1 + \int_D w_\epsilon |\bar{\partial}^* \alpha|^2 e^{-\varphi} d\mu \leq 2 \text{Re} \int_D w \bar{\partial}^* \alpha \cdot \bar{\alpha} e^{-\varphi} d\mu.$$  

By the same calculation as the first part and applying (3) to $0 < \delta < 1$, we have

$$\int_{\{z_1=0\} \cap D} |\alpha_1|^2 e^{-\varphi} d\mu_1 \leq \frac{1}{\pi} \int_D \log \frac{1}{|z_1|^2} |\bar{\partial}^* \alpha|^2 e^{-\varphi} d\mu + \frac{2}{\pi} \int_D |\bar{\partial}^* \alpha| \frac{|\alpha_1|}{|z_1|} e^{-\varphi} d\mu$$

$$\leq \frac{1}{\pi} \int_D \log \frac{1}{|z_1|^2} |\bar{\partial}^* \alpha|^2 e^{-\varphi} d\mu + \frac{2}{\pi} \int_D |\bar{\partial}^* \alpha|^2 e^{-\varphi} d\mu + \frac{1}{2\pi} \int_D |\bar{\partial}^* \alpha|^2 |z_1|^{2\delta} e^{-\varphi} d\mu$$

$$\leq \frac{1}{\pi} \int_D \log \frac{1}{|z_1|^2} |\bar{\partial}^* \alpha|^2 e^{-\varphi} d\mu + \frac{2}{\pi} \int_D |\bar{\partial}^* \alpha|^2 |z_1|^{2\delta} e^{-\varphi} d\mu$$

Using the fact that $x \left(\log \frac{1}{x} + 2\right) \leq 2$ for $0 < x \leq 1$, we have

$$\int_{\{z_1=0\} \cap D} |\alpha_1|^2 e^{-\varphi} d\mu_1 \leq \frac{2}{\pi} \int_D |\bar{\partial}^* \alpha|^2 |z_1|^{2\delta} e^{-\varphi} d\mu + \frac{1}{\pi \delta^2} \int_D |\bar{\partial}^* \alpha|^2 e^{-\varphi} d\mu$$

which completes the proof.

**Lemma 8.** Let $D$ be a pseudoconvex domain in $\mathbb{C}^n$ and $X = \{z \in D | z_1 = 0\}$. Let $f$ be a holomorphic function in $X$. If $H$ is locally integrable in $D$ and satisfies $\bar{\partial} H = f \bar{\partial} \left(\frac{1}{z_1}\right)$, then there exists a holomorphic function $\tilde{H}$ in $D$ such that $\tilde{H}(z) = z_1 H(z)$ a.e. and $\tilde{H}(z) = f(z)$ for $z \in X$. 
Proof. There exists a neighborhood $\omega$ of $X$ in $D$ such that $f$ can be extended to be holomorphic in $\omega$. Let $\chi \in C^\infty(D)$ be a function such that $\chi = 1$ in a neighborhood of $X$ in $\omega$, supp($\chi$) $\subset \omega$ and $0 \leq \chi \leq 1$ in $D$. We set

$$\omega = \frac{\bar{\partial} \chi}{z_1}.$$ 

Then $\omega$ satisfies that $\omega \in C^\infty_{(0,1)}(D)$, $\bar{\partial} \omega = 0$. Define

$$G = \frac{\chi f}{z_1} - H,$$

then $G$ is locally integrable. Since we have

$$\bar{\partial} G = \bar{\partial} (\chi f) - \frac{1}{z_1} \bar{\partial} \chi - \frac{1}{z_1} f \bar{\partial} \chi = \bar{\partial} \frac{f}{z_1},$$

there exists a smooth function $\tilde{G}$ in $D$ such that $\tilde{G} = G$ a.e. We set

$$\chi(z)f(z) - z_1 \tilde{G}(z) = \tilde{H}(z),$$

then we have $z_1 H(z) = \tilde{H}(z)$ a.e. and $\tilde{H}(z) = f(z)$ for $z \in X$. Moreover we have

$$\bar{\partial} \tilde{H}(z) = (\bar{\partial} \chi(z))f(z) - z_1 \bar{\partial} \tilde{G}(z) = (\bar{\partial} \chi(z))f(z) - z_1 \omega(z) = 0.$$ 

Hence $\tilde{H}(z)$ is holomorphic in $D$.

Lemma 9. Let $D$ be an open set in $\mathbb{C}^n$ and let $K \subset D$ be a compact set. Then there exists a constant $C$ such that for any holomorphic function $f$ in $D$ and any neighborhood $\omega$ of $K$

$$\sup_K |f| \leq C \|f\|_{L^1(\omega)}.$$ 

Lemma 10. Let $\{u_k\}$ be a sequence of holomorphic functions in $D$ which are uniformly bounded on any compact subset of $D$. Then there exists a subsequence $\{u_{k_j}\}$ of $\{u_k\}$ such that $\{u_{k_j}\}$ converges uniformly on any compact subset of $D$ to a holomorphic function in $D$.

Theorem 10. (Berndtsson[6]) Let $D$ be a bounded pseudoconvex domain in $\mathbb{C}^n$ and let $\varphi$ be a plurisubharmonic function in $D$. We set $X = \{z \in D|z_1 = 0\}$. Suppose that
$D \subset \{ z \in \mathbb{C}^n \mid |z_1| \leq A \}$. If $f$ is holomorphic in $X$, then there exists a holomorphic function $F$ in $D$ such that

$$F|_X = f, \quad \int_D |F|^2 e^{-\varphi} d\mu \leq 4A^2 \pi \int_X |f|^2 e^{-\varphi} d\mu_1,$$

where $d\mu$ and $d\mu_1$ are Lebesgue measures in $\mathbb{C}^n$ and $\mathbb{C}^{n-1}$, respectively.

**Proof.** Without loss of generality, we may assume that $A = 1$. There exists an increasing sequence of bounded strictly pseudoconvex domains in $\mathbb{C}^n$ with smooth boundary such that $\overline{D_n} \subset \subset D$ and $\bigcup_{n=1}^{\infty} D_n = D$. Let $\{ \varphi_n \}$ be a sequence of $C^\infty$ plurisubharmonic functions in $\overline{D_n}$ such that $\varphi_n \downarrow \varphi$. We set $g = f \overline{\partial} \left( \frac{1}{z_1} \right)$. Let $\alpha$ be a $\overline{\partial}$ closed $(0,1)$-form which satisfies the boundary condition on $\partial D_n$. From lemma 7, we have

$$| < g, \alpha >_{\varphi_n} |^2 = \lim_{\epsilon \to 0} \int_{D_n} f \frac{\varphi_n}{(|z_1|^2 + \epsilon)^2} \overline{\alpha_1} e^{-\varphi} d\mu = \pi f \overline{\alpha_1} e^{-\varphi} d\mu_1^2 \leq \pi^2 \int_{\{z_1 = 0\} \cap D_n} |f|^2 e^{-\varphi} d\mu_1 \leq 2\pi (1 + \frac{1}{\delta^2}) \int_{\{z_1 = 0\} \cap D_n} |f|^2 e^{-\varphi} d\mu_1.$$

From lemma 5, there exist integrable functions $u^n_{\delta}$ in $D_n$ such that

$$\overline{\partial} u^n_{\delta} = g, \quad \int_{D_n} |u^n_{\delta}|^2 |z_1|^{2\delta} e^{-\varphi} d\mu \leq 2\pi \left( 1 + \frac{1}{\delta^2} \right) \int_{\{z_1 = 0\} \cap D_n} |f|^2 e^{-\varphi} d\mu_1.$$

We set $F^n_{\delta} = u^n_{\delta} z_1$. Then, from lemma 8, $F^n_{\delta}$ are holomorphic in $D_n$ and satisfy $F^n_{\delta}|_{\{z_1 = 0\} \cap D_n} = f|_{\{z_1 = 0\} \cap D_n}$. Suppose that

$$\int_X |f|^2 e^{-\varphi} d\mu_1 = C < \infty,$$

then it holds that

$$\int_{D_n} |F^n_{\delta}|^2 e^{-\varphi} d\mu = \int_{D_n} |u^n_{\delta}|^2 |z_1|^{2\delta} e^{-\varphi} d\mu \leq \int_{D_n} |u^n_{\delta}|^2 |z_1|^{2\delta} e^{-\varphi} d\mu \leq 2\pi \left( 1 + \frac{1}{\delta^2} \right) C.$$

Therefore, for some fixed $n$, there exists a constant $C_1$ such that

$$\int_{D_n} |F^n_{\delta}|^2 d\mu \leq C_1.$$

\[\square\]
From lemma 9.10, there exists a sequence \( \{\delta_j\} \) with \( \delta_j \to 1 \) such that \( F^n_{\delta_j} \) converges uniformly on any compact subset of \( D_n \) to \( F^n \). Then \( F^n \) are holomorphic in \( D_n \) and satisfy \( F^n|_{\{z_1=0\} \cap D_n} = f|_{\{z_1=0\} \cap D_n} \). Moreover, we have
\[
\int_{D_n} |F^n|^2 e^{-\varphi_n} d\mu \leq 4\pi C.
\]
Let \( K \) be a compact subset of \( D \). There exists a natural number \( N \) such that \( K \subset D_n, (n \geq N) \). If we set
\[
M_n = \min_{D_n} e^{-\varphi_n},
\]
then, for \( n \geq N \), there exist a constant \( C_2 \) such that
\[
4\pi C \geq \int_{D_n} |F^n|^2 e^{-\varphi_n} d\mu \geq M_N \int_{D_n} |F^n|^2 d\mu \geq C_2 \sup_{K} |F^n|^2.
\]
Thus \( \{F^n\} \) are uniformly bounded on any compact subset of \( D \). Then we can find a subsequence \( \{F^{k_n}\} \) of \( \{F^n\} \) which converges uniformly on any compact subset of \( D \). We set \( \lim_{n \to \infty} F^{k_n} = F \). Then \( F \) is holomorphic in \( D \) and satisfies \( F|_X = f \). For any compact subset \( K \) of \( D \), we have
\[
\int_K |F|^2 e^{-\varphi} d\mu = \lim_{n \to \infty} \int_K |F^{k_n}|^2 e^{-\varphi_{k_n}} d\mu \leq 4\pi C,
\]
which completes the proof.

**Remark.** Siu[18] also obtained another proof of the theorem of Ohsawa-Takegoshi in which the constant \( C = \frac{64}{3} \pi A^2 \left(1 + \frac{4}{\pi} \right)^{1/2} \) provided \( D \subset \{ |z| \leq A \} \).

**References**


